EX ANTE INDIVIDUALLY RATIONAL TRADE

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Abstract. We study truthful implementation under an *ex ante* individual rationality constraint. In bilateral trade, we show that efficiency is not always achievable without running a deficit, but is achievable if the distribution of the buyer's values stochastically dominates that of the seller. Using similar ideas, we show that in partnership dissolution, efficiency is achievable without running a deficit regardless of the initial ownership shares.

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1. Introduction

Budget-constrained designers often face a tension between achieving efficient outcomes and ensuring that it is individually rational for agents to participate in a mechanism after learning their private information—i.e., at the *interim* stage.[1] Against this backdrop, a seminal contribution of d'Aspremont and Gérard-Varet (1979) shows that efficiency is always achievable (in the standard quasilinear setting) without running a deficit if agents must decide whether to participate before learning their private information—i.e., at the *ex ante* stage.

The resulting *ex ante* individual rationality constraint is an economically natural condition when agents commit to participating in a mechanism in the future. For example, in bilateral trade, it may be a reasonable property if a buyer and a seller commit to transacting using a particular bargaining protocol at a point in the future. Similarly, in partnership dissolution, partners may commit to a procedure for dissolving the partnership in the future before uncertainty in the investments of the partnership is resolved. In these cases, it may be costly for agents to renege on their participation commitment after they learn their values.

However, ensuring that the *ex ante* individual rationality constraint holds relies on knowledge of agents' higher-order beliefs. In particular, at the *ex ante* stage, an agent's participation decision depend on their prediction of how counterparties will behave during the mechanism, as counterparties' actions affect the payoffs that each agent can expect. These actions depend in turn on counterparties' hierarchy of beliefs after receiving their private information, so an agent's beliefs about counterparties' future hierarchies of beliefs affect their participation decision. If the designer is misspecified about these beliefs, then agents may not choose to participate in the mechanism—a version of the Wilson (1987) critique.[2]

Under private values, truthful (i.e., dominant strategy) mechanisms simplify the participation decision by making counterparties' interim beliefs irrelevant to their actions. Hence, each agent can decide whether to participate at the *ex ante* stage using only their first-order beliefs about their counterparties' payoff-relevant private information. From the perspective of the designer, the truthfulness of a mechanism ensures that its *ex ante* individual rationality (in addition to its equilibrium outcomes) is robust to misspecification of higher-order beliefs. Unfortunately, the efficient mechanisms proposed by d'Aspremont and Gérard-Varet (1979) are not truthful, and therefore their *ex ante* individual rationality is not robust.

In this paper, we study truthful implementation under an *ex ante* individual rationality constraint, focusing on the canonical mechanism design settings of bilateral trade and partnership dissolution. We provide a sufficient condition under which efficiency can be achieved in bilateral trade, and prove that it can always be achieved in partnership dissolution.

[1]See, e.g., Myerson and Satterthwaite (1983), Güth and Hellwig (1986), and Mailath and Postlewaite (1990).
[2]A similar concern applies to the interim individual rationality constraint. By contrast, *ex post* individually rational mechanisms ensure participation security regardless of agents' beliefs (Dasgupta et al., 1979).

	Interim IR	*Ex ante* IR
Bayesian IC	✗ Myerson and Satterthwaite (1983)	✓ d'Aspremont and Gérard-Varet (1979)
Truthful	✗ Myerson and Satterthwaite (1983)	✓ if $\text{Median}(F_B) \geq \text{Median}(F_S)$ This paper

Table 1. (Non)existence of efficient mechanisms for bilateral trade (that do not run deficits) under various combinations of incentive compatibility and individual rationality constraints.

In Section 2, we begin by investigating the canonical bilateral trade setting of Myerson and Satterthwaite (1983), in which a seller can sell a good to a buyer, and the agents attribute independent private values to the good. We restrict attention to mechanisms that do not run a deficit, in the sense that the payment of the buyer to the mechanism must always be weakly larger than the payment to the seller. Thus, we allow for the possibility that a broker (e.g., the government) may extract some money from the participants in the mechanism.

