Fiscal Multiplier: the Size of the Shock Matters *

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Preliminary

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Abstract

This paper studies the impact of the sign and magnitude of fiscal shocks on the fiscal multiplier. Through a theoretical examination, it highlights the significance of both the sign and magnitude of the shock in determining the multiplier. The study introduces a new empirical methodology, the Local Linear Local Projection, to detect complex non-linear patterns. When applied to US data, the methodology reveals that the degree of nonlinearity captured in the data varies with the identification strategy employed. Notably, zero does not appear to be a significant tipping point in the nonlinearity of the fiscal multiplier.

Keywords: Fiscal Multipliers, Local Projections, Non-linear Econometrics.

JEL Codes: E62, E32, C32, C52

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1 Introduction

The significance of the government spending multiplier for policy formulation is undeniable, and it remains a topic of vigorous debate. Defined as the output response to a one-dollar change in government spending, fiscal multipliers discussed in recent studies are predominantly state-dependent. Various states have been identified as potential sources of non-linearities in both theoretical and empirical research. In this study, I propose that the size —magnitude and/or sign— of the fiscal shock is a critical, yet often disregarded, factor contributing to non-linearity[1]. The potential existence of nonlinearities in government spending multipliers, relative to the size of fiscal shocks, has significant policy implications. For instance, following an example highlighted in Ben Zeev, Ramey, & Zubairy (2023), should larger spending initiatives consistently yield higher multipliers compared to smaller ones, adopting a policy of large-scale spending, coupled with future fiscal policies of lower contraction, might prove beneficial overall. Therefore, it is imperative to confirm the existence of such nonlinearities, and describe their nature in detail.

Through an examination of theoretical models rationalizing state-dependence in fiscal multipliers, I underscore the significance of the fiscal shock's size as a source of nonlinearities. Importantly, the size of the shock remains influential irrespective of the original state-dependence characteristics inherent in the model. Drawing upon this common feature in plausible theories of the state-dependent fiscal multiplier, this paper proposes a new empirical methodology —Local Linear Local Projection (LL-LP)— aiming to discern the range of effects caused by alterations in government spending, especially in terms of the size of these fiscal actions. To effectively detect nonlinearities, I build on local linear regression (LLR), which, unlike traditional econometric techniques commonly employed when estimating the fiscal multiplier, facilitates the discovery of more complex non-linear patterns beyond the scope of two or three distinct regimes.

Recent studies have analyzed the fiscal shock's role in creating nonlinearity within the fiscal multiplier. Barnichon, Debortoli, & Matthes (2022), employing Functional Approximations

[1] In this paper, size nonlinearity is conceptualized as encompassing both sign and magnitude nonlinearities. The term 'sign nonlinearity' refers to nonlinear effects that arise solely from the sign of the shock, while 'magnitude nonlinearity' refers to nonlinear effects that stem exclusively from the absolute value of the shock.

of Impulse Responses (FAIR Barnichon & Matthes (2018)), argue that negative government spending leads to a larger fiscal multiplier than positive spending. This conclusion, however, faces scrutiny from Ben Zeev, Ramey, & Zubairy (2023), who identified variations in individual Impulse Response Functions (IRFs) for positive versus negative shocks, but show that these do not translate into differences in multipliers. Their methodology, rooted in the work of Zeev (2020) and Forni, Gambetti, Maffei-Faccioli, & Sala (2023), introduces an absolute value and quadratic term to the shock within the framework of Ramey & Zubairy (2018), aiming to detect sign and magnitude nonlinear effects. However, as highlighted in Caravello & Bruera (2024, WP), sign and size[2] effects can be properly separated when the distribution of shocks is symmetric, which is not the case for military news shocks used in Ben Zeev et al. (2023).

The distinction between past work and this paper is two-fold. While past work focus on the theoretical relevance of the shock sign through specific models and channels, my research extends this view by demonstrating that nonlinearities due to shock size —encompassing both sign and magnitude— are almost universally present, regardless of the source of state-dependence in the fiscal multiplier. Furthermore, this paper introduces Local Linear Local Projections (LL-LP) aimed at capturing a broader range of nonlinearities than those depicted by a quadratic term in the shock or a small number of regimes. In addition, interpretations about the magnitude or the sign of the shock emerging from the proposed methodology are immune to the fact that the distribution of military news shocks is not symmetric.

I demonstrate the proficiency of the LL-LP in identifying a broader spectrum of nonlinear effects in simulation studies. Then, applying this methodology to data for the US, my findings depend on the identification strategy that is used. When using military news shocks to identify exogenous government spending, I uncover only mild nonlinearities while nonlinearities are more pronounced when relying on the recursive approach for identification. Overall, results suggest that zero is not a significant tipping point in the nonlinearity of the fiscal multiplier. Fiscal multipliers generated by small positive shocks or small negative shocks do not exhibit dramatically different effects, contrary to what might be suggested by an analysis focused solely on the sign of the shock as in Barnichon et al. (2022).

[2] Note that the definition of size nonlinearity in Caravello & Bruera (2024, WP) relates to magnitude nonlinearity in this paper. In this paper, size non-linearity relates to both sign and/or magnitude nonlinearity.

Section 2 reviews structural models with state-dependent fiscal multipliers, focusing on the model by Barnichon et al. (2022) to guide empirical methodology. Section 3 analyzes a simple dynamic system to illustrate key conditions under which the size of the shock matters. In Section 4, I introduce the LL-LP methodology and explain how to recover fiscal multipliers. In Section 5, I apply the methodology to US data. Section 6 presents a simulation study to evaluate the proposed methodology against other state-dependent estimators using two different data generating processes. Section 7 concludes.

2 State-dependent fiscal multiplier in structural models

In this section, I mention several structural models that feature state-dependent fiscal multipliers. In particular, I scrutinize the model developed in Barnichon, Debortoli, & Matthes (2022) as it will be used for simulating data in a section 5. The goal is to gain a deeper insight into the various forms of state-dependence that theoretical research has revealed. We are especially interested in exploring the different functional forms that fiscal multipliers assume in available theories. This theoretical perspective is vital to guide our empirical methodology for estimating fiscal multipliers, a necessary approach due to the limited macroeconomic data, which hampers a completely data-driven analysis of state-dependence.

2.1 Literature review on models of state-dependent fiscal multipliers

In recent decades, the concept of the fiscal multiplier has been increasingly scrutinized, particularly its state-dependent nature, within the structural modeling framework of macroeconomics. This body of research offers varied perspectives on the determinants and characteristics of the fiscal multiplier.

Christiano, Eichenbaum, & Rebelo (2011) provided a foundational understanding of the fiscal multiplier at the zero-lower bound (ZLB). Their research reveals a dynamic multiplier, fluctuating based on the probability of remaining at the ZLB, with a functional form resembling an exponential curve. This study mark a significant step in understanding the complex interplay between fiscal policy efficacy and macroeconomic conditions at the ZLB. Bilbiie, Monacelli, & Perotti (2013) shows, in a model with financial imperfection and heterogeneous agents, that a debt-financed tax cut may imply a multiplier that features a exponential functional form in the share of borrowers in the economy. Michaillat (2014) introduced a model wherein the fiscal multiplier is exponentially related to unemployment levels by modeling the labour market as a search-and-matching framework. Shen & Yang (2018) build business cycle-dependent fiscal multiplier as a model with involuntary unemployment due to binding downward nominal wage rigidity in recessions and full employment in expansions. These advancements in the literature underscored the critical role of labor market conditions in determining the impact of fiscal policy. Canzoneri et al. (2016) show that business cycle-dependent multipliers can arise from countercyclical changes in bank intermediation costs. They suggest that increased government spending has the effect of compressing interest rate spreads, which in turn enhances the ability of the private sector to secure loans during periods of recession. Further contributing to the discourse, Ghassibe & Zanetti (2020) posited that the source of economic fluctuations plays a pivotal role in the behavior of the fiscal multiplier. Their findings align with a polynomial relationship of the second degree with respect to the tightness in the goods market, broadening the understanding of market conditions on fiscal multipliers.

Most recently, Barnichon et al. (2022) developed a structural model that highlights the dependency of the fiscal multiplier on the nature of government spending shocks, either positive or negative. The subsequent section will delve into a detailed exploration of the state-dependent characteristics inherent in their model, setting the stage for the application of our methodology on simulated data.

2.2 Scrutiny of Barnichon, Debortoli, & Matthes (2022)

Barnichon et al. (2022) developed a heterogeneous agent business cycle model characterized by two main frictions: incomplete financial markets with borrowing constraints, hindering households' ability to buffer against unique income shocks, and downward nominal wage rigidities in the labor market, leading to fluctuating unemployment rates.

