Are Earnings Inequality and Firm Concentration Connected? Evidence from an Assignment Model

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Abstract

This paper studies the relationship between two secular trends: increasing firm concentration and rising earnings inequality. I propose an assignment model with heterogeneous firms and workers that jointly generates inequality in worker earnings and concentration in firm sales and employment. I characterize the equilibrium solutions for earnings and firm size measures, demonstrating that earnings inequality and firm concentration are inherently linked. Specifically, both earnings inequality and firm sales concentration increase with greater skill and productivity dispersion and higher price elasticity of demand, while firm employment concentration is primarily influenced by firm productivity dispersion. To quantify these dynamics, I calibrate the model using data on earnings inequality and firm concentration in the United States from the 1980s and 2010s. The quantitative analysis yields four main findings: (1) Rising price elasticity of demand has been the primary driver of earnings inequality and the second-largest driver of firm sales concentration. (2) Increasing dispersion in worker skills has contributed to greater concentration in both sales and earnings. (3) The dispersion of a technology affecting the productivity of each additional position within a firm has declined, leading to increased employment and sales concentration. (4) Firm productivity dispersion has decreased, suggesting that changes in firm and earnings concentration are not driven by widening underlying productivity differences between firms.
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JEL Classification:

1 Introduction

A growing body of literature has documented increased concentration in production and employment in the United States and other developed nations since the 1980s. This trend has been associated with declining labor shares of income and rising average markups (Gutiérrez & Philippon (2017); Grullon et al. (2019); Autor et al. (2020); Loecker et al. (2020)). Meanwhile, a largely separate line of research has examined the causes of growing earnings inequality over the same period (Katz & Autor (1999); Acemoglu & Autor (2011); Piketty & Saez (2014)). Intuitively, these two phenomena may be interconnected. First, firm size and earnings distributions exhibit similar functional forms, with significant density concentrated in the right tails, suggesting a joint distribution Rosen (1982). Second, as

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firms and workers share production rents, any shock that increases worker productivity can influence firm output, and vice versa.

This paper explores two key questions: What common factors could simultaneously drive increasing concentration in both earnings and production? And which of these factors have contributed most to the recent rise in concentration in the United States?

The first contribution of this paper is the development of a general equilibrium model that jointly explains inequality in worker earnings and concentration in firm sales and employment. Specifically, the model replicates observed earnings and firm size distributions. I construct a static assignment model with two-sided heterogeneity, where heterogeneous workers are assigned to hierarchical positions within heterogeneous firms and firms operate under monopolistic competition. I then characterize the assignment equilibrium that determines the allocation of workers to positions and derive expressions for worker earnings and firm size, measured in both employment and revenue.

The model integrates the canonical monopolistic competition framework (Krugman (1979); Melitz (2003)) with a static assignment model following Gabaix & Landier (2008) and Terviö (2008). Firms consist of heterogeneous positions, each operated by a worker who combines their skills with firm-specific technology to produce output. A firm's size, measured in employment, is determined by the number of positions (and workers) within the firm, while its size, measured in sales, is determined by the total revenue generated across all positions. Both measures of firm size are endogenously determined in equilibrium.

A firm consists of multiple job positions, each varying in productivity. While all positions share the firm $i's$ underlying productivity level, $A(i)$, individual position productivity decreases with its rank within the firm's hierarchy. This decrease in position productivity is determined by a sensitivity function $c(h)$, with $c\prime(h) < 0$, where h is the rank of the position within a firm. Across firms, all positions can be ranked by productivity. Firms compete to hire the most skilled workers for their positions. The complementarity between position productivity and worker skill results in positive assortative matching, where more productive positions are paired with more skilled workers. Moreover, competition for talent leads to earnings inequality that scales with firm size.

The model yields three main theoretical predictions. First, when both skill and productivity distributions follow a Pareto distribution, the earnings distribution also exhibits a Pareto-shaped upper tail, aligning with empirical observations. The shape parameter of the earnings distribution depends on firm productivity dispersion, worker skill dispersion, and price elasticity of demand. Specifically, earnings inequality rises as productivity or skill dispersion increases and/or as price elasticity of demand grows. Consequently, the model predicts greater earnings inequality in economies with more competitive product markets.

Second, the firm size distribution, measured by employment, also follows a Pareto distribution when the sensitivity function exhibits a Pareto distribution. The shape parameter of the employment size distribution depends on firm productivity dispersion and the dispersion of the sensitivity function. Increasing firm size concentration arises either from greater dispersion in firm productivity or lower dispersion in the sensitivity function. A lower dispersion in the sensitivity function implies that the marginal productivity of additional positions declines more gradually, encouraging firms to expand by creating more positions and increasing in size.

Finally, firm size concentration, measured by sales, increases with skill and productivity dispersion and price elasticity of demand, while it decreases with sensitivity dispersion. This reflects the fact that firm sales concentration is influenced not only by the distribution of skills and productivity but also by the price elasticity of substitution, as well as employment concentration.

These results indicate a strong connection between earnings inequality and firm sales concentration, as both are driven by skill and productivity dispersion, as well as price elasticity of demand. This relationship is intuitive: firms compete for heterogeneous talent by compensating workers based on their value to the firm. Due to the complementarity between firm productivity and worker skill, more productive firms can afford to pay higher wages, as they derive greater benefits from higher-skilled workers. Consequently, earnings depend on the firm productivity distribution, while firm sales depend on worker skill and its distribution. More competitive product markets further amplify these effects, as any increase in firms' productivity (which depends on both worker skill and firm productivity) leads to a disproportionately larger market share for the most productive firms.

In contrast, earnings inequality and firm size concentration, when measured by employment, are linked only through the firm productivity distribution. This suggests that while both employment and earnings have become more concentrated, their underlying drivers may not be entirely the same.

The second main contribution of this paper is quantitative. Using the model's predictions, I calibrate the model to assess which factors have contributed to the increase in both earnings inequality and firm size concentration. I select the model parameters by targeting a set of earnings inequality and firm concentration measures from the United States between the 1980s and 2010s.

Based on the calibration results, the skill distribution of workers has become more skewed between the 1980s and 2010s, with a shift toward higher-skill workers. The price elasticity of demand has also increased. These findings align with two strands of literature. First, the literature on skill-biased technological change suggests that ongoing technological advancements have primarily benefited high-skilled workers by increasing the market returns to their skills, consistent with the rise in price elasticity of demand. Second, research in organizational economics has shown that the increased use of information and communication technology (ICT) has reorganized production and job tasks, such that many tasks now require a combination of social and cognitive skills. If this combination of skills is rarer among workers but highly valued by employers, it could lead to increased skewness in the perceived skill distribution and higher market returns to skills. Additionally, the calibration points to a decrease in dispersion within the sensitivity function, suggesting that more productive firms are now able to grow larger. The firm productivity distribution has also become less dispersed, indicating that the productivity differences among active firms have narrowed.

I then assess the quantitative importance of the potential factors driving trends in earnings inequality and firm concentration through a counterfactual exercise. The results highlight the main factors contributing to increased concentration. An increase in the price elasticity of demand has been the largest driver of earnings inequality concentration and the second-largest driver of firm sales concentration, accounting for approximately 60% of the rise in earnings inequality and 23% of the increase in sales concentration. This reflects that consumers have become more sensitive to prices, which in turn increases sensitivity to skill and productivity differences, leading to a higher valuation of high skills and productivity.

The shift in the skill distribution of workers has been a significant driver not only of earnings inequality but also of sales concentration. The increased concentration of skills explains approximately 58% of the rise in earnings inequality and 19% of the increase in sales concentration. While changes in skills and returns to skills have been major drivers of earnings inequality over the past few decades (Acemoglu & Autor (2011)), these factors have not been linked to the increased concentration of firms in recent literature.

A decrease in the dispersion of the sensitivity function fully explains the rise in firm concentration, measured by employment, and also accounts for a significant portion of the increase in firm sales concentration. The reduction in sensitivity function dispersion indicates that more productive firms are able to grow larger, thus increasing concentration. In contrast, the calibrated decrease in productivity dispersion has moderated the rise in both earnings and firm concentration.

This paper contributes to the existing literature on the determinants of inequality and firm concentration by providing a unified explanation for both. Moreover, the model generates highly skewed earnings and firm size distributions with similar functional forms as observed in data. While the main suspect identified in this paper, a technological change that has made certain skills of workers more scarce and thus highly-demanded and highly-paid, has been emphasized as a source of earnings inequality, this paper shows that it can also lead to increased firm concentration. This result can provide a complementary explanation for the rise of superstar firms: while Autor et al. (2020) and Cortes & Tschopp (2020) emphasize the changes in demand-side through changes in price elasticity of demand, I show that the changes in skill distribution can also lead to concentration in both production and earnings along with price elasticity of demand. Because of the complementarities between firms' productivity and workers' skills, the best firms can attract highly skilled but scarce workers by paying them the highest wages. The firms then benefit from these scarce talents and can grow larger relative to other firms.

2 Related literature

To be added

3 Stylized facts

This section provides empirical evidence on the long-run patterns of employment, sales, earnings, and income concentration and their correlations in the United States. I measure firm size concentration—both in employment and sales—by using the commonly used top 1 percent shares. For example, the top 1 percent employment share measures the share of workers that work in the firms on the 99th percentile of the employment size distribution. Similarly, I measure earnings and income concentration using the top 1 percent earnings or income share, the share of total earnings received by the 99th percentile of the earnings distribution.

I obtain data on the top 1 percent employment and sales concentration of U.S. firms from Kwon et al. (2024). They digitize historical publications of the Statistics of Income from the Internal Revenue Service to construct measures of U.S. firm size distributions measured in sales, assets, and net income. Data on size bins based on receipts (sales) is available after 1959. A caveat of Kwon et al. (2024) analysis is that they only have comprehensive data on corporations (both C- and S-corporations). However, for the years for which they have data on noncorporations, they find similar results of rising concentration. Kwon et al. (2024) also estimate the top firm employment shares using data from the census database on Business Dynamics Statistics. These data are available starting from 1978.

I use data on the top 1 percent earnings share from Gould & Kandra (2022), available starting from 1979. I focus on earnings inequality rather than more widely used income inequality to match more closely with the model presented in section 4. Specifically, the model focuses on solving the labor income distribution, rather than the capital income distribution. While the top income inequality measures include both the labor and capital income, the earnings inequality measure includes only labor income, making it more comparable measure to map with the model.[1] For consistency, I also report the top 1 percent income share obtained from the World Inequality Database.

Panels A, B, C, and D in Figure 1 display trends in employment, sales, earnings, and income concentration, respectively, between 1979 and 2018. As have been documented by Kwon et al. (2024), Autor et al. (2020), and Covarrubias et al. (2020), concentration in firm sales and employment has increased over the past three decades. The top 1 percent employment share has increased from 54 percent in 1987 (the lowest value in the sample) to 60 percent in 2018 (the highest value in the sample). In other words, the top 1 percent

[1] It is worth noting that workers are increasingly compensated using equity-based compensation, especially at the top of the earnings distribution Eisfeldt et al. (2023). One could thus interpret the labor compensation in the model to be consisted of both typical cash pay and equity compensation. In that case, an inequality measure based on total income could be more accurate.

of the firms in the employment distribution employed three workers out of five in 2018. Concentration is firm sales in even more pronounced: in 1986, the top 1 percent of the firms were responsible for 69 percent of total firm sales, and this share has increased to 81 percent by 2018.