Our first main result shows that under the assumption that the median value of the buyer is higher than that of the seller, full efficiency can be achieved by a truthful, *ex ante* individually rational mechanism. By the Green–Laffont–Holmström Theorem, the mechanism must be a Groves mechanism for an appropriate participation charge. The participation charge we construct turns out to have a simple form: the participation charge of each agent is a piecewise linear function of the value of the other agent. These participation charges are large enough to cover the deficit of the Vickrey–Clarke–Groves (VCG) mechanism if trade does occur, but small enough to preserve *ex ante* individual rationality.

However, we also show that for general distributions of values, truthful, *ex ante* individually rational mechanisms cannot achieve full efficiency in general. Intuitively, as participation charges must be paid regardless of whether trade occurs, and must be large enough to cover the VCG deficit if trade does occur, the *ex ante* probability of trade must be sufficiently large to be able to achieve *ex ante* individual rationality. The hypothesis that the distribution of the buyer's median value is higher than than the buyer's is one way of ensuring this property, as it implies that trade is efficient with probability at least $\frac{1}{4}$. Table 1 connects our result to previous results on the (non)existence of efficient mechanisms under various combinations of incentive compatibility and individual rationality constraints.

In Section 3, we extend our analysis beyond the case of bilateral trade to the canonical partnership dissolution model of Cramton et al. (1987). Consider N agents who initially share ownership of a partnership, and have independent and identically distributed values for the partnership. Cramton et al. (1987) provide a necessary and sufficient condition for full efficiency to be achievable by a Bayesian incentive compatible, interim individually rational mechanism—intuitively, sufficiently symmetric initial ownership is necessary and sufficient for efficient disolvability. By contrast, we show that regardless of the initial ownership shares, full efficiency can be achieved by a truthful, *ex ante* individually rational mechanism.

To construct an efficient, *ex ante* individually rational, truthful mechanism, we first suppose that the partnership is initially owned by one agent. Thus, the design problem boils down to one with a single seller and several buyers with independent and identically distributed values. As there are several buyers among whom the partnership (if traded) will be allocated to the one with the highest value, it is as if the buyers as a whole have stochastically higher values than the seller. We can then construct a Groves mechanism by extending our construction from the bilateral trade case. We can then conclude by first randomly allocating ownership of the entire partnership in proportion to the initial ownership shares.

The closest papers to ours are Hagerty and Rogerson (1987) and Athey and Miller (2007), who studied truthful mechanisms for bilateral trade. Hagerty and Rogerson (1987) showed that if the mechanism can run neither a deficit nor a surplus and is *ex post* individually rational, then it must be a (randomized) posted price. As we allow for surpluses and impose only *ex ante* individual rationality, we can move beyond (random) posted prices and, under certain conditions, achieve full efficiency. Athey and Miller (2007) allowed for surpluses, but excluded any profits for the designer from welfare. They showed that full efficiency cannot then be achieved, and characterized the second best. We allow for surpluses and consider efficient allocation of the good, thereby implicitly assuming that revenue for the designer is weighted equally to transfers in the hands of the traders.

2. Ex-ante individually rational bilateral trade

We consider a bilateral trade setting, where buyer B has a private value θ_B for buying the good, and seller S has a private cost θ_S for selling the good, distributed independently according to F_B and F_S, respectively. For simplicity, assume both distributions have support in $[0, 1]$ and admit strictly positive densities f_B and f_S.

By the revelation principle, we restrict attention to direct revelation mechanisms. A mechanism (q, t) specifies the probability of trade $q(\theta) = q(\theta_B, \theta_S) \in [0, 1]$, and payments $t(\theta) = t(\theta_B, \theta_S) = (t_B(\theta_B, \theta_S), t_S(\theta_B, \theta_S))$.[3] The utility derived by the buyer when type vector θ is reported into the mechanism is $u_B(\theta_B, \theta_S) = \theta_B q(\theta) - t_B(\theta)$ and that of the seller is $u_S(\theta_B, \theta_S) = -\theta_S q(\theta) - t_S(\theta)$, where we normalize the utility of the outside option to 0.

We next recall the four standard properties of mechanisms that feature in our analysis.