For an intuitive understanding of how the fiscal multiplier varies with the nature of government spending shocks, let us refer to their analytical example illustrated in Figure 1[3]. In their model, the equilibrium output level is initially set at the intersection of the Aggregate

[3] This figure is directly taken from Barnichon et al. (2022), corresponding to Figure 9.

Demand (AD) curve and the Aggregate Supply (AS) at point (A). The incomplete markets result in asymmetric shifts in the AD curve in response to positive and negative government spending shocks. Additionally, the AS curve's convexity, a consequence of downward wage rigidity, plays a crucial role. A positive (or negative) government spending shock shifts the AD curve to the right (or left), with the extent of the impact contingent upon the AS curve's steepness. In the presented example, the multiplier effect of a negative shock surpasses that of a positive shock.

However, the model goes beyond a mere dependency on the shock's direction. It incorporates a state-dependence linked to both the initial output level and the shock's magnitude. The starting output level is pivotal as it determines the point on the steep AS curve where the economy stands. Similarly, the shock's size is critical as it dictates how far the AD curve shifts along the AS curve.

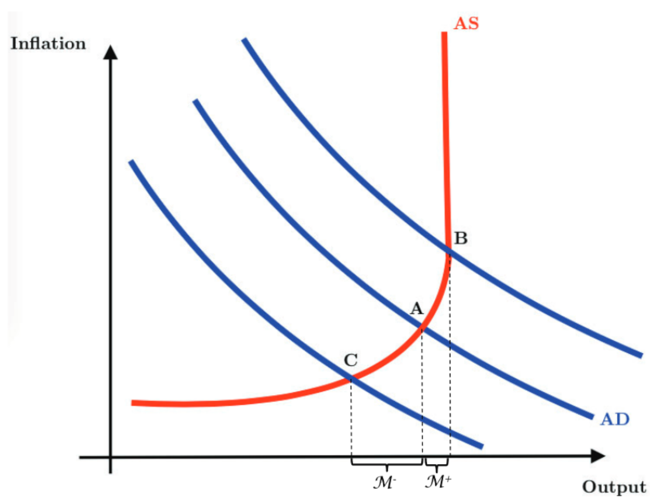


Figure 1: Illustration of demand shocks in theoretical model Barnichon et al. (2022)

Figure 2 illustrates the varying theoretical fiscal multipliers for different magnitudes of government spending shocks for given initial output levels. This illustrate the model's production of non-linear fiscal multipliers. The complexity of the model extends beyond simply categorizing fiscal stimuli as positive or negative; it shows a continuous variation of the fiscal multiplier in response to the shock's magnitude (g_t) and the initial output level (z_t). Notably, the initial output level plays a crucial role: under certain conditions, minor positive shocks might result in a fiscal multiplier not substantially lower than that produced by small negative shocks, challenging rationals based solely on the shock's direction. Additionally, Figure 3 presents the theoretical multiplier for various government spending shocks at a specific initial output level. These visual representations support the notion that the state-dependence of the fiscal multiplier could gradually fluctuate based on the considered state-dependence and the size of the shock.

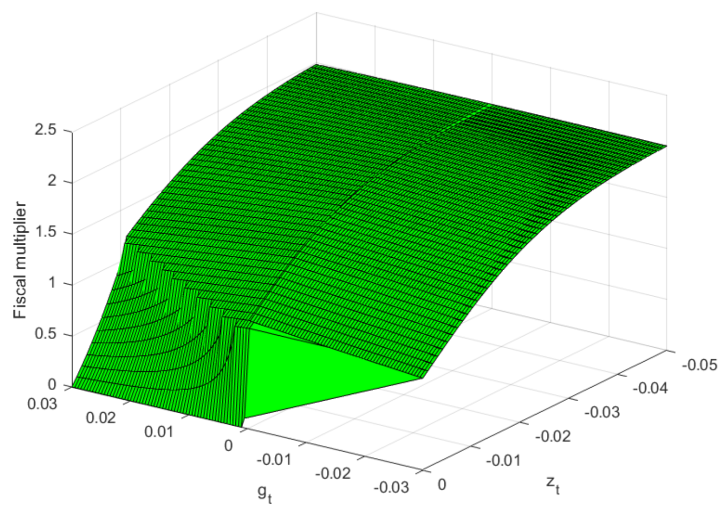


Figure 2: Theoretical fiscal multiplier as a function of the shock size, and initial output level

Therefore, from a theoretical perspective, the formulation of the state-dependent fiscal multiplier is likely to exhibit pronounced non-linearities. Similar to the findings in Ghassibe & Zanetti (2020), the multiplier could mirror a second-order polynomial. As illustrated in Figure 3, its functional form may also comprehend a mixture of functional forms. This suggests the necessity for econometric methodologies capable of discerning such intricate functional forms within data. Specifically, reducing these non-linearities to merely few distinct regimes or a quadratic term may prove overly simplistic, hence underscoring the need for a more granular approach to accommodate these non-linearities.

Another key insight emerging from the scrutinized model (Barnichon, Debortoli, & Matthes (2022)), which has seemingly been underappreciated in the literature about the fiscal multipliers, is the importance of the shock's magnitude in state-dependent models. Notably,

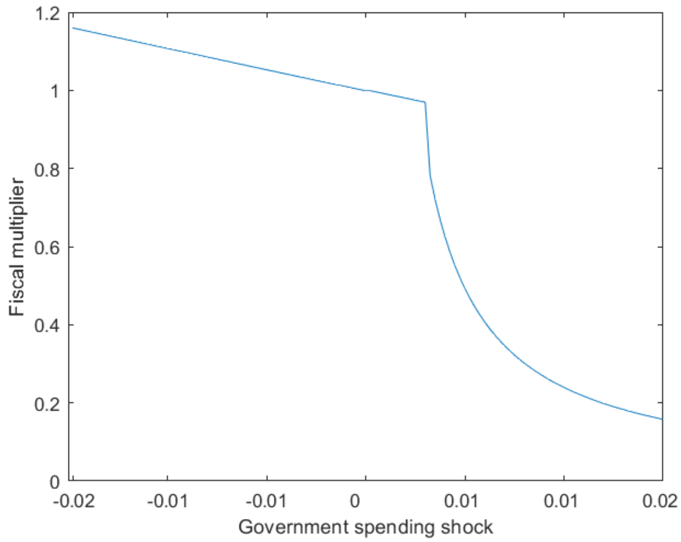


Figure 3: Theoretical fiscal multiplier as a function of the shock size given a specific initial level of output

although not their primary focus, the models developed by Christiano et al. (2011), Michaillat (2014), Shen & Yang (2018), Canzoneri et al. (2016) also present fiscal multipliers that vary according to the magnitude or orientation of the shock. Consequently, it becomes apparent that formulating a state-dependent fiscal multiplier, irrespective of the endogenous state under consideration, *almost invariably* involves a shock dependency. The subsequent section will provide a more thorough examination of this premise.

3 Endogenous state-dependence: the size of the shock matters

In this section, I explore a class of models that generate state-dependent fiscal multipliers with state-dependence endogenous to government spending. Crucially, these models inherently feature a source of nonlinearity, excluding those that are merely linearized. Within this class, I demonstrate that the magnitude of the fiscal shock is a critical determinant of the fiscal multiplier, except under specific restrictive conditions. We focus on identifying these conditions - *that render the shock size irrelevant* - for the fiscal multiplier. Most importantly, some conditions persist even in the models whose laws of motion do not initially incorporate any non-linearities related to the shock itself. Overall, this analysis aim at highlighting the fragility of linearity of the fiscal multiplier to the magnitude of the shock.

3.1 Non-linear state space model

As a starting point to formalize this idea, let us consider a simple non-linear state space model characterized by these two equations:

$$y_t = a_y(s_{t-1})y_{t-1} + \tilde{a}_g(s_{t-1}, g_t)$$
$$s_t = b_1 s_{t-1} + b_2 g_t$$

where the constant coefficients b_1, b_2 differ from zero, and the varying coefficients $a_y(\cdot), \tilde{a}_2(\cdot,\cdot)$ are not null functions and are continuously differentiable[4]. For ease of understanding, this model is based on a single-variable state and outcome, but it has the capacity to be extended to encompass multiple variables. Crucially, this dynamic system's non-linear aspects are limited to the output's behavior over time, considered a fundamental condition for producing state-dependent fiscal multipliers. Moreover, consistent with contemporary theoretical discussions on fiscal multipliers, the state is assumed to be influenced by government spending decisions.