The top 1 percent earnings share follows a similar pattern. The share has almost doubled from 7 percent in 1979 to 13 percent in 2018.

I then analyze the correlations between the concentration measures. The results are summarized in Figure 2, where charts show correlations between different concentration measures using both binned and raw scatter plots and Pearson correlation coefficients (r).

The results show that all concentration measures are strongly positively correlated, raw Pearson correlation coefficients ranging from 0.68 to 0.96. Not surprisingly, firm employment and sales concentration measures show a tight positive relationship (Panels A and B). The correlation coefficient using raw data is 0.96. Maybe more surprisingly, the top 1 percent earnings share is strongly positively correlated with both the top 1 percent employment and sales shares.

4 A static assignment model with heterogeneous firms and workers

4.1 Preliminaries

I consider a static assignment problem with two-sided heterogeneity where heterogeneous workers sort into positions within heterogeneous firms. A worker g differs in their skills $T(g)$ and a firm i in their productivity $A(i)$. Production in a given position requires technology measured by productivity of a position in a given firm and human capital, or skills, of a worker. Firms compete monopolistically under constant elasticity of product substitution and produce separate varieties $\omega \in \Omega$, where Ω consists of a continuum of varieties.

While canonical assignment models typically assume one-to-one matching (Becker (1973); Sattinger (1993); Teulings (1995); Gabaix & Landier (2008); Terviö (2008)), where each firm is matched with one worker, I extend the one-to-one matching framework as in Gabaix & Landier (2008): A firm can hire multiple workers in the spirit of many-to-one matching.

A firm is built out of job positions. Each job position within a firm varies in its productivity. While each position shares underlying firm productivity $A(i)$, the overall productivity of each position is decreasing in the position's rank within the firm hierarchy. For example, the position of Chief Executive Officer (CEO) has a higher productivity than any lower-ranked position and, thus, contribute more to the overall output of the firm. Alternatively, job positions can be be thought of as job levels. Job levels are characterized by the degree of autonomy, complexity, and responsibility in which job tasks are carried out (Bayer & Kuhn (2023)), with higher job levels requiring a higher degree of these characteristics. The variation of these required characteristics across job levels is then captured by differences in productivity. The assumption of within-firm job positions with different

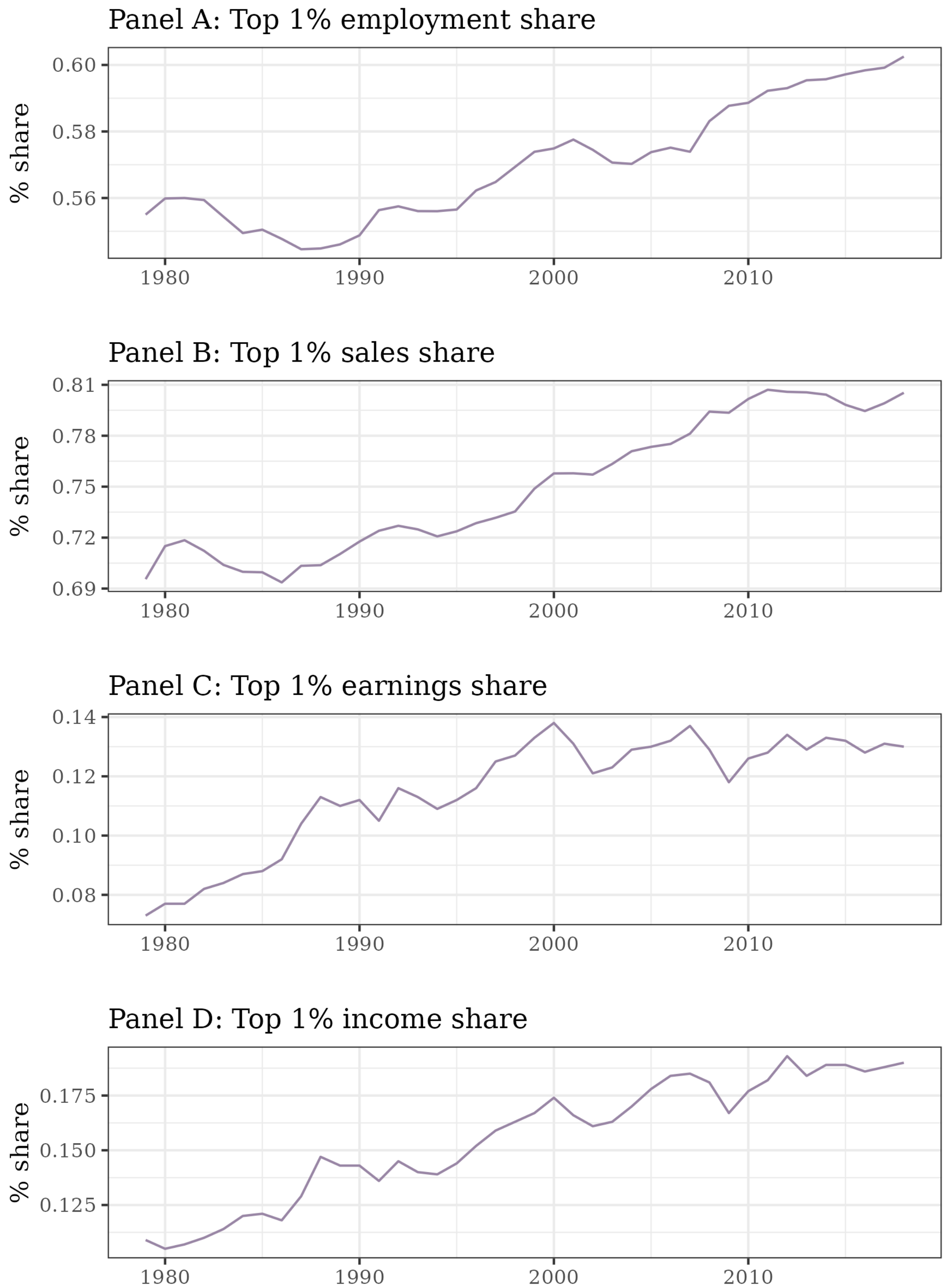


Figure 1: Employment, sales, and earnings concentration between 1979 and 2018 in the United States.

Note:
Sources: Gould & Kandra (2022); Kwon et al. (2024).

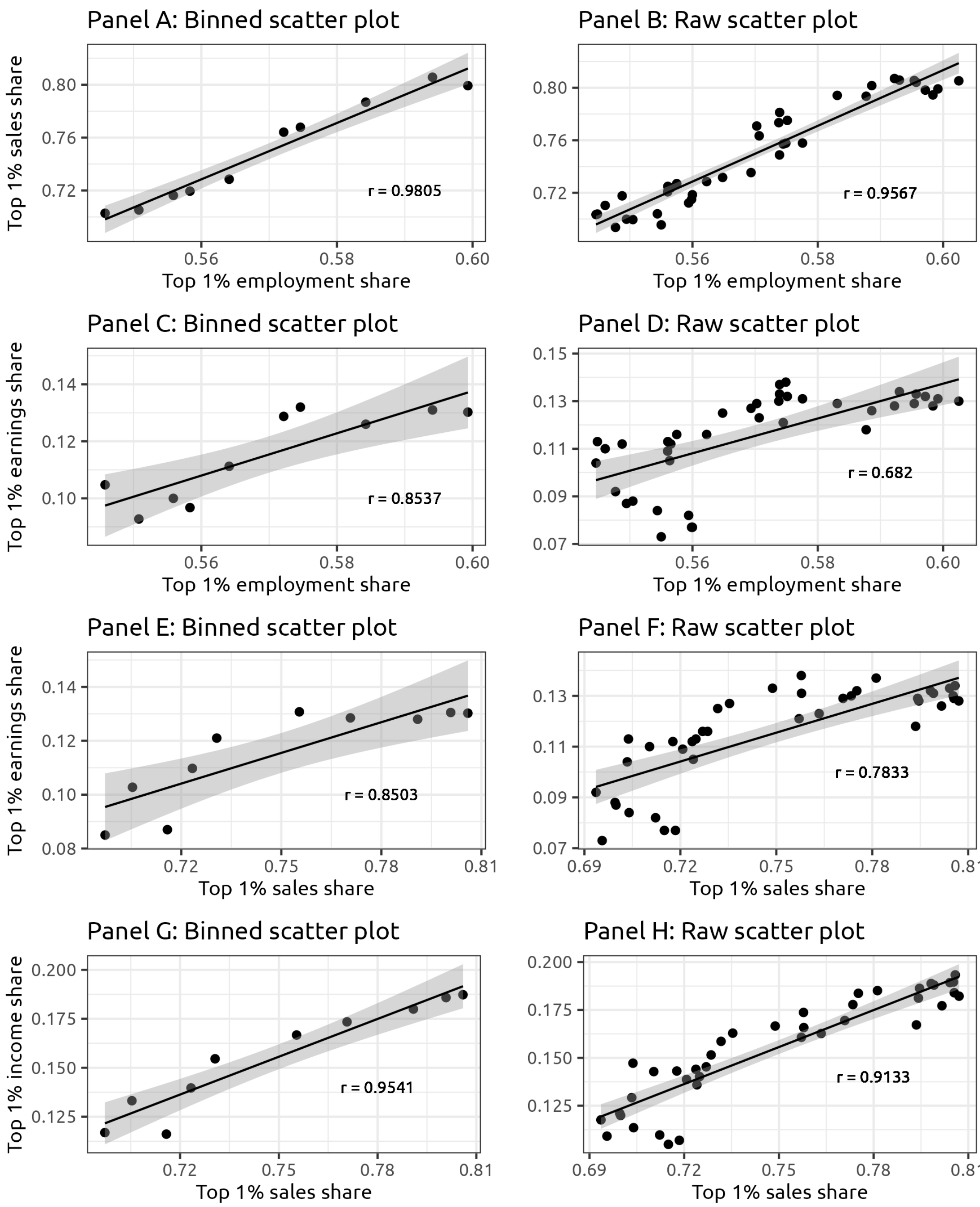


Figure 2: Correlation between employment, sales, and earnings concentration between 1979 and 2018 in the United States.

Note: r corresponds to Pearson correlation measure.
Sources: Gould & Kandra (2022); Kwon et al. (2024); Author's calculations.

productivity is consistent with the fact that the highest productivity firm does not hire all the highest skilled workers. This feature also restrains the size of firms.