Definition 1. A mechanism (q, t) is *truthful* if for all types θ_B, θ_S and reports $\hat{\theta}_B, \hat{\theta}_S$, we have

$$q(\theta_B, \theta_S) - t_B(\theta_B, \theta_S) \geq \theta_B q(\hat{\theta}_B, \theta_S) - t_B(\hat{\theta}_B, \theta_S)$$
$$-\theta_S q(\theta_B, \theta_S) - t_S(\theta_B, \theta_S) \geq -\theta_S q(\theta_B, \hat{\theta}_S) - t_S(\theta_B, \hat{\theta}_S).$$

[3]We assume that $q(\theta)$ and $t(\theta)$ are measurable, and that $t(\theta)$ is uniformly bounded, so expected utilities exist.

Definition 2. A mechanism (q, t) is *efficient* if for all types θ_B, θ_S with $\theta_B > \theta_S$ (resp. $\theta_B < \theta_S$), we have that $q(\theta_B, \theta_S) = 1$ (resp. $q(\theta_B, \theta_S) = 0$).

Definition 3. A mechanism (q, t) has *no deficit* if for all types θ_B, θ_S, we have that

$$t_B(\theta_B, \theta_S) + t_S(\theta_B, \theta_S) \geq 0.$$

Definition 4. A mechanism (q, t) is *ex ante individually rational* if $\mathbb{E}_{\theta_B,\theta_S}[u_B(\theta_B, \theta_S)] \geq 0$ and $\mathbb{E}_{\theta_B,\theta_S}[u_S(\theta_B, \theta_S)] \geq 0$.

Our first main result shows that these four properties are simultaneously achievable if the median value of the buyer is higher than that of the seller.

Theorem 1. *If* $\text{Median}(F_B) \geq \text{Median}(F_S)$*, there exists an efficient, truthful,* ex ante *individually rational mechanism that has no deficit.*

As discussed in the introduction, the results of Myerson and Satterthwaite (1983) imply that *ex ante* individual rationality cannot be strengthened to interim individual rationality in Theorem 1. The results of d'Aspremont and Gérard-Varet (1979) imply that Theorem 1 holds if truthfulness is relaxed to Bayesian incentive compatibility. Our contribution here is to provide a positive result for truthful mechanisms under a mild restriction on distributions.

To prove Theorem 1, note that by the Green–Laffont–Holmström Theorem, we must use a Groves mechanism with participation charges for the buyer (resp. seller) that depend only on the seller's (resp. buyer's) type. We construct the participation charges to be piecewise linear. Specifically, let p^* be the market-clearing price in a replica economy with infinitely many buyers and sellers—i.e., p^* satisfies $F_B(p^*) + F_S(p^*) = 1$. Our participation charges have a kink point at type p^* and require the buyer (resp. seller) to make payments to the mechanism if the seller has a low value (resp. buyer has a high value). These participation charges eliminate the *ex post* deficit of the VCG mechanism, which would be large whenever the *ex post* gains from trade are large.

Proof. Let p^* be the large market market-clearing price, that is p^* is the unique solution to $F_B(p^*) + F_S(p^*) = 1$. We consider a Groves (1973) mechanism with $q(\theta) = \mathbb{1}(\theta_B \geq \theta_S)$ and $t_B(\theta_B, \theta_S) = \theta_S q(\theta) + h_B(\theta_S)$ and $t_S(\theta_B, \theta_S) = -\theta_B q(\theta) + h_S(\theta_B)$, where the participation charges are of the form

$$h_B(\theta_S) = \max(p^* - \theta_S, 0) + \kappa$$
$$h_S(\theta_B) = \max(\theta_B - p^*, 0) - \kappa$$

with a constant lump-sum transfer κ to be chosen. We will show that this mechanism is efficient, truthful, *ex ante* individually rational, and has no deficit. For all κ, efficiency holds by construction, and truthfulness holds because the mechanism is a Groves mechanism.