3.2 Fiscal multipliers

Fiscal multipliers are quantifiable through various approaches, typically defined as the ratio of an alteration in output to a discretionary adjustment in government spending. This paper will focus on a prevalent approach for measuring fiscal multipliers. We define the multiplier at horizon h as follows:

$$\mathcal{M}_h = \frac{\partial y_{t+h}}{\partial g_t}$$

[4]The functions $a_y(\cdot), \tilde{a}_2(\cdot,\cdot)$ are assumed to be continuously differentiable to ease mathematical derivation. However, a similar result can be derived with functionas that are piecewise differentiable.

We define the cumulative multiplier at horizon h as

$$\mathcal{CM}_h = \frac{\partial \sum_{j=0}^{h} y_{t+j}}{\partial G_{t+h}} \text{ where } G_{t+h} = \sum_{j=0}^{h} g_{t+j}$$

Note the relation between $\mathcal{CM}_h$ and $\mathcal{M}_h$. For an infinitesimal change in g_t, assuming that government spending is not autocorrelated, we get[5]:

$$\frac{\partial G_{t+h}}{\partial g_t} = 1 \implies \mathcal{CM}_h = \sum_{j=0}^{h} \frac{\partial y_{t+j}}{\partial g_t} \frac{\partial g_t}{\partial G_{t+h}} = \sum_{j=0}^{h} \mathcal{M}_j$$

As such, any nonlinearity arising in the multiplier $\mathcal{M}_j$ for any j translates, in almost all the cases, into a nonlinearity in the cumulative multiplier $\mathcal{CM}_h$. It would not be the case if nonlinearities in $\mathcal{M}_k$ for $k \neq j$ exactly cancel the ones of $\mathcal{M}_j$, which is a knife-edge case.

In the next two sections, I focus on the contemporary multiplier $\mathcal{M}_0$ and the one period horizon (h=1) fiscal multiplier $\mathcal{M}_1$ to understand under which conditions they are linear in the shock.

3.3 Contemporary multiplier ($\mathcal{M}_0$)

In this section, I want to understand under what circumstances the contemporary multiplier does not feature size dependence. Formally, this is the case when:

$$\frac{\partial \mathcal{M}_0}{\partial g_t} = \frac{\partial^2 y_t}{\partial g_t^2} = 0 \iff \frac{\partial^2 \tilde{a}_g(s_{t-1}, g_t)}{\partial g_t^2} = 0 \tag{1}$$

In the absence of this condition (C1), dependence in the size of the shock would be inherently factored into the law of motion. As highlighted in the previous section, this condition is often violated in the fiscal literature, as demonstrated in works like Barnichon et al. (2022) and Faria e Castro et al. (2023) among others.

Let us assume that condition (C1) holds and $\frac{\partial^2 \tilde{a}_g(s_{t-1}, g_t)}{\partial g_t^2} = 0$. Thus, I now focus on the class of model that can generate state-dependent fiscal multiplier, but which law of motion for output is linear in the government spending shock. Under this assumption, the system

[5] The case in which government spending is autocorrelated is under investigation.

simplifies as follows:

$$
\begin{aligned}
y_t &= a_y(s_{t-1})y_{t-1} + a_g(s_{t-1})g_t \\
s_t &= b_1 s_{t-1} + b_2 g_t
\end{aligned}
$$

where the constant coefficients b_1, b_2 differ from zero, and the varying coefficients $a_y(\cdot), a_g(\cdot)$ are not null functions.

Importantly, this framework encompass some empirical setups used in the literature such as the one used in Ramey & Zubairy (2018), Gonçalves, Herrera, Kilian, & Pesavento (2024), Cloyne, Jordà, & Taylor (2023).

3.4 One period ahead (h=1) multiplier ($\mathcal{M}_1$)

Let us now understand under what circumstances the multiplier at horizon h=1 does not feature size dependence. Formally, this is the case when:

$$
\begin{aligned}
& \forall y_{t-1}, s_{t-1}, g_t, g_{t+1}, \quad \frac{\partial \mathcal{M}_1}{\partial g_t} = \frac{\partial^2 y_{t+1}}{\partial g_t^2} = 0 \\
\iff & \forall y_{t-1}, s_{t-1}, g_t, g_{t+1}, \quad b_2[a_y^{''}(s_t)y_t + a_g^{''}(s_t)g_{t+1}] = -2a_g(s_{t-1})a_y^{'}(s_t)
\end{aligned}
$$

This equation holds true for all values of g_{t+1}. Specifically, it remains valid for both g_{t+1} and $-g_{t+1}$, leading to the condition: $a_g^{''}(s_t) = 0$ for all s_t (C2). Intuitively, this suggests that the shock loadings ($a_g(\cdot)$) cannot exhibit nonlinearities with respect to the endogenous state, as there is no mechanism within the dynamic system that can adapt to future exogenous variations in g_{t+1} in a way that ensures the multiplier is independent of the shock's size g_t. Consequently, the condition simplifies to:

$$
b_2 a_y^{''}(s_t)y_t = -2a_g(s_{t-1})a_y^{'}(s_t), \qquad \forall y_{t-1}, s_{t-1}, g_t
$$

The equation's right side delineates the progression of the contemporary government spending shock's effect from time t to t+1, as it influences the persistence of output within the state. To preempt the emergence of nonlinear dynamics at t+1 due to the shock at time t, this influence necessitates an exact counteraction. This counterbalance is embodied on the left side of the equation. It encapsulates the interaction between the state-induced acceleration in the persistence of output and the prevailing economic output. In other words, to neutralize the potential nonlinear impact of the shock at time t on output in future time periods, the immediate and evolving response of output persistence to changes in the state, interacted with the current economic conditions, must accurately counterbalance the ongoing effect of the previous period's government spending shock. To delve deeper into the consequences of this equation, I will examine it under two scenarios: first, when $a_y^{''}(s_t) = 0$; and second, when this condition does not hold.

Case 1 - Suppose that for all s_t, $a_y^{''}(s_t) = 0$. Under this condition, I have $a_y^{'}(s_t) = 0$ given that I am considering $a_g(s_{t-1}) \neq 0$ to ensure that the policy shock impact is relevant. As a result, the function $a_y(\cdot)$ is required to be constant (C3-1). Importantly, it should not show linear dependence on the state s_{t-1}, an approach frequently adopted in empirical models for state-dependence analysis (Cloyne, Jordà, & Taylor (2023), Huidrom, Kose, Lim, & Ohnsorge (2020)). From an intuitive standpoint, this is because $a_y(\cdot)$ would lack the necessary flexibility to offset the nonlinear consequences of previous government spending shocks passing through the persistence of output, as represented on the right side of the equation.

Case 2 - Suppose there exist s_t such that $a_y^{''}(s_t) \neq 0$. Then I can substitute y_t and write

$$
\begin{aligned}
y_t = a_y(s_{t-1})y_{t-1} + a_g(s_{t-1})g_t &= -\frac{2}{b_2}\frac{a_g(s_{t-1})a_y^{'}(s_t)}{a_y^{''}(s_t)}, \qquad \forall y_{t-1}, g_t, \exists s_{t-1} \\
\iff a_y(s_{t-1})y_{t-1} + a_g(s_{t-1})\underbrace{\left[\frac{1}{b_2}(s_t - b_1 s_{t-1})\right]}_{g_t} &= -\frac{2}{b_2}\frac{a_g(s_{t-1})a_y^{'}(s_t)}{a_y^{''}(s_t)} \\
\iff a_y^{''}(s_t)\left[\tilde{s}_{t-1} - s_t\right] = 2a_y^{'}(s_t), &\quad \forall y_{t-1}, g_t, \text{ where } \tilde{s}_{t-1} \text{ is predetermined}
\end{aligned}
$$

Consequently, I solve a second-order differential equation treating $\tilde{s}_{t-1}$ as a constant. The solution, which incorporates integration constants C_1 and C_2, is as follows (a plot of the solution is provided in the appendix):

$$a_y(s_t) = C_1 + \frac{C_2}{(s_t - \tilde{s}_{t-1})} = C_1 + \frac{C_2}{b_2\left(g_t - y_{t-1}\dfrac{a_y(s_{t-1})}{a_g(s_{t-1})}\right)}, \qquad \forall y_{t-1}, g_t, \exists s_{t-1}$$

Therefore, I have pinpointed a unique form of $a_y(\cdot)$ rendering the fiscal shock's size inconsequential for the fiscal multiplier at the one-period horizon (C3-2). It illustrates the precise nature of the law of motion needed to offset any nonlinear impacts that may arise from the magnitude of the shock through nonlinearities in the persistence of output when the state is endogenous.