Specifically, decreasing productivity of each position, given its rank, is measured by a sensitivity function $c(h)$, $c'(h) < 0$, where h is the rank of a position in a firm. Before-worker-match productivity of a position h in a firm i is then $c(h)A(i)$, which I call effective productivity of a position. The total number of active positions in each firm is equal to the number of workers hired by the firm. To produce, each position pays a fixed cost f_e, so the total number of positions and firms is pinned down by f_e and $c(h)$. After a position is matched with a worker, final productivity of a position depends on both productivity of the position (before match) and skills of the hired worker. The total output of a firm equals the summed-up output of the positions.

Wages of workers in each firm and position are endogenously determined in a competitive market. The wage-setting mechanism follows Gabaix & Landier (2008) and Terviö (2008) assignment models used to study CEO wages: There exists an assortative matching between positions and workers based on positions' effective productivities and workers' skills. In the equilibrium, there will be a positive assortative matching between workers and positions, implying that the highest-skilled workers will be hired by the positions with the highest effective productivity.

Lastly, I assume that there are no complementarities between workers. This assumption provides tractability on the solution as it implies that each assignment problem between a given firm and each of its workers can be solved independently in the same way as in a one-to-one matching case.

4.2 Workers

4.2.1 Skill distribution of workers

Workers have heterogeneous skills that are drawn from a given distribution. Each worker $g \in (0, 1]$ has a skill (or human capital) level determined by a function $T(g)$, and the skill function has the following properties: a worker g can be ranked based on her skill $T(g)$, and $T(g)$ is a decreasing function of g, $T'(g) < 0$. This property means that a lower value of g implies a more skilled worker. Workers supply their labor inelastically, so $T(g)$ can be interpreted as a worker g's efficiency units of labor.

Assumption 1: The skill distribution is of a form $T(g) = Bg^{-\beta}$, where B is the scale parameter, and β is the shape parameter of the distribution. This distribution corresponds to a Pareto distribution with a scale parameter equal to $B^{1/\beta}$ and a shape parameter equal to $1/\beta$.

The benefit from assuming that the skill distribution of workers follows a Pareto is that it provides analytical tractability. However, the assumption may appear strong at a first glance. For example, many traditional skill measures, like intelligence quotient (IQ), tend to follow a normal distribution. However, in this context, skills capture the whole range of experiences and skills workers have accumulated over their life cycle. Consider a snapshot of workers in a labor market. Workers are characterized by their age, experience, skills, education, innate talent, and so on. Some workers are early in their careers with limited experience while others have climbed up the career ladder and accumulated a variety of skills. While innate intelligence distribution of workers may not follow a Pareto, this is not necessarily true for the distribution of accumulated experience and skills. For example, it is well know that proportional random growth is a basic mechanism generating Pareto distributions (Champernowne (1953); Simon (1955)). If worker skills grow and shrink according to proportional random growth over the life cycle, Gibrat's law holds (each worker's skills have the same expected growth rate and the same standard deviation of growth rate), and there exists a lower bound for a skill level, a steady state distribution will be a Pareto.

There exists a threshold level of skills, G, determined in terms of the worker ranking such that workers with $g \leq G$ will be matched with a firm and produce, while everyone else will produce at home and consume a value of home production, w. There are no information asymmetries, so both productivity of firms and positions, and skills of workers are known by each worker and each firm. Compensation of a worker will depend on their skill, $T(g)$—in the equilibrium, she will earn a wage rate $w_g(T)$. Wages are bounded below by a wage rate $w_g(T) \geq w$. The level of w represents the minimum wage level a worker will accept, which I assume to be equal to the value of home production.

4.2.2 Preferences and utility maximization problem

I assume that workers' preferences follow the Pollak's additive utility functions (Pollak (1971)) and focus on the special case of constant elasticity of substitution (CES) preferences. All workers have same preferences but have different total incomes, W_g. I abstract away from the consumption-savings problem by assuming that workers consume all their income. The general form of Pollak's preferences is given by an additive utility function of the form

$$U_g = \int_{\omega \in \Omega_g} \alpha_\omega (q_g(\omega) - \gamma_g)^{1-\frac{1}{\sigma}} d\omega,$$

where $q_g(\omega)$ is a worker g's consumption of a variety ω, γ_g is a constant, and $q(\omega) > \gamma_g$. The parameter α_ω is a variety-specific demand shifter, which I from now on assume to equal 1. I assume that $\Omega \subseteq \bar{\Omega}$, where $\bar{\Omega}$ is a compact set containing all potential varieties in the economy, while Ω contains all varieties produced by active firms. The CES form of Pollak's preferences is obtained by setting $\gamma_g = 0$,

$$U_g = \int_{\omega \in \Omega_g} q_g(\omega)^{1-\frac{1}{\sigma}} d\omega,$$

and the maximization problem of a worker g buying varieties $\omega \in \Omega$ becomes

$$\max_{q_g(\omega) \geq 0} U_g = \int_{\omega \in \Omega} q_g(\omega)^{1-\frac{1}{\sigma}} \quad s.t. \quad W_g \geq \int_{\omega \in \Omega} p(\omega) q_g(\omega) d\omega,$$

where $p(\omega)$ denotes the price of a variety ω. The first order conditions can be written as

$$q_g(\omega) : \ (1 - \frac{1}{\sigma})q_g(\omega)^{-\frac{1}{\sigma}} = \lambda_g p(\omega) \ \ \forall \ q_g(\omega) > 0$$

$$\lambda_g : \ W_g = \int_{\omega \in \Omega} p(\omega) q_g(\omega) d\omega,$$

where λ_g is a Lagrange multiplier. The demand function for each variety, ω, by a worker g is then[2]

$$q_g(\omega) = W_g p(\omega)^{-\sigma} \int_{\Omega} p(\omega^{'})^{-1} p(\omega^{'})^{\sigma} d\omega^{'} = p(\omega)^{-\sigma} \frac{W_g}{P},$$

where $\omega^{'}$ is taken from a set of all positively consumed varieties in the economy and $P = \int_{\Omega} p(\omega^{'})^{1-\sigma} d\omega^{'}$ is the aggregate price index in the economy. The demand for each variety is a decreasing function of its own price and an increasing function of a worker's income W_g. The aggregate demand for each variety depends on the aggregate income in the economy, $W = \int_g W_g \, dg$,

$$q(\omega) = \frac{1}{P} p(\omega)^{-\sigma} \int_g W_g dg = p(\omega)^{-\sigma} \frac{W}{P}. \tag{1}$$

Given the CES preferences, the price elasticity of demand is constant and equal to $\varepsilon(\omega) = -\sigma$.

4.3 Firms

4.3.1 Production technology and productivity distribution of firms

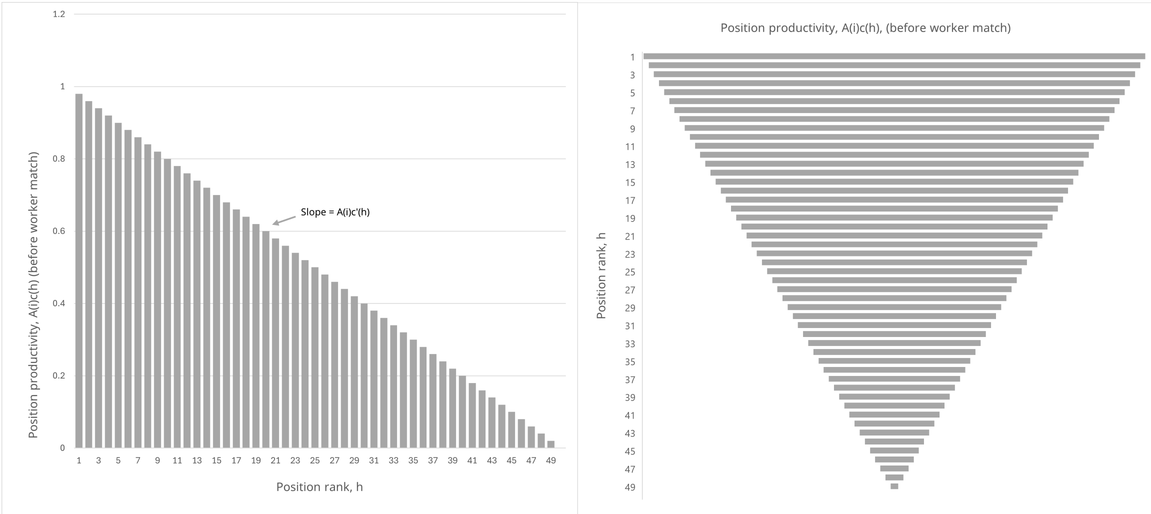
Firms draw their productivity from a given distribution. Each firm indexed by $i \in (0, 1]$ has a productivity $A(i)$. In the same way as workers, firms can be ranked based on their productivity, where i is the rank of a firm. The lower rank i implies higher productivity, and $A(i)$ is a decreasing function of i.

Assumption 2: The productivity distribution is of a form $A(i) = Di^{-\delta}$, where D is the scale parameter, and δ is the shape parameter of the distribution. This distribution corresponds to a Pareto distribution with a scale parameter equal to $D^{1/\delta}$ and a shape parameter equal to $1/\delta$.

As mentioned above, each firm is made off heterogeneous job positions. Each position within a firm i differs from the firm along two dimensions. First, each additional open position h in i has a decreasing productivity and, thus, a decreasing positive impact on the firm's total output. As in Gabaix & Landier (2008), this impact is measured using a sensitivity function $c(h) > 0$, $c'(h) < 0$, which implies that $c(1) \geq c(2) \geq ... \geq c(h_i^*)$ where h_i^* is the last active position a firm i, determined in the equilibrium. The larger the firm, the smaller is the output that an additional worker generates, given the firm productivity.

[2]Refer to the appendix for algebra.

The described organizational structure of a firm is illustrated in Figure 3. Panel (a) plots effective productivity of each position h for a given firm with productivity $A(i)$ and a given functional form of $c(h)$. In this example, effective productivity of a position decreases linearly in its rank h. Panel (b) illustrates the same organizational hierarchy, but in a common pyramid shape. The organization is consisted of positions, or job levels, and each position contributes to the total output of a firm by a varying degree determined by $c(h)$.



(a) Productivity of position by rank

(b) Organizational hierarchy, ranked by productivity

Figure 3: Firm structure and productivity of positions before worker match

Second, each position is matched with a worker with a varying skill $T(g)$ so the total productivity of a position depends on its worker's skill. The total productivity of a position, $c(h)A(i)T(g)$, depends then on the skill of the worker, a firm's productivity $A(i)$, and the rank h of the position within a firm i. Each position with a worker is indexed by a triple (i, g, h).

The production function for each position is of the following form: $y(i, g, h) = c(h)A(i)T(g)$. The output of a position is increasing in a firm's and its worker's rank (i.e., decreasing in i and g) as well as a position's rank in the firm hierarchy. The marginal cost of each position (i, g, h) is of the form $c(i, g, h) = \frac{1}{c(h)A(i)T(g)}$.

4.3.2 Effective productivity of each position

As in Gabaix & Landier (2008), I will need to determine the effective productivity distribution of positions to solve the assignment problem between workers and positions.[3] The firm productivity distribution is given by $A(i) = Di^{-\delta}$, which provides a one-to-one mapping between a firm's rank i and its productivity $A(i)$. The effective productivity of a position is given by $\tilde{A}(i, h) = c(h)A(i)$, where $c(h)$ is a function that measures productivity

[3]As a position's productivity will determine the size, measured in sales, of each position, I will use the "size" and "productivity" of a position interchangeably.

of a position ranked h^{th} in a firm.