To show that the mechanism has no deficit regardless of the choice of κ, consider four cases in turn, noting that $t_B(\theta_B, \theta_S) + t_S(\theta_B, \theta_S) = h_B(\theta_S) + h_S(\theta_B) - \mathbb{1}(\theta_B \geq \theta_S)(\theta_B - \theta_S)$:

(1) If $\theta_S \geq p^*$ and $\theta_B \geq p^*$, then irrespective of the relative values of θ_B and θ_S, we have

$$t_B(\theta_B, \theta_S) + t_S(\theta_B, \theta_S) = \theta_B - p^* - \mathbb{1}(\theta_B \geq \theta_S)(\theta_B - \theta_S) \geq 0.$$

(2) If $\theta_S \geq p^*$ and $\theta_B < p^*$, then $t_B(\theta_B, \theta_S) + t_S(\theta_B, \theta_S) = 0$

(3) If $\theta_S < p^*$ and $\theta_B < p^*$, then irrespective of the relative positions of θ_B, θ_S, we have

$$t_B(\theta_B, \theta_S) + t_S(\theta_B, \theta_S) = p^* - \theta_S + \mathbb{1}(\theta_B \geq \theta_S)(\theta_S - \theta_B) \geq 0.$$

(4) If $\theta_S < p^*$ and $\theta_B \geq p^*$, then

$$t_B(\theta_B, \theta_S) + t_S(\theta_B, \theta_S) = p^* - \theta_S + \theta_B - p^* - \mathbb{1}(\theta_B \geq \theta_S)(\theta_B - \theta_S) = 0.$$

It remains to choose κ and show that the mechanism is *ex ante* individually rational. Note that $u_B(\theta_B, \theta_S) = (\theta_B - \theta_S)q(\theta) - h_B(\theta_S)$ and $u_S(\theta_B, \theta_S) = (\theta_B - \theta_S)q(\theta) - h_S(\theta_B)$. Hence, the *ex ante* individual rationality constraint requires that

$$\begin{aligned} \mathbb{E}_{\theta_B, \theta_S}\left[\mathbb{1}(\theta_B \geq \theta_S)(\theta_B - \theta_S)\right] &\geq \mathbb{E}_{\theta_S}\left[h_B(\theta_S)\right] \\ \mathbb{E}_{\theta_B, \theta_S}\left[\mathbb{1}(\theta_B \geq \theta_S)(\theta_B - \theta_S)\right] &\geq \mathbb{E}_{\theta_B}\left[h_S(\theta_B)\right]. \end{aligned} \tag{1}$$

To derive these conditions, we compare the expected participation charges to the *ex ante* gains from trade. The *ex ante* gains from trade satisfy

$$\begin{aligned} GT &= \int_0^1 \int_0^1 \mathbb{1}(\theta_B \geq \theta_S)(\theta_B - \theta_S)\, dF_S(\theta_S)\, dF_B(\theta_B) \\ &= \int_0^1 \int_0^{\theta_B} (\theta_B - \theta_S)\, dF_S(\theta_S)\, dF_B(\theta_B) \\ &= \int_0^1 \left[\theta_B F_S(\theta_B) - \int_0^{\theta_B} \theta_S\, dF_S(\theta_S)\right] dF_B(\theta_B) \\ &= \int_0^1 \left[\int_0^{\theta_B} F_S(\theta_S)\, d\theta_S\right] dF_B(\theta_B) = \int_0^1 F_S(y)(1 - F_B(y))\, dy, \end{aligned}$$

where the first three inequalities are standard rewriting, the fourth is obtained by integration by parts of the inner integral, and the fifth is obtained by integration by parts of the outer integral. Similar calculations show that the expected participation charges are given by

$$\begin{aligned} \mathbb{E}_{\theta_B}\left[h_S(\theta_B)\right] &= \int_{p^*}^1 (\theta_B - p^*)\, dF(\theta_B) - \kappa = (1 - p^*) - \int_{p^*}^1 F_B(\theta_B)\, d\theta_B - \kappa \\ &= \int_{p^*}^1 (1 - F_B(\theta_B))\, d\theta_B - \kappa \\ \mathbb{E}_{\theta_S}\left[h_B(\theta_S)\right] &= \int_0^{p^*} F_S(\theta_S)\, d\theta_S + \kappa. \end{aligned}$$