To sum up, I have derived precise conditions under which the dynamic system considered generates one period ahead fiscal multipliers that *do not* depend on the size shock. This finding can only originate from a dynamic model that does not incorporate initial state-dependence in response to the shock (C1), a situation that could realistically arise, as suggested explicitly by the research in Faria e Castro et al. (2023), or implicitly in Barnichon et al. (2022) among others. The shock loading cannot exhibit nonlinearities with respect to the endogenous state (C2). Moreover, the potential nonlinear impacts of the government spending shock are exactly counterbalanced by other nonlinear factors present in the model (C3). Finally, our analysis has so far focused solely on nonlinearities in the theoretical multiplier over a one-period horizon (h=1). The exploration of nonlinearities that manifest over longer timeframes following the shock is likely to present even more intricated restrictions to guarantee that the size of the shock is not relevant for the fiscal multiplier[6] . This underscores the fact that when considering a theory incorporating state-dependent fiscal multipliers, the magnitude of the shock is *almost always* going to be a significant factor. If this were not the case, the dynamic system would be in a knife-edge case where conditions (C1), (C2) and (C3) are verified.

Therefore, given the widespread existence of this feature in plausible theories of state-dependent fiscal multipliers, and its potential policy implications, it is imperative to formulate and use econometric models that effectively capture and measure this phenomenon in real-world data. In essence, econometric models that do not account for shock nonlinearity in

[6]Nonetheless, this area of inquiry remains relevant, particularly in addressing the findings of Ben Zeev et al. (2023). Their research observed variations in individual Impulse Response Functions (IRFs) for positive versus negative shocks, yet it did not report any differences in the multipliers themselves.

estimating fiscal multipliers are prone to misspecification. Conversely, instances where well-specified econometric models do not identify any nonlinear effects in the shock are equally telling. Such a scenario would imply that, based on the available data, the endogenous state being analyzed could be considered exogenous, mitigating concerns about potential biases as outlined in Gonçalves, Herrera, Kilian, & Pesavento (2024).

This section, from an alternative viewpoint, clarifies that the state-dependent VAR employed in Gonçalves, Herrera, Kilian, & Pesavento (2024) inherently produces nonlinearities in the policy shock, as it fails to satisfy conditions (C2) and (C3).[7] Consequently, the state-dependence framework posited by Ramey & Zubairy (2018) is not adequately specified to accurately estimate the correct impulse response functions (IRFs).

[7]The implications of this is self-evident in simulations presented in section 6.

4 Econometric approach

In this section, I introduce an empirical methodology to detect significant nonlinearities potentially present in the fiscal multiplier. Our approach integrates the Local Projection method (Jorda, 2005) with a narrative identification strategy, a combination well-suited for handling nonlinearities. Within this framework, I permit the key term to vary in response to the state-dependence and the shock, thereby connecting Local Projections with semi-parametric models. These models are linear regarding some regressors, yet allow certain terms to vary smoothly in relation to other variables. Consequently, I aim to extract the state-dependent fiscal multiplier from the ensuing h-step ahead semi-parametric predictive regressions model:[8]

$$y_{t+h} = \phi_h^2(L) z_{t-1} + m_h(g_t, s_{t-1}) + u_t$$

where $m_h(\cdot)$ is a continuously differentiable and possibly nonlinear function of the state-dependence s_{t-1}, g_t is the policy shock, z_{t-1} includes control variables. I restrict the function $m_h(g_t, s_{t-1})$ to only account for nonlinearities in the shock ($m_h(g_t, s_{t-1}) = m_h(g_t)$).[9] In this framework, I write the fiscal multiplier $\beta_h(g_t)$ as:

$$\frac{\partial y_{t+h}}{\partial g_t} = \frac{dm_{t+h}(g_t)}{dg_t} = \beta_h(g_t)$$

4.1 Local linear regression

Typically, semi-parametric models are estimated using either kernel methods or the Sieve approach, as outlined in Hansen (2022). In the following analysis, I adopt the kernel method for estimating y_{t+h}, specifically employing local linear regression. This means that I develop a semi-parametric estimator for y_{t+h}, which is grounded in a local linear approximation framework,

[8] More broadly, our analysis can be framed in terms of the conditional expectation function (CEF) $\mathbb{E}[Y_{t+h}|Z_{t-1} = z_{t-1}, G_t = g_t, S_{t-1} = s_{t-1}]$. To simplify the model and manage its complexity, I posit that the CEF is separable between the control variables Z and the nonparametric factors of state-dependence and shock size, and I further assume that the CEF exhibits linearity with respect to Z. Consequently, I examine $\mathbb{E}[Y_{t+h}|Z_{t-1} = z_{t-1}, G_t = g_t, S_{t-1} = s_{t-1}] = \phi_h^2(L) z_{t-1} + m_h(g_t, s_{t-1})$.

[9] This restriction can be seen as an approximation, integrating out s_{t-1}, such that $\mathbb{E}\left[m_h(G_t, S_{t-1})|G_t = g_t\right] = m_h(g_t)$, $\mathbb{E}\left[\phi_h^1(G_t, S_{t-1})|G_t = g_t\right] = \phi_h^1(g_t)$.

as follows:

$$\begin{aligned}
y_h(z_{t-1}, G) &\approx y_h(z_{t-1}, g) + \frac{dm_h(g)}{dg}(G - g), \text{ for } G \approx g \\
&\approx -\frac{dm_h(g)}{dg} g + m_h(g) + \phi_{2,h}(L) z_{t-1} + \frac{dm_h(g)}{dg} G \\
&\approx \underbrace{\phi_{1,h}(g)}_{\text{local constant}} + \phi_{2,h}(L) z_{t-1} + \underbrace{\frac{dm_h(g)}{dg}}_{\beta_h(g)\text{: local FM}} G
\end{aligned}$$

From a conceptual standpoint, I introduce a local restriction on the fiscal multiplier for its identification, postulating that it is constant locally around a given shock g, and equal to $\beta_h(g) = m'_h(g)$. Utilizing local linear regression, the function $\beta_h(\cdot)$ is determined exclusively by the subsample where government spending (G) is proximate to a specific value g, within a defined bandwidth h. Observations outside this range do not directly influence the estimation. Figure 4 illustrates this concept using a purely illustrative dataset:

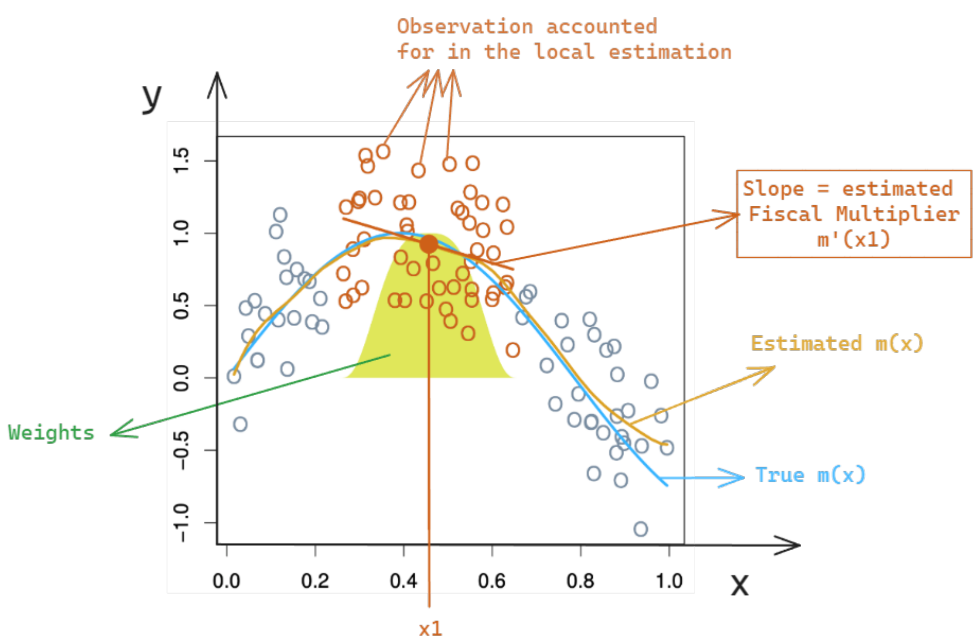


Figure 4: Illustration of the local linear regression

This technique bears similarity to the state-dependent Local Projections method of Ramey & Zubairy (2018), as well as the smooth transitions Local Projections approach of Auerbach

& Gorodnichenko (2012), but it distinguishes itself through the utilization of weights specific to the nonparametric literature.[10] In particular, I will use a Gaussian kernel such that the weighting function is defined as:[11]

$$K(G-g) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(G-g)^2}{2b^2}\right)$$

In this equation, b represents the bandwidth parameter, which I determine through cross-validation. Specifically, our selection of b is aimed at minimizing the integrated mean-square error (IMSE) of $\hat{y}_h(z_t, g_t)$ as an estimator of $y_h(z_t, g_t)$. The bandwidth, being globally applicable, means that the overall behavior of $y_h(z_t, g_t)$ influences the estimator of the fiscal multiplier $\hat{\beta}_h(\cdot)$.