All positions in the economy can be ranked based on their effective productivity. I need to define a mapping between effective productivity and an effective rank. Assume that i is the upper quantile (or the rank) of a firm with productivity a. Then, i satisfies $i = P(\hat{A} > a)$. Using $A(i) = Di^{-\delta}$, we have $i = P(A > a) = D^{1/\delta}A^{-1/\delta}$. Following the same logic, it follows that

$$i = P(\tilde{A} > a) = P(c(h)A_i > a) = P(A_i > a/c(h))$$

$$= E(P(A_i > a/c(h)|c(h)) = E(D^{1/\delta}(a/c(h))^{-1/\delta})$$

$$= D^{1/\delta}E(c(h)^{1/\delta})a^{-1/\delta}.$$

This result implies that effective productivity at a quantile i is $\tilde{A}(i,h) = \tilde{D}i^{-\delta}$ with $\tilde{D} = DE\left[c(h)^{1/\delta}\right]^{\delta}$. The average sensitivity, the term $\bar{c} = E[c(h)^{1/\delta}]^{\delta}$ will be an average sensitivity over all positions in the economy, $\bar{c} = \left(I^{-1}\int_{i=I}^{0} h_i^{*-1}\left(\sum_{h=1}^{h_i^*} c_{h_i}^{1/\delta}\right) di\right)^{\delta}$, where I is the total number of firms in the economy.

4.3.3 A position's profit-maximization problem

Given the ranked effective productivity, the profit-maximization problem of positions can now be solved using backward induction, as in Jung & Subramanian (2017). I can first solve an active position's optimal pricing rule for all (i, g, h), taking each position's productivity as given. I can then solve the optimal assignment problem of workers into positions, and last, a position's entry problem.

As mentioned above, in order to solve the profit-maximization problem separately for each position, I need to assume that there are no complementarities between workers. Skill of one worker has no effect of a skill of a co-worker. In Appendix (B), I expand on the potential implications of this assumption. Specifically, I compare a solution of firm revenue and worker wage in a case where a firm chooses one price depending on the total productivity of its workers. In that case, the solution for total revenue differs from this baseline solution via additional, positive cross-terms between coworkers skills, implying higher total revenues. By ignoring these positive cross-terms, the baseline solution effectively shuts down all these positive complementarities. While these complementarities are interesting and worth of studying, I abstract away from complementarities in order to keep the model tractable and to obtain closed-form solutions.

Pricing rule. An active position with an effective size $\tilde{A}(i,h)$ will be choosing its price to maximize its profits. Its total productivity is $\tilde{A}(i,h)T(g)$, where $T(g)$ is the skill of a worker matched with a position ranked i. Conditional on g, the optimal pricing rule is then

$$p_{i,g,h} = \frac{\varepsilon(\omega)}{1+\varepsilon(\omega)} \frac{1}{\tilde{A}(i,h)T(g)} = \frac{\sigma}{\sigma-1} \frac{1}{\tilde{A}(i,h)T(g)}, \tag{2}$$

which follows the typical markup pricing rule in monopolistic competition models.

The output and the maximized profits of each position with an effective size $\tilde{A}(i,h)$ are then

$$q_{i,g,h} = \left(\frac{\sigma-1}{\sigma}\right)^{\sigma} \left[\tilde{A}(i,h)T(g)\right]^{\sigma} \frac{W}{P}, \tag{3}$$

$$\pi^*_{i,g,h} \equiv p_{i,g,h}q_{i,g,h} - w_g - f_e = \frac{W}{P}\left(\frac{\sigma-1}{\sigma}\right)^{\sigma-1} \left[\tilde{A}(i,h)T(g)\right]^{\sigma-1} - w_g - f_e, \tag{4}$$

where w_g is the wage of a worker with ranking g, determined in the equilibrium, and f_e is fixed cost of production.

4.3.4 Assignment problem

There will now be one-to-one matching between a worker with a rank g and a position with an effective rank of i. It is easy to see that the supermodularity (i.e., complementarity between worker and position rank) condition required for positive assortative matching holds. Formally, a firm chooses a worker to maximize

$$\underset{g}{max}\ \pi^*_{i,g,h} = \frac{W}{P}\left(\frac{\sigma-1}{\sigma}\right)^{\sigma-1} \left[\tilde{A}(i,h)T(g)\right]^{\sigma-1} - w_g - f_e.$$

The total revenue that is generated by a position ranked i and a worker g equals

$$R_{i,g,h} = \frac{W}{P}\left(\frac{\sigma-1}{\sigma}\right)^{\sigma-1} \left[\tilde{A}(i,h)T(g)\right]^{\sigma-1},$$

and it is easy to show that the marginal revenue of i is increasing in g,

$$\frac{\partial^2 R}{\partial i \partial g} = \frac{W}{P}\left(\frac{\sigma-1}{\sigma}\right)^{\sigma-1} (\sigma-1)\left[\tilde{A}(i,h)T(g)\right]^{\sigma-2} \tilde{A}'(i,h)T'(g) > 0, \tag{5}$$

as $\tilde{A}'(i,h)$, $T'(g) < 0$. This implies that the supermodularity condition holds, and as shown in the previous literature (Sattinger (1993); Legros & Newman (2007)), the matching equilibrium is then unique and positive assortative matching holds. This implies that higher-skilled workers will work in higher productivity positions.

The first-order condition of a position's choice of a worker then equals

$$\frac{\partial \pi^*_{i,g,h}}{\partial g} = \frac{W}{P}\frac{(\sigma-1)^{\sigma}}{\sigma^{\sigma-1}}\tilde{A}(i,h)\left[\tilde{A}(i,h)T(g)\right]^{\sigma-2} T'(g) - w'_g = 0$$

$$\Leftrightarrow w'_g = -\beta \frac{W}{P}\frac{(\sigma-1)^{\sigma}}{\sigma^{\sigma-1}}(B\tilde{D})^{\sigma-1} i^{-\delta(\sigma-1)} g^{-\beta(\sigma-1)-1}. \tag{6}$$

Equation (6) is the assignment equation shown in Sattinger (1993) and Gabaix & Landier (2008). As the positive assortative matching (PAM) now holds between a worker and a position with the same effective ranking, I can set $i = g$, and write the assignment equation as

$$w_i' = -\beta \frac{W}{P} \frac{(\sigma-1)^\sigma}{\sigma^{\sigma-1}} (B\tilde{D})^{\sigma-1} i^{-(\sigma-1)(\beta+\delta)-1}.$$

The assignment equation states that the marginal cost of hiring a slightly better worker, w_g', is equal to the marginal benefit of hiring a slightly better worker, $-\beta \frac{W}{P} \frac{(\sigma-1)^\sigma}{\sigma^{\sigma-1}} (B\tilde{D})^{\sigma-1} i^{-(\sigma-1)(\beta+\delta)-1}$. Taking integral both sides with respect to i leads to

$$
\begin{aligned}
w_i &= -\beta \frac{W}{P} \frac{(\sigma-1)^\sigma}{\sigma^{\sigma-1}} (B\tilde{D})^{\sigma-1} \int_{i=0}^{i^*} i^{-(\sigma-1)(\beta+\delta)-1} di \\
&= \frac{\beta (B\tilde{D})^{\sigma-1}}{(\beta+\delta)} \frac{W}{P} \left(\frac{\sigma-1}{\sigma}\right)^{\sigma-1} /_{i=i^*}^{i} i^{-(\sigma-1)(\beta+\delta)} + w \\
&= \frac{\beta (B\tilde{D})^{\sigma-1}}{(\beta+\delta)} \frac{W}{P} \left(\frac{\sigma-1}{\sigma}\right)^{\sigma-1} \left[i^{-(\sigma-1)(\beta+\delta)} - i^{*-(\sigma-1)(\beta+\delta)} \right] + w, \qquad (7)
\end{aligned}
$$

where i^* is the rank of the last active position as well as the worker in the economy. Wages are increasing in a common multiplier $\frac{\beta (B\tilde{D})^{\sigma-1}}{(\beta+\delta)} \frac{W}{P} \left(\frac{\sigma-1}{\sigma}\right)^{\sigma-1}$ and in the total number of positions i^* in the economy, and decreasing in i, implying that higher-skilled workers with lower i will be paid more.

If I assume that there exists a sufficiently large number of positions in the economy, $i^{*-(\sigma-1)(\beta+\delta)}$ approaches to zero, and the wage distribution simplifies to

$$w_i = \frac{\beta}{\beta+\delta} \left(\frac{\sigma-1}{\sigma}\right)^{\sigma-1} \frac{W}{P} \left(B\tilde{D}\right)^{\sigma-1} i^{-(\sigma-1)(\beta+\delta)} + w. \qquad (8)$$

Equation (8) shows that the wage level of each worker depends on four factors: first, wages are proportional to a common factor $\frac{\beta}{\beta+\delta} \left(\frac{\sigma-1}{\sigma}\right)^{\sigma-1} \frac{W}{P}$, which captures the overall size of the economy through $\frac{W}{P}$. Second, the wage level depends on the effective size of the position a worker is matched with, $\tilde{D} i^{-\delta}$, to the power of $\sigma - 1$. Third, the wage level depends on the skill of the worker, $B i^{-\beta}$, to the power of $\sigma - 1$. Finally, the wage level of a worker depends on a minimum wage level, w.

If I assume that $w = 0$, or focus on the top wages implying that w has a diminishing effect on the total wage level, the overall wage distribution follows an inverse Pareto, consistent with empirical literature. The shape parameter of the distribution is $-(\sigma-1)(\beta+\delta)$, implying that the degree of wage inequality depends on the price elasticity of demand, σ, and the shape parameters of the skill and productivity distributions, β and δ, respectively.

Similarly, I can characterize the total sales of each position as follows:

$$R_i = \left(\frac{\sigma-1}{\sigma}\right)^{\sigma-1} \frac{W}{P} \left[\tilde{A}(i,h) T(i)\right]^{\sigma-1} = \left(\frac{\sigma-1}{\sigma}\right)^{\sigma-1} \frac{W}{P} \left(B\tilde{D}\right)^{\sigma-1} i^{-(\sigma-1)(\beta+\delta)}. \qquad (9)$$

The total sales of each position are proportional to the common factor, $\left(\frac{\sigma-1}{\sigma}\right)^{\sigma-1}\frac{W}{P}$, the effective size of the position, $\tilde{D}i^{-\delta}$, to the power of $\sigma-1$, and the skill of the worker, $Bi^{-\beta}$, to the power of $\sigma-1$.

4.3.5 Number of positions in each firm, number of firms, and a firm size

Number of positions and firm size measured in employment. I next define the optimal number of positions in each firm—determining a firm size measured in employment—the number of firms, and the firm size measured in sales for each firm. In order to do so, following Gabaix & Landier (2008), I first re-write the total sales of each position in terms of the underlying firm size.

Proposition 1. *In equilibrium, a worker of rank i is matched with a position whose effective size $c(h)A_n$ is ranked i, and the total sales of a position ranked h within a firm n with size A_n can be written as*

$$R_i \equiv w_{n,h} = \underbrace{\left(\frac{\sigma-1}{\sigma}\right)^{(\sigma-1)}\frac{W}{P}}_{\Xi}\left[B\bar{i}^{-\beta}\right]^{(\sigma-1)}\left[\tilde{D}\bar{i}^{-\delta}\right]^{-(\sigma-1)\frac{\beta}{\delta}}\left[\tilde{D}n^{-\delta}\right]^{(\sigma-1)(1+\frac{\beta}{\delta})}$$

$$= \Xi\left[B\bar{i}^{-\beta}\right]^{(\sigma-1)}\left[\bar{c}A\left(\bar{i}\right)\right]^{-(\sigma-1)\frac{\beta}{\delta}}\left[c(h)A_n\right]^{(\sigma-1)(1+\frac{\beta}{\delta})}$$

where $\bar{i}$ is the reference ranking of skill [4].

Proof. As $\tilde{A}\left(i\right) \equiv \tilde{D}i^{-\delta} \equiv c(h)A\left(i\right) \equiv c(h)Di^{-\delta}$; $\tilde{A}(\bar{i}) \equiv \tilde{D}\bar{i}^{-\delta} \equiv \bar{c}A(\bar{i}) \equiv \bar{c}D\bar{i}^{-\delta}$, $T(\bar{i}) \equiv B\bar{i}^{-\beta}$, and $\tilde{A}_n \equiv c(h)A_n$, I can write (8) as

$$\left[\left(\frac{\sigma-1}{\sigma}\right)^{-(\sigma-1)}\frac{P}{W}w_i\right]^{\frac{1}{\sigma-1}} = (B\tilde{D})i^{-(\beta+\delta)} = B\tilde{D}\left(i^{-\delta}\right)^{\beta/\delta-1} = B\tilde{D}^{-\beta/\delta}\left(\tilde{D}i^{-\delta}\right)^{1+\beta/\delta}$$

$$= B\bar{i}^{-\beta}\left[\tilde{D}\bar{i}^{-\delta}\right]^{-\frac{\beta}{\delta}}\left[\tilde{D}i^{-\delta}\right]^{1+\frac{\beta}{\delta}}$$

$$\Leftrightarrow R_i = \Xi\left[B\bar{i}^{-\beta}\right]^{(\sigma-1)}\left[\bar{c}A(\bar{i})\right]^{-(\sigma-1)\frac{\beta}{\delta}}\left[c(h)A(i)\right]^{(\sigma-1)(1+\frac{\beta}{\delta})}. \tag{10}$$

As equation (10) holds for any size of a firm, A_n, it can be written as

$$\Leftrightarrow R_i \equiv R_{n,h} = \Xi\left[B\bar{i}^{-\beta}\right]^{(\sigma-1)}\left[\bar{c}A(\bar{i})\right]^{-(\sigma-1)\frac{\beta}{\delta}}\left[c(h)A_n\right]^{(\sigma-1)(1+\frac{\beta}{\delta})}. \tag{11}$$

□

An equilibrium sales of each position h in a firm n with size A_n thus depends on a common factor $\Xi\left[B\bar{i}^{-\beta}\right]^{(\sigma-1)}\left[\bar{c}A\left(\bar{i}\right)\right]^{-(\sigma-1)\frac{\beta}{\delta}}$ and a position-specific factor $\left[c(h)A_n\right]^{(\sigma-1)(1+\frac{\beta}{\delta})}$.

[4] A reference skill could be, for example, the skill level of a median worker.

The sales of each position in the economy is proportional to Ξ, which captures the effect of aggregate income and the price index on wages, the skill level of the worker in a reference firm, $\left[B\bar{i}^{-\beta}\right]$, to the power of $\sigma - 1$, and productivity of the reference firm, $[\bar{c}A(\bar{i})]$, to the power of $-(\sigma-1)\frac{\beta}{\delta}$. Moreover, the sales of a position h in a firm n are proportional to the effective size of the position, to the power of $(\sigma-1)(1+\frac{\beta}{\delta})$.

Following Proposition 1, I can write the wage level of each position as

$$w_{n,h} = \left(\frac{\beta}{\beta+\delta}\right)\left(\frac{\sigma-1}{\sigma}\right)^{\sigma-1}\frac{W}{P}\left[B\bar{i}^{-\beta}\right]^{(\sigma-1)}[\bar{c}A(\bar{i})]^{-(\sigma-1)\frac{\beta}{\delta}}\left[c(h)A_n\right]^{(\sigma-1)(1+\frac{\beta}{\delta})}. \tag{12}$$

Then, I can write the profits for each position in terms of $c(h)$ and n as follows

$$\pi_{n,h} = R_{n,h} - w_{n,h} - f_e$$

$$= \left(\frac{\delta}{\beta+\delta}\right)\left(\frac{\sigma-1}{\sigma}\right)^{(\sigma-1)}\frac{W}{P}\left[B\bar{i}^{-\beta}\right]^{(\sigma-1)}\left[\bar{c}D\bar{i}^{-\delta}\right]^{-(\sigma-1)\frac{\beta}{\delta}}\left[c(h)Dn^{-\delta}\right]^{(\sigma-1)(1+\frac{\beta}{\delta})} - f_e \geq 0.$$

Using the above expression, I can determine the number of positions in a given n as:

$$\left(\frac{\delta}{\beta+\delta}\right)\underbrace{\left(\frac{\sigma-1}{\sigma}\right)^{(\sigma-1)}\frac{W}{P}\left[B\bar{i}^{-\beta}\right]^{(\sigma-1)}\left[\tilde{D}\bar{i}^{-\delta}\right]^{-(\sigma-1)\frac{\beta}{\delta}}D^{(\sigma-1)(1+\frac{\beta}{\delta})}}_{\Gamma}\left[c(h^*)n^{-\delta}\right]^{(\sigma-1)(1+\frac{\beta}{\delta})} = f_e$$

$$\Leftrightarrow c(h^*)^{(\sigma-1)(1+\frac{\beta}{\delta})}n^{-(\sigma-1)(\beta+\delta)} = \left(\frac{\beta+\delta}{\delta}\right)\frac{f_e}{\Gamma}$$

$$\Leftrightarrow c(h^*)^{(\sigma-1)(1+\frac{\beta}{\delta})} = \left(\frac{\beta+\delta}{\delta}\right)\frac{f_e}{\Gamma n^{-(\sigma-1)(\beta+\delta)}} \Leftrightarrow c(h_n^*) = \left(\frac{\beta+\delta}{\delta}\frac{f_e}{\Gamma}\right)^{\frac{1}{(\sigma-1)(1+\frac{\beta}{\delta})}}n^{\delta}. \tag{13}$$

The number of positions in each firm ranked n is pinned down by fixed cost f_e. As $c(h)$ is decreasing in h, the above expression shows that the $c(h_i^*)$ is decreasing in n: as the right-hand side of the equation is increasing in n, it must be that the number of positions h_n^* is decreasing in n. This implies that the number of positions (and workers) will be higher in a high-productivity firm with low n.

If I further assume that the specific functional form of $c(h)$ is a Pareto, $c(h) = Ch^{-\rho}$, the firm size distribution measured by the number of workers (or positions) is also a Pareto:

$$C{h_n^*}^{-\rho} = \left(\frac{\beta+\delta}{\delta}\frac{f_e}{\Gamma}\right)^{\frac{1}{(\sigma-1)(1+\frac{\beta}{\delta})}}n^{\delta} \Leftrightarrow h_n^* = \left(\frac{1}{C}\right)^{-\frac{1}{\rho}}\left(\frac{\beta+\delta}{\delta}\frac{f_e}{\Gamma}\right)^{-\frac{1}{\rho(\sigma-1)(1+\frac{\beta}{\delta})}}n^{-\frac{\delta}{\rho}}. \tag{14}$$

Equation (14) states that the firm size distribution is an inverse Pareto with a shape parameter $-\frac{\delta}{\rho}$. This result is consistent with the empirical literature estimating that Pareto distribution is a good approximation on the size distribution. The shape parameter $-\frac{\delta}{\rho}$

indicates that the firm size distribution becomes more skewed (concentration increases) when δ increases or ρ decreases. On the one hand, concentration in firm size—measured in employment—increases in the skewness of the firm productivity distribution measured by δ. The higher the concentration in productivity, the higher the concentration in employment. On the other hand, firm size concentration is decreasing in ρ, measuring the shape of the organizational hierarchy. Lower ρ implies a lower hierarchy with each position within a firm having more similar productivity. Thus, a lower ρ increases concentration in firm size as the most productive firms will higher a larger number of workers.

Note that the shape parameter of the firm size distribution, where size is measured as the number of workers, is independent of β and σ and only depends on the firm technology captured by firm productivity and hierarchical structure.

Firm size measured in sales. Using (11), I can write the total sales of a firm ranked n as follows

$$R_n = \sum_{k=1}^{h_n^*} R_{n,k} = \sum_{k=1}^{h_n^*} \frac{W}{P} \left(\frac{\sigma-1}{\sigma}\right)^{\sigma-1} \left[B\bar{i}^{-\beta}\right]^{(\sigma-1)} \left[\tilde{D}\bar{i}^{-\delta}\right]^{-(\sigma-1)\frac{\beta}{\delta}} \left[c_k D n^{-\delta}\right]^{(\sigma-1)(1+\frac{\beta}{\delta})}$$

$$= \underbrace{\frac{W}{P} \left(\frac{\sigma-1}{\sigma}\right)^{\sigma-1} \left[B\bar{i}^{-\beta}\right]^{(\sigma-1)} \left[\tilde{D}\bar{i}^{-\delta}\right]^{-(\sigma-1)\frac{\beta}{\delta}} D^{(\sigma-1)(1+\frac{\beta}{\delta})}}_{\Gamma} n^{-(\sigma-1)(\beta+\delta)} \sum_{k=1}^{h_n^*} c_k^{(\sigma-1)(1+\frac{\beta}{\delta})}$$

$$= \Gamma i^{-(\sigma-1)(\beta+\delta)} \sum_{k=1}^{h_i^*} c_k^{(\sigma-1)(1+\frac{\beta}{\delta})}. \tag{15}$$

Total number of firms. The total number of firms, I, will be determined as the lowest productivity firm that can have at least one worker, produce, and generate non-negative profits. Formally, using (13), I can write

$$c_1 = \left[\frac{\beta+\delta}{\delta}\frac{f_e}{\Gamma}\right]^{\frac{1}{(\sigma-1)(1+\frac{\beta}{\delta})}} I^{\delta}. \tag{16}$$

If I further assume that $c(h) = Ch^{-\rho}$, then the number of firms equals

$$I = C^{\frac{1}{\rho}} \left[\frac{\beta+\delta}{\delta}\frac{f_e}{\Gamma}\right]^{-\frac{1}{(\sigma-1)(\delta+\beta)}}.$$

The number of firms is increasing in C and the aggregate size of the market Γ, while decreasing in the fixed cost of production, f_e.