4.2 Cumulative fiscal multiplier

In accordance with the approach suggested by Ramey & Zubairy (2018), our focus is on calculating the cumulative fiscal multiplier. This involves determining the ratio of the total change in output to the total change in government spending over a specified period after a exogenous shock happened. To this end, I look at the variation in government spending in subsequent periods to an exogenous shock such that:

$$g_{t+h} = \delta_{1,h} + \delta_{2,h}(L)z_{t-1} + m_{g,h}(\text{shock}_t) + u_t$$

We define the multiplier of government spending to the shock as $m'_{g,h}(\text{shock}_t) = \beta_{g,h}(\text{shock}_t)$. Consequently, this leads to a definition of the cumulative fiscal multiplier, which remains locally constant around a specific government spending level g, and is expressed as:

$$\mathcal{CM}_h(\text{shock}) = \frac{\sum_{i=0}^{h} \beta_{y,i}(\text{shock})}{\sum_{i=0}^{h} \beta_{g,i}(\text{shock})}$$

[10] Ramey & Zubairy (2018) reweight observations of interest with 1 or 0. As such, they assume a constant fiscal multiplier in two locations of the data. Auerbach & Gorodnichenko (2012) builds a continuum of weights between 0 and 1 through a logistic function of the state.

[11] Hansen (2022) recommends the use of the Gaussian or Epanechnikov kernels. In addition, he mentions that either will give similar results. According to Loader (1999), the weight function does not have much effect on the bias-variance tradeoff, but it influences the visual quality of the fitted regression curve.

Standard errors for the cumulative multiplier are computed through the delta method. Whether it is possible to estimate the cumulative multipliers in one step through the methodology proposed in Ramey & Zubairy (2018) is under investigation.

We use the delta method to approximate the variance of the specific function $f(\beta^y, \beta^g) = \frac{\sum \beta_{y,i}}{\sum \beta_{g,i}}$ for a given shock magnitude, the gradient ∇f with respect to each $\beta_{y,i}$ and $\beta_{g,i}$ is calculated assuming independence among $\beta_{y,i}$ and $\beta_{g,i}$ for different i. The variance-covariance matrix Σ will be diagonal, with the variances of $\beta_{y,i}$s and $\beta_{g,i}$s along the diagonal.

The partial derivatives of f with respect to β_i and γ_i are:

$$\frac{\partial f}{\partial \beta_{y,i}} = \frac{1}{\sum \beta_{g,i}}$$

$$\frac{\partial f}{\partial \beta_{g,i}} = -\frac{\sum \beta_{y,i}}{\left(\sum \beta_{g,i}\right)^2}$$

These derivatives are used to construct the gradient vector ∇f. Given the independence of parameters, the off-diagonal elements of Σ are zero, and the diagonal elements are the variances of $\beta_{y,i}$s and $\beta_{g,i}$s. The variance of f, and thus the standard error, is computed as:

$$\mathrm{Var}(f) = \nabla f^T \cdot \Sigma \cdot \nabla f$$

4.3 Bandwidth selection

So far, the bandwidth is selected in an automated fashion by generalized cross-validation (GCV) at each horizon h for real GDP and government spending. This methodology has been chosen because of the high number (h $\times$ 2) of bandwidth that needs to be selected to estimate the fiscal multiplier with the current approach. However, in a future version of this work I plan to reduce the number of bandwidth parameter to 1 by designing a one step estimation method for the multiplier. This will allow me to choose the bandwidth more carefully, and to potentially consider adaptative bandwidth that are appropriate in this context Gonçalves et al. (2024).

Criterion such as GCV and AIC have been introduced as perform automatic bandwidth and model selection: an algorithm that takes the data as input and produces the best local polynomial fit as output. Unfortunately, as argued in Loader (1999), this goal is unattainable, since there is often considerable uncertainty in data and it is unclear what the best fit should be. Just minimizing a criterion discards significant information provided by the whole profile of the curve provided by each criterion and the bias-variance tradeoff that these statistics provide. Hence, Loader (1999) argues that the whole profile of GCV and AIC should be used as a graphical aid in choosing smoothing parameters, in addition to a careful examination of the noise in the fit, and of the residuals to detect bias.

5 Empirical Analysis [Results subject to changes]

The empirical literature on fiscal multipliers has revolved around two main identification strategies: recursive identification (Blanchard & Perotti, 2002; Auerbach & Gorodnichenko, 2012) and narrative identification (Ramey, 2011; Ramey & Zubairy, 2018). Therefore, I estimate the model

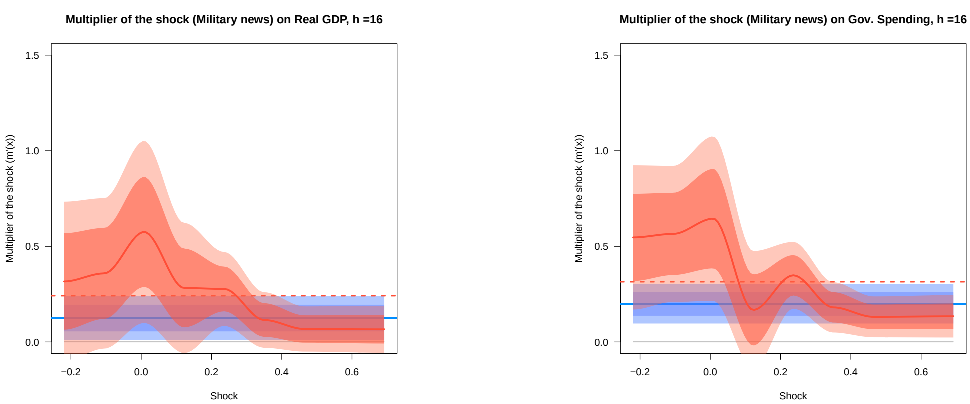
$$x_{t+h} = \delta_{1,h} + \delta_{2,h}(L)z_{t-1} + m_{g,h}(\text{shock}_t) + u_t$$

where the shocks are either the military news shocks (Ramey & Zubairy (2018)) or the BP shocks (Blanchard & Perotti (2002)). The outcome variable x_{t+h} is either real GDP or government spending forwarded by h periods. Our vector of control variables, z, contains real GDP and government spending, each divided by trend GDP. In addition, z includes lags of the news variable to control for any serial correlation in the news variable when this shock is used. The term $\delta_{2,h}(L)$ is a polynomial of order 4.

5.1 Military news shocks

For military news shocks, I examine the step-by-step estimates. Figures 6 and 7 showcase the shock multipliers on real GDP (a) and government spending (b) over 2-year and 4-year horizons, respectively. Consistent with the discoveries in Ben Zeev et al. (2023), which highlight non-linear patterns in the impulse response functions (IRFs) of real GDP and government spending due to military news shocks, our analysis reveals that negative shocks trigger a significantly larger reaction in both Real GDP and government spending compared to positive shocks. While for positive shocks, the multiplier obtained is broadly in line with the linear estimates (blue, Ramey & Zubairy (2018)), negative shocks generate significantly higher impact. Importantly, the average multiplier implied by the nonlinear estimate (dashed red line) are higher than the linear estimate.

The Local Linear Local Projections (LL-LP) method reveals that the nonlinearity in fiscal multipliers is not just about positive or negative shocks. It shows that multipliers start to increase when the shock is less than 0.3% of GDP. This insight suggests that, as understood during the analysis of the theoretical models of fiscal multipliers, the distinction between negative and positive shocks conceals a more intricate pattern of nonlinearities, which is relevant for the policy debate.