4.4 Model predictions

Given the model results presented in the previous section, I can draw information on the model predictions related to earnings inequality and firm concentration.

Proposition 2. *Overall earnings inequality is increasing in σ, β, and δ.*

Proof. Using (8), the wage ratio between workers with different rankings equals

$$\frac{w_i}{w_{i'}} = \left(\frac{i'}{i}\right)^{(\sigma-1)(\beta+\delta)}.$$

Without a loss of generality, assume that $i' > i$. The overall wage differentials between workers are thus increasing in σ, δ and β, indicating that more skewed skill and productivity distributions along with a higher price elasticity of substitution lead to higher earnings inequality. □

The prediction that a higher σ increases earnings inequality arises from a notion that a higher σ implies a higher price sensitivity of consumers. A higher σ means that an increase in productivity and, thus, a decrease in price increases demand and revenue for a product relatively more compared with an environment with a lower σ. As any increase in productivity is valuable for firms, firms are willing to pay more to higher-productive workers, leading to higher inequality. Note also that a higher σ implies that firms can charge a lower markup on their products in the product market. The model thus predicts that markets with more competitive product markets (with lower markups) have higher inequality compared to less competitive product markets, all else equal.

Wage inequality also depends on the skewness of both the skill and productivity distributions. The more unequal skill or productivity leads to higher wage inequality.

Proposition 3. *Concentration of the firm size distribution, measured in sales, depends on σ, β, δ, and ρ.*

Proof. Using (15), the sales ratio between firms with different rankings equals

$$\frac{R_n}{R_{n'}} = \left(\frac{n'}{n}\right)^{(\sigma-1)(\beta+\delta)} \frac{\sum_{k=1}^{h_n^*} c_k^{(\sigma-1)(1+\frac{\beta}{\delta})}}{\sum_{k=1}^{h_{n'}^*} c_k^{(\sigma-1)(1+\frac{\beta}{\delta})}}.$$

The first part of the sales ratio, $\left(\frac{n'}{n}\right)^{(\sigma-1)(\beta+\delta)}$ shows that, similar to earnings inequality, sales concentration depends on the shapes of the skill and productivity distributions and the price elasticity of demand. In addition, sales concentration also depends on the relative number of workers in each firm, shown by the second term. As $h_n^* > h_{n'}^*$, the ratio $\frac{\sum_{k=1}^{h_n^*} c_k^{(\sigma-1)(1+\frac{\beta}{\delta})}}{\sum_{k=1}^{h_{n'}^*} c_k^{(\sigma-1)(1+\frac{\beta}{\delta})}} > 1$, and the ratio is increasing in the skewness of the employment size distribution. The skewness of the employment size distribution is increasing in δ and decreasing in ρ, as shown in equation (14). □

Propositions 2 and 3 imply that concentration in both earnings and sales distribution depends partly on the same parameters. This result makes intuitively sense as firms are competing for heterogeneous talent by paying the workers based on their value for the firm. Because of the complementarities between firm productivity and worker skill, more productive firms can pay more (as they benefit more from a higher skill), making the pay dependent of the firm productivity distribution and the sales dependent of the worker skills and its distribution. More competitive product markets with a high σ further amplify these effects as any increase in total productivity of firms (which depend on both worker skill and firm productivity) will increase the market share of the firm relatively more.

Proposition 4. *Concentration in the firm size distribution, measured in employment, is increasing in δ and decreasing in ρ.*

Proof. Using (13), the worker ratio between firms with different rankings equals

$$\frac{h_i^*}{h_{i'}^*} = \left(\frac{i'}{i}\right)^{\frac{\delta}{\rho}}.$$

Again, assume that $i' > i$, and the worker ratio, and thus concentration in the number of workers, is increasing in δ and decreasing in ρ. □

The concentration in the firm size, now measured in employment, only depends on the skewness of the firm productivity distribution and the organizational structure. The number of workers is determined by the zero-profit condition, given the fixed cost of hiring a new worker, f_e. More productivity firms can hire more workers as they can sustain a lower-productivity worker and still cover the fixed cost. Also, if the organization structure, $c(h)$, is less skewed (lower ρ), the benefits of hiring an additional worker are decaying at a lower rate, increasing the number of workers hired. As higher-productivity firms are benefiting more from lower ρ, the overall concentration in employment is decreasing in ρ.

Finally, the model has implications on wage markdowns. A wage markdown is the ratio between the wage level and the marginal revenue product of labor. This marginal revenue product of labor measures the change in revenue that results from employing an additional unit of labor. In this model, the marginal revenue product for hiring each additional worker is exactly equal to the position revenue, defined in equation (11). Thus, a proposition follows:

Proposition 5. *The wage markdown for each worker equals $\frac{\beta}{\beta+\delta}$.*

Proof. Take the ratio of the wage and the marginal revenue product of a worker by using equations (12) and (11). It is easy to see that ratio is exactly $\frac{\beta}{\beta+\delta}$. □

This result implies that firms are willing to pay a higher share of every additional revenue generated to workers when the worker skill distribution is more skewed relative to the firm productivity distribution.

4.5 Equilibrium

Let's solve the total income in the economy. In equilibrium, the total income is a sum of the total wage income and total profits.[5] First, simplify the price index[6] as

$$P = \int_{i=i^*}^{0} p(i)^{1-\sigma} di = \int_i \left[\left(\frac{\sigma}{\sigma-1} \frac{1}{(B\tilde{D})i^{-(\beta+\delta)}} \right) \right]^{1-\sigma} di$$

$$= \left(\frac{\sigma-1}{\sigma}\right)^{\sigma-1} \int_i (B\tilde{D})^{\sigma-1} i^{-(\sigma-1)(\beta+\delta)} di'.$$

The total income is then defined as

$$W = \int_{i=i^*}^{0} (w_i + \pi_i)\, di + \int_{i=i^{max}}^{i^*} w\, di$$

$$= \int_{i=i^*}^{0} \left(\frac{W}{P} \left(\frac{\sigma-1}{\sigma}\right)^{\sigma-1} \left(B\tilde{D}\right)^{\sigma-1} i^{-(\sigma-1)(\beta+\delta)} - f_e \right) di + \int_{i=i^{max}}^{i^*} w\, di.$$

Equilibrium. Given σ, B, β, D, δ, f_e, C, ρ, and w, a stationary equilibrium is characterized by the share i^* of positions who produce, the share of active firms I^*, the number of positions in each firm i, h_i^*, aggregate price index P^*, total income W^*, average sensitivity of positions, $\bar(c)^*$, total profits of each position i, $\pi^*(i)$, wage rates of workers, $w^*(i)$, total output of position, $q^*(i)$, and position pricing rule, $p^*(i)$ such that

- *position profit maximization*: Each active position i maximizes its profits by producing $q^*(i)$ at price $p^*(i)$, where

$$p^*(i) = \frac{\sigma}{\sigma-1} \frac{1}{(B\tilde{D}^*)i^{-(\beta+\delta)}};\ q^*(i) = \left(\frac{\sigma-1}{\sigma}\right)^{\sigma} \left[(B\tilde{D}^*)i^{-(\beta+\delta)}\right]^{\sigma} \frac{W^*}{P^*}.$$

- *Worker-position matching, and wages and profits:* A worker with rank i is matched with the equally ranked position i. The profits of positions and wages of workers satisfy

$$\pi^*(i) = \frac{W^*}{P^*} \left(\frac{\sigma-1}{\sigma}\right)^{\sigma-1} \left[(B\tilde{D}^*)i^{-(\beta+\delta)}\right]^{\sigma-1} - w_i^* - f_e$$

$$w_i^* = \frac{\beta}{\beta+\delta} (B\tilde{D}^*)^{\sigma-1} \frac{W^*}{P^*} \left(\frac{\sigma-1}{\sigma}\right)^{\sigma-1} \left[i^{-(\sigma-1)(\beta+\delta)} - i^{*-(\sigma-1)(\beta+\delta)}\right] + w.$$

- *Number of firms:*

$$I^* = C^{\frac{1}{\rho}} \left(\frac{\beta+\delta}{\delta} \frac{f_e}{\Gamma}\right)^{-\frac{1}{(\sigma-1)(\delta+\beta)}}.$$

- *Number of positions in each firm i:*

[5] The way profits are distributed across workers will not change the main findings as preferences are homothetic.

[6] I now use the effective ranking i_h of each position instead of a variety ω to define each firm. Those can be used interchangeable since each position produces its own variety (or the same variety as the mother firm but with a different quality).

$$h_i^* = C^{\frac{1}{\rho}} \left(\frac{\beta+\delta}{\delta} \frac{f_e}{\Gamma} \right)^{-\frac{1}{\rho(\sigma-1)(1+\frac{\beta}{\delta})}} i^{*-\frac{\delta}{\rho}}.$$

- *Entry of positions*: The free entry condition determines the share of positions that produce, i^*

$$\frac{\delta}{\beta+\delta}(B\tilde{D}^*)^{\sigma-1}\frac{W^*}{P^*}\left(\frac{\sigma-1}{\sigma}\right)^{\sigma-1} i^{*-(\sigma-1)(\beta+\delta)} = f_e + w.$$

- *Aggregate price index*: Aggregate price index satisfies

$$P^* = \int_{i=i^*}^{0} p(i)^{1-\sigma} di.$$

- *Weighted average sensitivity of positions:*

$$\bar{c}^* = \left[\int_{i=I^*}^{0} \bar{h}_i^{-1} \left(\int_{k=1}^{\bar{h}_i} c_{k_i}^{1/\delta} dk \right) di \right]^{\delta}.$$

- *Total income:*

$$W^* = \int_{i=i^*}^{0} \left(\frac{W^*}{P^*} \left(\frac{\sigma-1}{\sigma} \right)^{\sigma-1} \left(B\tilde{D}^* \right)^{\sigma-1} i^{*-(\sigma-1)(\beta+\delta)} - f_e \right) di + \int_{i=i^{max}}^{i^*} w \, di.$$

5 Quantitative analysis

5.1 Data and calibration of the model parameters

In this subsection, I describe how I set the model parameters for the quantitative exercises. I specifically focus on calibrating parameters that drive inequality and concentration according to the model. I first calibrate model parameters to the U.S. data such that the model matches a set of inequality and firm concentration measures in the 1980s, using data from 1980 to 1986 and then re-calibrate parameters such that the model can match the same moments in the 2010s, using data from 2007 to 2013.

The main question of this paper is to evaluate the drivers of the joint changes in firm concentration and earnings inequality. The first step to start the evaluation is to analyze how the model parameters have needed to change over time the model to generate the observed changes in the earnings variance components. As it is challenging if not impossible to directly measure changes in underlying productivity or skill distribution of firms and workers without using matched microdata on employers and employees, I use the structure of the model to help to identify changes in the six key parameters of the model, σ, β, δ, ρ, w, and f_e, reported in Table 1. Direction of changes in these parameters then helps to pin down the potential underlying reasons that are driving the joint concentration of earnings and firm size.

I calibrate β_{1980} such that the model-generated moment matches the total impact of worker fixed effect (WFE) on the earnings variance in the data (Song et al. (2019)). Matching the worker fixed effect share of the total wage variance helps to discipline the relative effect of β compared with relative effect of δ in driving inequality and concentration, as both affect these concentration measures. The WFE in Song et al. (2019) is interpreted as a combination of the skill level of a worker and the returns to the skill. In my model, this corresponds to $WFE_t = Bi^{-(\sigma_t - 1)\beta_t}$, where β_t controls the skill level of the worker whereas $(\sigma_t - 1)$ can be thought of as a market return for the skill. I calibrate β_{1980} such that share of the earnings variance arising from the worker fixed effects, $\frac{var(WFE)_{1980}}{var(w_i)_{1980}}$ in the model matches the one observed in the data between years 1980–86, consistent with the time intervals used in Song et al. (2019).

I calibrate δ_t and ρ_t such that the model matches two measures of firm size concentration in periods $t \in \{1980 - 86, 2007 - 13\}$: First, I calibrate δ_t such that the model matches the share of total sales by the top 10 percent of firms. Second, I calibrate ρ_t such that the model matches the share of total employment in the top 1 percent of the firms. I obtain data on both measures from Kwon et al. (2024).

As σ_t affects the overall wage dispersion in the model, and specifically the shape of the wage distribution at the top of the distribution, I set σ_t such that the model matches the top 10 percent earnings share in the data. I obtain the data from Kopczuk et al. (2007) and Gould & Kandra (2022).

I set w_t such that the model-generated mean-to-median wage ratio matches the ratio in the data. I calculate the ratio using U.S. Census Bureau data on U.S. personal mean and median incomes (U.S. Census Bureau (2024a), U.S. Census Bureau (2024b)). Lastly, $f_{e,t}$ is set such that I target the sales share of the top 50 percent of firms using data from Kwon et al. (2024).

The remaining parameters are set outside the model as the scale parameters of the productivity, skill, and sensitivity distributions, B, D, and C, respectively, mainly affect the levels of the endogenous variables rather than their dispersion, the main interest of this paper. B and D cannot be easily identified separately, as they enter each equation of the model as a product, BD. The best representation of these parameters is aggregate total factor productivity. For the calibration, I set the product BD to be equal to 10,000 since the choice of the Pareto scale parameter mainly affects the size of the economy. Finally, I normalize C to equal 1.

5.2 Calibration results

Calibrated parameter values. I present the parameters, their values, and their descriptions in Table 1. To match the calibration targets measuring earnings and firm concentration, I find that all six parameters have changed over time.

According to the calibration, β has increased by 14.7% between the 1980s and 2010s, indicating that worker skills have become more dispersed and more concentrated toward highest-skill workers. Given the structure of the model, the change in the skill distribution can be separated from the change in the returns to skills measured by σ. The change in β could indicate changes in how various types of skills are perceived and valued in the labor market, and how the valuation has changed over time.

Table 1: Model parameters and their descriptions.

Parameter, set outside the model	**Description**	**Parameter value** $t = 1980$	$t = 2007$
BD	scale parameters	10,000	10,000
C	scale parameter	1.0	1.0

Parameter, calibrated	**Description**	**Parameter value** $t = 1980$	$t = 2007$	$\%\Delta$
σ_t	price elasticity of demand	2.3705	2.5040	5.6%
β_t	shape parameter, skill distribution	0.2288	0.2625	14.7%
δ_t	shape parameter, productivity distribution	0.1982	0.1881	−5.1%
ρ_t	shape parameter, sensitivity distribution	0.1735	0.1440	−17.0%
$f_{e,t}$	fixed cost of a position	0.3944	0.2390	−39.4%
w_t	threshold wage of workers	2.4600	3.500	42.3%

This kind of a change in the perceived skill distribution could arise from at least two sources. First, increased inequality in both amount and quality of schooling could shift skill distribution: if high-quality schooling is becoming more concentrated toward a smaller number of workers and if skills learned through high-quality education are becoming more highly valued in the labor market, the perceived skill distribution of workers can become more concentrated. Second, increasingly skewed skill distribution could reflect largely studied skill-biased technological change (Acemoglu & Autor (2011)). Both potential explanations for the shifted skill distribution reflect the well-studied phenomenon that technological change has led to a labor market where high-skilled jobs require an increasingly complex skill set. These skills are harder to find among workers, either because they are harder to teach, harder to learn, or both. Thus, any underlying change in technology has made workers' skills more unequal.

According to the calibration results, the productivity dispersion of firms has decreased slightly between 1980s and 2010s. The calibrated value of δ has decreased by -5.1% between the 1980s and 2010s. This means that firms have become more similar in terms of their underlying productivity. While it is not clear why firms might have become more equal, the finding is consistent with Bloom et al. (2018) who study why the large firm wage premium has decreased since the 1980s in the United States and find that firm fixed effects capturing the wage differentials arising from the differences between firms has decreased.

The price elasticity of demand measured by σ has increased by 5.6%. This result implies that consumers have become more sensitive to prices and that higher-skilled workers and higher-productivity firms are the beneficiaries of this change. This result is in line with the literature documenting the increasing returns to skills (Acemoglu & Autor (2011)) and the role of technological change and globalization increasing the price elasticity of demand (Autor et al. (2020)).

I find that the parameter controlling for the dispersion in sensitivity, or sensitivity to skill, ρ has decreased by 17% between 1980s and 2010s. This result implies that it is easier for firms to grow bigger since marginal product of each additional position is decreasing

Table 2: Model fit

Parameter	Target description	$t = 1980$		$t = 2007$		$\%\Delta$	
		Target	**Moment**	**Target**	**Moment**	**Data**	**Model**
σ_t	Top 10% worker earnings share	0.32	0.31	0.40	0.39	24.1%	25.7%
β_t	WFE share of total variance	0.47	0.46	0.52	0.50	10.3%	8.4%
δ_t	Top 10% firm sales share	0.89	0.88	0.93	0.92	5.0%	4.7%
ρ_t	Top 1% firm employment share	0.55	0.56	0.59	0.59	6.1%	6.4%
$f_{e,t}$	Top 50% firm sales share	0.989	0.979	0.995	0.989	0.6%	1.0%
w_t	Mean-to-median earnings ratio	1.39	1.39	1.48	1.61	6.6%	15.7%

at a slower pace.

Model fit. Table 2 shows the model fit in terms of how well the model matches the calibration targets. Overall, the model closely matches the targets in both periods.

First, the model matches well the various measures of firm concentration: the model perfectly matches the employment share of the top 1 percent of the firms in 2010s, while overestimates the share slightly in 1980s. The model closely matches the sales shares of the top 10 percent and the top 50 percent of firms, slightly underestimating these concentration measures in both periods.

Second, the model generates earnings inequality moments similar to the ones in the data. While the model somewhat underestimates the earnings share of the top 10 percent of the workers in both periods and somewhat overestimates the mean-to-median earnings ratio in 2010s, the model-generated moments are close to the values in the data. Finally, the model generates a similar worker fixed effects (WFE) share of the total earnings variance as observed in the data.

In addition, I also compare the untargeted, model-generated growth rates in these concentration measures. As shown in the last two columns of Table 2, the model generates very similar growth rates in each measure as observed in the data.

5.3 Model validation

To be added

6 Counterfactual analysis: Drivers of earnings inequality and firm concentration

I next quantify the importance of changes in the skill distribution, productivity distribution, price elasticity of demand, and sensitivity distribution in explaining the joint trends in the firm and earnings concentration. I show in Table 3 how much of the changes in the top 10% earnings share, the top 10% sales share, and the top 1% employment share are attributed to changes in these factors.

I quantify the importance of each of these factors in explaining the changes in the concentration measures by fixing the model parameters to their estimated levels in 1980s and

Table 3: Decomposing the changes in concentration measures between 1980s and 2010s.

	Top 10% earn. share		Top 10% sales share		Top 1% empl. share	
Parameters	%	Norm., %	%	Norm., %	%	Norm., %
β	53.6	57.7	17.6	18.9	1.7	4.1
δ	-9.0	-9.7	-44.4	-47.5	-83.5	-208.0
ρ	-1.4	-1.5	93.7	100.4	161.6	402.9
σ	55.7	59.9	21.4	23.0	-30.3	-75.6
w	-2.6	-2.7	2.4	2.5	-5.2	-10.5
f_e	-3.4	-3.7	2.6	2.7	-5.2	-12.9
Total explained	92.9	100.0	93.3	100.0	40.1	100.0

changing the value of each corresponding parameter (β, δ, ρ, and σ) from its value in the 1980s to its value in the 2010s one at the time. That way, I assess how much each parameter is driving the changes in each concentration measure.

Table 3 concludes the results from the counterfactual exercises. Each row shows how large is the contribution of each parameter in explaining the changes in concentration measures shown in columns. For example, the increase β can explain 53.6 percent of the total change in the top 10% earnings share. To put it differently, the increase in β alone would have increased the top 10% earnings share by 53.6 percent if all other parameters in the model had stayed constant. As the total impacts do not sum up to exactly 100%, I normalize the impact of each parameter in explaining the change in the concentration measure by dividing its impact by the total impact. For example, I calculate the normalized impact of β, 57.7%, on the change in the top 10% earnings share as $100 * (53.6\%/92.9\%) = 57.7\%$.

Skill distribution. An increased concentration in workers' skills, measured by β, has been an important driver of both earnings inequality and sales concentration. I find that the rise in β explains around 58% and 19% of the total rise in earnings inequality and sales concentration between 1980s and 2010s, respectively.

This result implies that the same suspect—more skewed skill distribution—that has increased earnings inequality has also led to a concentration of production toward the highest-productivity firms. In the model, the positive assortative matching between the most productive positions and workers leads to an increased concentration of sales toward the top positions (and top firms) as productivity of a position is increasing in its worker's skill. Competition over the skilled workers lead to a rent-sharing between workers and positions, which then guarantees that the wage of a worker is increasing proportionally with the increase in sales of a position, increasing the concentration of earnings.

Price elasticity of demand. I find that an increase in σ—the price elasticity of demand—accounts a large fraction of the rise in both earnings inequality and sales concentration (60% and 23%, respectively). The role of σ is of similar magnitude as β. As mentioned earlier, σ can also be interpreted as returns to skill and productivity. Using this interpretation, the results show that increasing returns to skill and productivity can generate increases in inequality and concentration. This finding is consistent with Autor et al. (2020) who argue that the increased concentration of production could be driven by changes in price elasticity of substitution, potentially because of globalization and technological change. This finding is also in line with Cortes & Tschopp (2020) who find a positive relationship between price elasticity of demand and firm and earnings concentration in Europe.