(a) Impact of Military News Shock on Real GDP

(b) Impact of Military News Shock on Government Spending

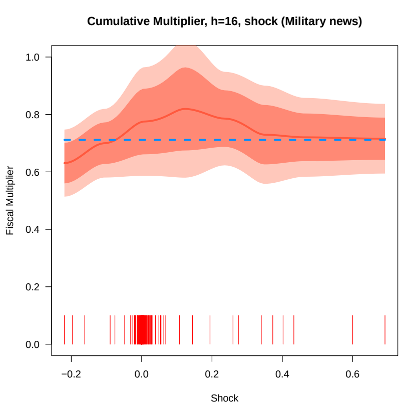
Figure 5: Multiplier at horizons 4Y (h=16)

5.2 Cumulative multipliers

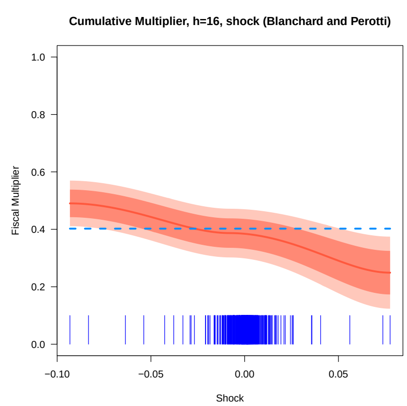
Using military news shocks Ben Zeev et al. (2023) points out that nonlinear IRFs do not lead to nonlinear cumulative fiscal multipliers. In our analysis using military news shocks and BP shocks, I both corroborate and question this assertion. The 2-year cumulative multipliers, depicted in Figures 5(a) for military news shocks and 5(b) for BP shocks, along with the

4-year cumulative multipliers shown in Figure 6, further elaborate on this point. Notably, Figures 5(a) and 6(a) demonstrate minimal nonlinearity when military news shocks serve as the identification strategy, corroborating Ben Zeev et al. (2023)'s finding that the cumulative fiscal multiplier does not depend on the shock. On the contrary, the use of BP shocks reveals a different pattern, as indicated in Figures 5(b) and 6(b), where the cumulative fiscal multiplier exhibits a linear decline with increasing shock size, resonating with the findings of Barnichon et al. (2022). In other words, the identification of nonlinearities is contingent on the identification strategy employed.

An important takeaway from this analysis is that zero does not appear to be a significant tipping point in the nonlinearity of the fiscal multiplier. Fiscal multipliers generated by small positive shocks or small negative shocks do not exhibit dramatically different effects, contrary to what might be suggested by an analysis focused solely on the sign of the shock. This finding goes beyond the binary view of the world, which posits that negative spending initiatives consistently yield higher multipliers compared to positive ones as in Barnichon et al. (2022). It suggests that adopting a policy of large-scale spending, coupled with future fiscal policies of lower contraction, might not always be detrimental for the economy.



(a) Military News shocks



(b) Blanchard and Perotti shocks

Figure 6: Cumulative Multiplier at horizons 4Y (h=16)

6 Simulation study

In this section I carry out a simulation study to benchmark the performance of our proposed methodology against state-dependent LP (Ramey & Zubairy (2018)) and state-dependent LP with a squared term in the shock as in Forni et al. (2023), Ben Zeev et al. (2023) or Faria e Castro et al. (2023). In addition, I evaluate our proposed methodology using a different simulation framework, one where traditional state-dependent local projections have shown limitations.

6.1 DGP 1: Barnichon, Debortoli, & Matthes (2022)

Utilizing the model from Barnichon et al. (2022), I simulate 500 data points of output Y_t based on exogenous government spending shocks. The model's parameterization is chosen to align with the theoretical fiscal multiplier depicted in Figure (3). To introduce errors in the fiscal multiplier, I simulate preference shocks among agents. Our objective is to investigate the state-dependence on government spending shocks within this model. To achieve this, I will estimate the following models (M1), (M2), (M3) on the simulated data, aiming to assess and compare their efficacy:

$$(M1) \qquad y_t = \mathbb{1}(g_t > 0)(\phi_p + \beta_p g_t) + \mathbb{1}(g_t < 0)(\phi_n + \beta_n g_t) + u_t \tag{2}$$

$$(M2) \qquad y_t = \phi + \beta_y g_t + \beta_g g_t^2 + u_t \tag{3}$$

$$(M3) \qquad y_t = \phi + m(g_t) + u_t \tag{4}$$

As highlighted in earlier sections, the inferred fiscal multiplier corresponds to the first derivative with respect to the policy shock g_t. In model (M1), the fiscal multiplier $\mathcal{M}_{M1}(g_t)$ is imposed to be constant across locations of the shock. In model (M2), it is linear, expressed as $\mathcal{M}_{M2}(g_t) = \beta_y + 2\beta_g g_t$[12]. In our proposed model (M3), the multiplier has the potential for nonlinearity, represented as $\mathcal{M}_{M3}(g_t) = m'(g_t)$, and standard errors are computed by bootstrap.

[12]We compute standard errors based on $\mathrm{SE}\left(\beta_y + 2\beta_g g_t\right) = \sqrt{\mathrm{Var}\left(\beta_y\right) + 4g_t^2 \, \mathrm{Var}\left(\beta_g\right) + 4g_t \, \mathrm{Cov}\left(\beta_y, \beta_g\right)}$.

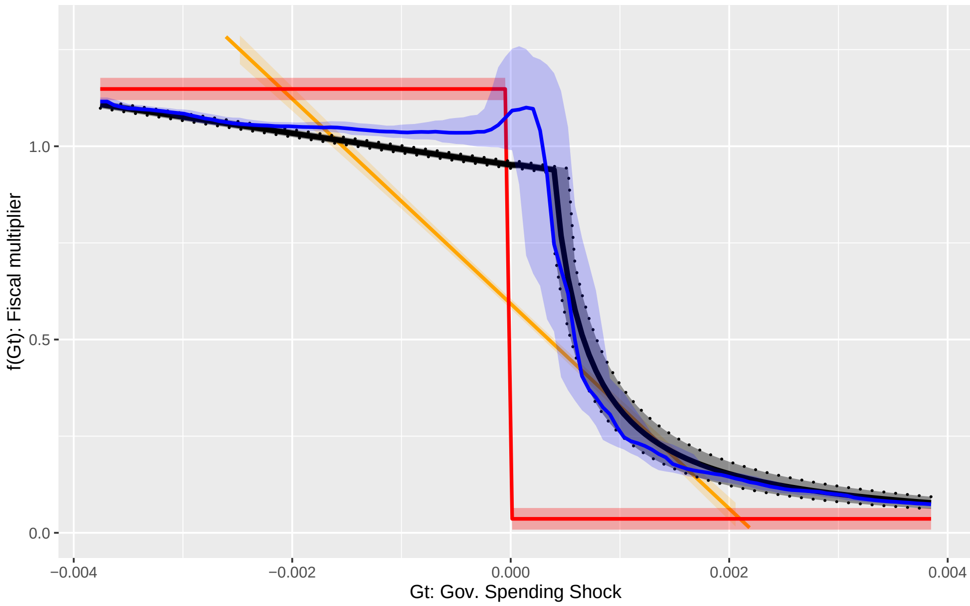


Figure 7: Estimation of fiscal multipliers on simulated data

Black: theoretical fiscal multiplier implied by Barnichon et al. (2022). Red: $\mathcal{M}_{M1}(g_t)$ estimated with Ramey & Zubairy (2018) framework. Orange: $\mathcal{M}_{M2}(g_t)$ estimated as in Faria e Castro et al. (2023) (results are truncated). Blue: $\mathcal{M}_{M3}(g_t)$ estimated with, the proposed methodology, LLR.

The example illustrates the shortcomings of standard approaches in addressing state-dependence in fiscal multipliers, especially in missing out on critical nonlinearities for shocks near $g_t = 0$, which are vital for informed policymaking.

6.2 DGP 2: Gonçalves, Herrera, Kilian, & Pesavento (2024)

In this subsection, I evaluate our proposed methodology using a different simulation framework, one where traditional state-dependent local projections have shown limitations, as demonstrated in Gonçalves et al. (2024). We employ their first data generating process (DGP) to simulate data:

$$y_t = \beta_{t-1} g_t + \gamma_{t-1} y_{t-1} + \varepsilon_{2t},$$

where

$$\beta_{t-1} = \beta_E H_{t-1} + \beta_R (1 - H_{t-1})$$
$$\gamma_{t-1} = \gamma_E H_{t-1} + \gamma_R (1 - H_{t-1})$$

Figure (8) displays the horizon one (h=1) multiplier ($\mathcal{M}_1 = \frac{\partial y_{t+1}}{\partial g_t}$), illustrating variations across different shock sizes and initial states as generated by the DGP. As outlined in Gonçalves et al. (2024), the magnitude of the shock is a key factor in creating significant nonlinearities. This revelation is consistent with the discussions in section 3 where I emphasized that this DGP fails to fulfill the criteria of conditions (C2) and (C3). However, this analysis yields two additional insights: firstly, the sign of the shock appears to be just as, if not more, influential than the shock's size (Figure 9a); and secondly, while state-dependence arises solely from the sign of the outcome variable y_{t-1}, the nonlinearities present in the multiplier offer a more nuanced picture than a simple examination of the sign of y_{t-1} (Figure (9b)). These findings reinforce our rationale to concentrate on the shock as a likely source of nonlinearities and to approach these nonlinearities with greater granularity than what has been typical in existing literature.