Sensitivity distribution. I find that the decrease in ρ has been fully driving the rise in employment concentration. The decrease in ρ generates an increase on 403% in the top 1% employment share. As shown in section 4.4, concentration in employment should be mainly driven by the shape parameter of the firm size distribution, δ/ρ. A decline in ρ drives up the value of the shape parameter and, thus, concentration in employment. As the marginal product of each additional position is decreasing at a slower rate when ρ decreases, more productive firms hire more workers relative to less productive firms and increase their share of the total employment.

As employment concentration also contributes to firm sales concentration, the decrease in ρ has also been the largest driver of firm concentration measured in sales, explaining alone 100% of the increase.[7]

Productivity distribution. I find that the decline in the firm productivity dispersion, measured by δ, has decreased the concentration in employment, sales, and to a lesser extent, in earnings inequality.

To conclude, the counterfactual exercises show that joint forces have been driving concentration in both earnings and firm sales. An increase in β has been a large contributor to the increase in both. While it is well known that changes in skill distribution and returns to skills have been driving the increased earnings inequality in the past couple of decades, the recent literature studying the increased firm concentration has been silent about the potential role of β. Moreover, in line with previous research, increased price elasticity of demand, σ, has been an important driver of both earnings inequality and firm concentration.

7 Conclusions

In this paper, I investigate the connection between the secular increase in earnings inequality and firm concentration. First, I build an assignment model of earnings inequality and firm concentration that can generate realistic distributions of earnings and firm concentration, both in sales and employment. Second, I show using the model that concentration trends in earnings and firm size are intertwined. According to the model, both earnings inequality and firm concentration measured by sales are increasing in three common factors: skill and productivity dispersion of workers and firms, respectively, and price elasticity of demand. In addition, earnings inequality and firm concentration, measured in employment, are both increasing in firm productivity dispersion.

I then calibrate the model to match earnings inequality and firm concentration measures in the United States in the 1980s and 2010s and quantify the importance of the pa-

[7]While β and σ do not enter in the shape parameter of the firm employment distribution, they end up contributing somewhat to the employment concentration as they change the firm entry decisions at the left-tail of the size distribution. The same reasoning applies to w and f_e.

rameter changes as drivers of increased concentration. The quantitative exercise results in four main findings: first, a higher dispersion in worker skill has played an important role in explaining concentration: workers' skills have become more skewed, shifting both production and earnings towards the best workers and firms. Second, the increases in both earnings and sales concentration are strongly driven by the increase in the price elasticity of demand. Third, decreased dispersion in the sensitivity distribution has been the main driver of both the employment and sales concentration. Last, the firm productivity dispersion has somewhat decreased, so I conclude that the changes in firm and earnings concentration have not been driven by a growth in the underlying differences between firms.

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Appendix A: Demand functions

The CES form of Pollak's preferences is obtained by setting $\gamma_g = 0$,

$$U_g = \int_{\omega \in \Omega_g} \alpha_\omega q_g(\omega)^{1-\frac{1}{\sigma}} d\omega,$$

and the maximization problem of a worker g buying varieties $\omega \in \Omega$ becomes

$$\underset{q_g(\omega)\geq 0}{max}\, U_g = \int_{\omega \in \Omega} \alpha_\omega q_g(\omega)^{1-\frac{1}{\sigma}} \;\; s.t. \;\; W_g \geq \int_{\omega \in \Omega} p(\omega) q_g(\omega) d\omega.$$

First-order conditions can be written as

$$q_g(\omega): \;\; (1-\frac{1}{\sigma})\alpha_\omega q_g(\omega)^{-\frac{1}{\sigma}} = \lambda_g p(\omega) \;\; \forall \;\; q_g(\omega) > 0$$

$$\lambda_g: \;\; W_g = \int_{\omega \in \Omega} p(\omega) q_g(\omega) d\omega,$$

where λ_g is a Lagrange multiplier. Any pair of varieties, ω, $\omega^{'} \in \Omega$ gives

$$\frac{\alpha_\omega q_g(\omega)^{-\frac{1}{\sigma}}}{\alpha_{\omega^{'}} q_g(\omega^{'})^{-\frac{1}{\sigma}}} = \frac{p(\omega)}{p(\omega^{'})} \Leftrightarrow p(\omega^{'})\alpha_\omega q_g(\omega)^{-\frac{1}{\sigma}} = p(\omega)\alpha_{\omega^{'}} q_g(\omega^{'})^{-\frac{1}{\sigma}}$$

$$\Leftrightarrow \frac{p(\omega^{'})}{\alpha_{\omega^{'}}} q_g(\omega^{'})^{\frac{1}{\sigma}} = \frac{p(\omega)}{\alpha_\omega} q_g(\omega)^{\frac{1}{\sigma}}$$

$$\Leftrightarrow q_g(\omega^{'}) = \frac{\left(\frac{p(\omega^{'})}{\alpha_{\omega^{'}}}\right)^{-\sigma}}{\left(\frac{p(\omega)}{\alpha_\omega}\right)^{-\sigma}} q_g(\omega).$$

Multiply both sides by $p(\omega^{'})$ and integrate over all $\omega^{'} \in \Omega$ to get

$$\int_\Omega p(\omega^{'}) q_g(\omega^{'}) d\omega^{'} = \int_\Omega \frac{p(\omega^{'})\left(\frac{p(\omega^{'})}{\alpha_{\omega^{'}}}\right)^{-\sigma}}{\left(\frac{p(\omega)}{\alpha_\omega}\right)^{-\sigma}} q_g(\omega) d\omega^{'}$$

$$\Leftrightarrow W_g = \frac{q_g(\omega)}{\left(\frac{p(\omega)}{\alpha_\omega}\right)^{-\sigma}} \int_\Omega p(\omega^{'}) \left(\frac{p(\omega^{'})}{\alpha_{\omega^{'}}}\right)^{-\sigma} d\omega^{'}$$

$$\Leftrightarrow q_g(\omega) = \frac{\left(\frac{p(\omega)}{\alpha_\omega}\right)^{-\sigma}}{\int_\Omega p(\omega^{'}) \left(\frac{p(\omega^{'})}{\alpha_{\omega^{'}}}\right)^{-\sigma} d\omega^{'}} W_g,$$

where the budget constraint of a worker g implies that $W_g = \int_\Omega p(\omega^{'}) q_g(\omega^{'}) d\omega^{'}$.

Appendix B: Firm problem with complementarities between workers

appendix B)

Assume firms have the same hierarchical organizational structure as in the baseline profit-maximization problem. A firm's objective is to choose the total number of positions, h^*, a worker with a given skill $T(g)$ for each of its position h, and its price, (ω), to maximize profits. A firm's problem can be solved backwards.

Given the total number of positions, h^* and the worker talent in those positions, $T(g_h)$, a firm chooses its price to maximize profits. Define $\Psi(i) \equiv q^h(i) \equiv \sum_{h=1}^{h^*} c(h)T(g_h)A(i)$, which is the total productivity of a firm; $Q(\omega) \equiv p(\omega)^{-\sigma}\frac{W}{P}$, which is the demand function for a variety ω with a price $p(\omega)$. Define firm technology using its cost function. Labor used for production can be written as $h^* = \frac{Q}{\frac{1}{h^*}\Psi}$, where $\Psi = \sum_{h=1}^{h^*} c(h)T(g_h)A(i)$ and $\frac{1}{h^*}\Psi$ is the average productivity of a worker in a firm. Thus, firm cost function can be written as $1 = \frac{Q}{\Psi}$.

A firm then chooses its price to maximize

$$\underset{p}{max}\, \pi_{i,g,h} = p(\omega)Q(\omega) - \frac{Q(\omega)}{\Psi(\omega)} - \sum_{h=1}^{h^*} w_{i,h,g},$$

where $w_{i,h,g}$ are the wages that a firm i needs to pay for a worker g in a position h.

First order condition solves

$$p(\omega) : \frac{\partial \pi}{\partial p} = Q + (p - \frac{1}{\Psi})\frac{\partial Q}{\partial p} = 0$$

$$\Leftrightarrow p^{-\sigma}\frac{W}{P} - \sigma(p - \frac{1}{\Psi})p^{-\sigma-1}\frac{W}{P} = 0$$

$$\Leftrightarrow 1 - \sigma + \frac{\sigma}{\Psi}p^{-1} = 0 \Leftrightarrow p^* = \left(\frac{\sigma}{\sigma-1}\right)\frac{1}{\Psi},$$

showing that the optimal price follows the typical monopolistic competition pricing rule.

Given p^*, I can write firm revenue as

$$\widehat{R_i} = p^*(\omega)Q(\omega) = p^*(\omega)^{1-\sigma}\frac{W}{P} = \left(\frac{\sigma-1}{\sigma}\right)^{\sigma-1}\Psi^{\sigma-1}\frac{W}{P}$$

$$= \left(\frac{\sigma-1}{\sigma}\right)^{\sigma-1}\left[\sum_{h=1}^{h^*} c(h)T(g_h)A(i)\right]^{\sigma-1}\frac{W}{P}.$$

In contrast, the baseline model implies firm revenue of a form

$$R_i = \left(\frac{\sigma-1}{\sigma}\right)^{\sigma-1}\sum_{h=1}^{h^*}\left[c(h)T(g_h)A(i)\right]^{\sigma-1}\frac{W}{P}.$$

When comparing these expression, a couple of observations arise. First, when $\sigma = 2$, the revenues are equal, $R_i = \widehat{R_i}$. It follows that the solutions for the assignment problems will also be exactly the same. In contrast, $\widehat{R_i}$ is higher than R_i when $\sigma > 2$ because of the positive cross-terms between position productivities. Thus, the baseline problem which excludes these positive complementarities between workers.

I illustrate these differences using simple example. Assume that $\sigma = 3$ and a firm has two workers, $h^* = 2$. I can then write revenues as

$$\widehat{R_i} = \left(\frac{2}{3}\right)^2 \left[\sum_{h=1}^{2} c(h)T(g_h)A(i)\right]^2 \frac{W}{P}$$

$$= \left(\frac{2}{3}\right)^2 \frac{W}{P} \left[c(1)T(g_1)A(i) + c(2)T(g_2)A(i)\right]^2$$

$$= \left(\frac{2}{3}\right)^2 \frac{W}{P} A(i)^2 \left[(c(1)T(g_1))^2 + c(1)c(2)T(g_1)T(g_2) + (c(2)T(g_2))^2\right].$$

The positive cross-term $c(1)c(2)T(g_1)T(g_2) > 0$ differentiates $\widehat{R_i}$ from R_i. This additional term also implies that that marginal benefit of hiring a worker of skill g is higher than in the baseline problem:

$$\frac{\partial \pi}{\partial g_1} : \left(\frac{2}{3}\right)^2 c(1)A(i)^2 \frac{W}{P} T\prime(g_1) \left[c(1)T(g_1) + c(2)T(g_2)\right] = w\prime(g_1).$$

The marginal benefit of hiring a worker with a slightly higher skill (lower g), shown on the left-hand side, is now higher compared to the baseline problem. In particular, the expression for the marginal benefit includes an additional positive term $c(2)T(g_2)$, whose impact is further scaled up by $\left(\frac{2}{3}\right)^2 c(1)A(i)^2 \frac{W}{P} T\prime(g_1)$.