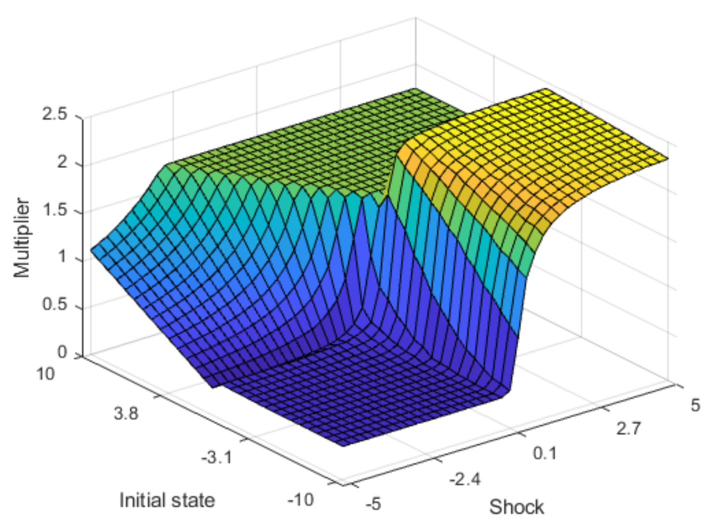
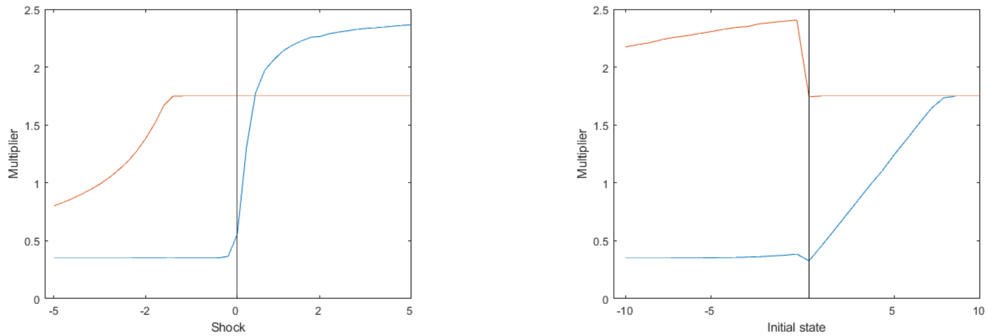


Figure 8: Fiscal multiplier at h=1 as a function of the shock size, and initial state y_{t-1}. Generated by DGP 1 (Gonçalves et al. (2024)).



(a) Multiplier as a function of the shock y_{t-1} for an initial state of +7 (orange line) or -7 (blue line)

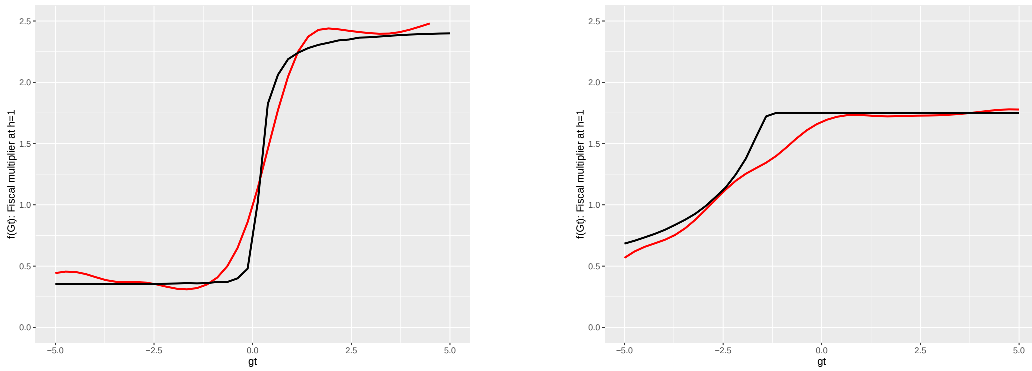
(b) Multiplier as a function of initial state y_{t-1} for a shock of +2 (orange line) or -2 (blue line)

Figure 9: Nonlinearities generated by DGP 2 (Gonçalves et al. (2024))

Subsequently, I apply the outlined methodology to deduce the fiscal multiplier at horizon 1 from 1,000,000 simulated observations. To incorporate the initial state y_{t-1} I use an indicator function. While our previous analysis indicates that the implied nonlinearity is more intricate, for the sake of simplicity, I proceed as follows:

$$(M4) \qquad y_t = \mathbb{1}(y_{t-1} > 0)\,[\phi_+ + m_+(g_t)] + \mathbb{1}(y_{t-1} < 0)\,[\phi_- + m_-(g_t)] + u_t \tag{5}$$

Figure (10a), and (10b) report the comparison of the fiscal multiplier implied by the DGP and the estimated fiscal multiplier at horizon one. We average the IRFs for different initial state of the same sign generated by the DGP to build the comparison. Overall, the results show that the proposed methodology can provide a good approximation of the continuous nonlinearities generated by the dynamic system.



(a) Negative state ($y_{t-1} < 0$). Fiscal multiplier at horizon 1 implied by the theoretical model (black), estimated on simulated data (red).

(b) Positive state ($y_{t-1} > 0$). Fiscal multiplier at horizon 1 implied by the theoretical model (black), estimated on simulated data (red).

Figure 10: Matching nonlinearities generated by DGP 2 (Gonçalves et al. (2024))

7 Conclusion

This study provides a comprehensive exploration of the fiscal multiplier, emphasizing the crucial impact of the size —magnitude and/or sign— of fiscal shocks on its magnitude. By delving into various structural models, particularly the one by Barnichon et al. (2022), I highlight that the nature of fiscal shocks — whether positive or negative — and their magnitude significantly influence the fiscal multiplier. This finding appears to be prevalent in plausible theories of state-dependent multipliers, underscoring the importance of identifying nonlinearities in the fiscal multiplier with respect to the shock.

With this goal in mind, I introduce a new methodology, the local linear local projection (LL-LP), which enables to capture the nuanced nonlinearities inherent to the fiscal multiplier. The LL-LP methodology has demonstrated its proficiency in identifying a broader spectrum of nonlinear effects in simulation studies. When applied to US data, the findings from this methodology vary depending on the identification strategy used. Using military news shocks to identify exogenous government spending, I uncover only mild nonlinearities. Conversely, nonlinearities are more pronounced with the recursive approach for identification.

Overall, results suggest that zero does not appear to be a significant tipping point in the nonlinearity of the fiscal multiplier. Fiscal multipliers generated by small positive shocks or small negative shocks do not exhibit dramatically different effects, contrary to what might be suggested by an analysis focused solely on the sign of the shock. This finding goes beyond the binary view of the world, which posits that negative spending initiatives consistently yield higher multipliers compared to positive ones as in Barnichon et al. (2022). It suggests that adopting a policy of large-scale spending, coupled with future fiscal policies of lower contraction, might not always be detrimental for the economy.

8 Appendix

8.1 The size of the shock

We seek to identify the specific conditions under which the multiplier at a one-period horizon (h=1) *remains unaffected* by the magnitude of the fiscal shock.[13] Intuitively, this scenario occurs when the derivative of the multiplier with respect to the shock at time t (g_t) equals zero ($\frac{\partial \mathcal{M}_1}{\partial g_t} = 0$).

To understand this, I proceed by iterating the system forward by one period, leading us to the following findings:

$$\begin{aligned} y_{t+1} &= f(y_t, s_t, g_{t+1}) \\ &= f(f(y_{t-1}, s_{t-1}, g_t), h(s_{t-1}, g_t), g_{t+1}) \end{aligned}$$

Following this line, I show that if (A1), (A2), (A3) and (C1) are verified, then the size of the shock *does not* matter for the multiplier at horizon t+1 if and only if the dynamic system verifies[14]

$$\frac{\partial^2 y_{t+1}}{\partial g_t^2} = \frac{\partial \mathcal{M}_1}{\partial g_t} = 0 \qquad \iff \qquad \begin{aligned} & A^2 F_{yy} + B^2 F_{ss} = -2ABF_{ys}, \forall y_{t-1}, s_{t-1}, g_t, g_{t+1} \\ & \text{where } A = \frac{\partial f(y_{t-1}, s_{t-1}, g_t)}{\partial g_t}, B = \frac{\partial h(s_{t-1}, g_t)}{\partial g_t} \\ & F_{XY} = \frac{\partial^2 f(y_t, s_t, g_{t+1})}{\partial X_t \partial Y_t} \end{aligned}$$

This criterion precisely characterizes the set of laws of motion where the fiscal multiplier for the next period (horizon 1) is not influenced by the size of the shock. In the following subsection, it will become evident that the class of law of motion I have derived tend to be somewhat limiting in practical applications.

[13] A similar analysis with the *sign* of the shock is left for a future version of this project.

[14] Note that A and B are only functions of variables at t-1 because of (C1). We consider them as constants.

8.2 General condition for the size of the shock to matter

Our aim is to understand whether the size of the shock at time t matters. Hence, let us look at the derivative of y_{t+1} with respect to g_t and see whether it is depending on g_t or not.

$$\begin{aligned}
\frac{\partial y_{t+1}(s_{t-1}, g_t, g_{t+1})}{\partial g_t} &= \frac{\partial f_y(y_t, s_t, g_{t+1})}{\partial y}\frac{\partial f_y(y_{t-1}, s_{t-1}, g_t)}{\partial g} + \frac{\partial f_y(y_t, s_t, g_{t+1})}{\partial s}\frac{\partial f_s(s_{t-1}, g_t)}{\partial g} \\
&= A_{t-1}\frac{\partial f_y(y_t, s_t, g_{t+1})}{\partial y} + B_{t-1}\frac{\partial f_y(y_t, s_t, g_{t+1})}{\partial s} \\
&= A_{t-1}\frac{\partial f_y(y_t, s_t)}{\partial y} + B_{t-1}\frac{\partial f_y(y_t, s_t)}{\partial s} + g_{t+1}\left[A_{t-1}\frac{\partial a(y_t, s_t)}{\partial y} + B_{t-1}\frac{\partial a(y_t, s_t)}{\partial s}\right] \\
&\text{where } A_{t-1} = \frac{\partial f_y(y_{t-1}, s_{t-1}, g_t)}{\partial g}, B_{t-1} = \frac{\partial f_s(s_{t-1}, g_t)}{\partial g}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 y_{t+1}(s_{t-1}, g_{t:t+1})}{\partial g_t^2} &= \left[\frac{\partial^2 f_y(y_t, s_t, g_{t+1})}{\partial y^2}\frac{\partial f_y(y_{t-1}, s_{t-1}, g_t)}{\partial g} + \frac{\partial^2 f_y(y_t, s_t, g_{t+1})}{\partial s\partial y}\frac{\partial f_s(s_{t-1}, g_t)}{\partial g}\right]\frac{\partial f_y}{\partial g} \\
&\quad + \left[\frac{\partial^2 f_y(y_t, s_t, g_{t+1})}{\partial s^2}\frac{\partial f_s(s_{t-1}, g_t)}{\partial g} + \frac{\partial^2 f_y(y_t, s_t, g_{t+1})}{\partial s\partial y}\frac{\partial f_y(y_{t-1}, s_{t-1}, g_t)}{\partial g}\right]\frac{\partial f_s}{\partial g} \\
&= A_{t-1}^2\frac{\partial^2 f_y(y_t, s_t, g_{t+1})}{\partial y^2} + B_{t-1}^2\frac{\partial^2 f_y(y_t, s_t, g_{t+1})}{\partial s^2} + A_{t-1}B_{t-1}\frac{\partial^2 f_y(y_t, s_t, g_{t+1})}{\partial s\partial y} \\
&= A_{t-1}^2\frac{\partial^2 f_y(y_t, s_t)}{\partial y^2} + B_{t-1}^2\frac{\partial^2 f_y(y_t, s_t)}{\partial s^2} + A_{t-1}B_{t-1}\frac{\partial^2 f_y(y_t, s_t)}{\partial s\partial y} \\
&\quad + g_{t+1}\left[A_{t-1}^2\frac{\partial^2 a(y_t, s_t)}{\partial y^2} + B_{t-1}^2\frac{\partial^2 a(y_t, s_t)}{\partial s^2} + A_{t-1}B_{t-1}\frac{\partial^2 a(y_t, s_t)}{\partial s\partial y}\right]
\end{aligned}$$

8.3 The sign of the shock matters

Let's also explore the conditions under which the sign of the fiscal shock is irrelevant for the fiscal multiplier at a one-period horizon (h=1). Essentially, I aim to establish the conditions that make the multiplier at h=1 an even function. This would mean $M_1(g_t) = M_1(-g_t)$, implying specific requirements for the derivatives. Notably, the first derivative should be an odd function, satisfying $\frac{\partial M_1(g_t)}{\partial g} = -\frac{\partial M_1(-g_t)}{\partial g}$.

Formally, our analysis indicates that if:

1. the impact of government spending g_t varies based on the state s_{t-1},

2. and this state-dependence itself is influenced by the government spending shock,

then a necessary condition for the shock's sign to be inconsequential for the multiplier at h=1 is that the following holds true:

$$A(F_y - F_y^*) + B(F_s - F_s^*) = 0, \text{for all } y_{t-1}, s_{t-1}, g_t, g_{t+1}$$
$$\text{where } A = \frac{\partial f_y(y_{t-1}, s_{t-1})}{\partial g}, B = \frac{\partial f_s(s_{t-1})}{\partial g}$$
$$F_X = \frac{\partial f_y(y_t, s_t, g_{t+1})}{\partial X}$$

This stipulation precisely delineates the category of laws of motion where the fiscal multiplier at a one-period horizon is independent of the shock's direction. In the following subsection, it will become evident that the class of law of motion I have derived in the last two subsections tend to be somewhat limiting in practical applications.

8.4 Chart of the knife-edge case function found in section 3

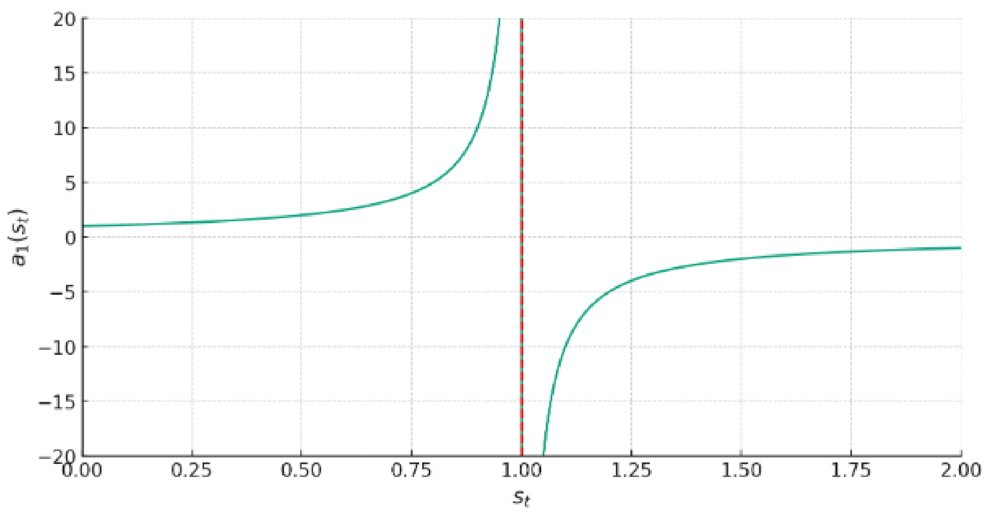


Figure 11: Function $a_y(s_t)$

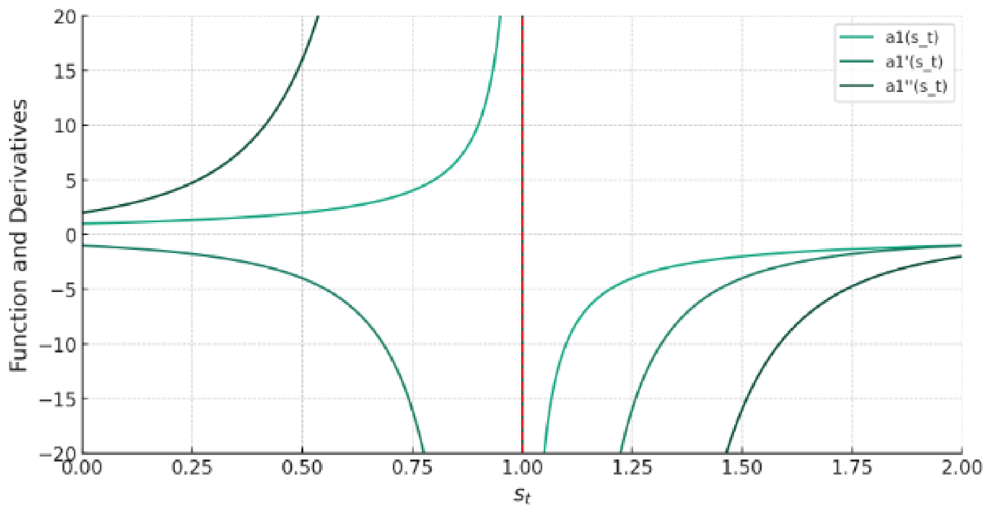


Figure 12: Function $a_y(s_t)$ and its derivatives

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