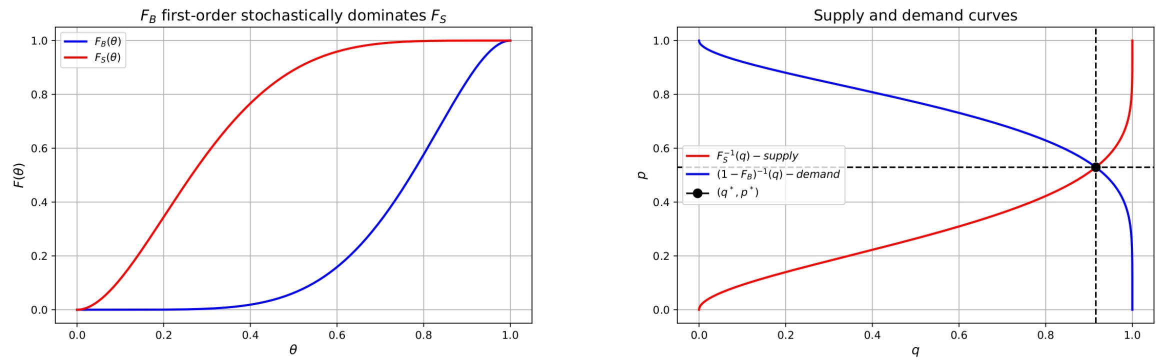

FIGURE 1. Example of the determination of the replica economy market-clearing price, with $F_B \sim \text{Beta}(6,2)$, $F_S \sim \text{Beta}(2,5)$.

Now, observe that

$$GT = \int_0^1 F_S(y)(1-F_B(y))\,dy = \int_0^{p^*} F_S(y)(1-F_B(y))\,dy + \int_{p^*}^1 F_S(y)(1-F_B(y))\,dy$$

$$\geq (1-F_B(p^*)) \int_0^{p^*} F_S(y)\,dy + F_S(p^*) \int_{p^*}^1 (1-F_B(y))\,dy.$$

By the definition of the market clearing price and because $\text{Median}(F_B) \geq \text{Median}(F_S)$, we have $F_S(p^*) = 1 - F_B(p^*) \geq \frac{1}{2}$. Hence, we obtain

$$2GT \geq \int_0^{p^*} F_S(y)\,dy + \int_{p^*}^1 (1-F_B(y))\,dy. \tag{2}$$

To ensure individual rationality, we define the lump-sum transfer κ by

$$\kappa = \begin{cases} 0 & \text{if } GT \geq \max\left(\mathbb{E}_{\theta_S}\left[\max(p^*-\theta_S,0)\right], \mathbb{E}_{\theta_B}\left[\max(\theta_B-p^*,0)\right]\right) \\ GT - \int_0^{p^*} F_S(y)\,dy & \text{if not and } \int_0^{p^*} F_S(y)\,dy \leq \int_{p^*}^1 (1-F_B(y))\,dy \\ \int_{p^*}^1 (1-F_B(y))\,dy - GT & \text{otherwise.} \end{cases}.$$

By (2) and the formulae for expected participation charges, we have that $\mathbb{E}_{\theta_S}[h_B(\theta_S)] \leq GT$ and $\mathbb{E}_{\theta_B}[h_S(\theta_B)] \leq GT$, which ensures that (1) holds and hence that the mechanism is *ex ante* individually rational. □

Figure 1 depicts how the replica economy market-clearing price p^* is determined. We plot two cumulative distribution functions F_B, F_S such that F_B first-order stochastically dominates F_S. We then draw the supply and demand curves in the large market—these are obtained by thinking of the type space as the price space, and the set to which types are mapped by distributions as the quantity space. We plot the supply curve $F_S^{-1}(.)$ and the demand curve $(1-F_B)^{-1}(.)$, with quantities on the x-axis and prices on the y-axis. The market clearing quantity-price pair is then the (unique) intersection of these two curves. In the figure, this reads $F_S^{-1}(q^*) = (1-F_B)^{-1}(q^*) = p^*$. Hence, p^* is the point in [0,1] satisfying $F_B(p^*) + F_S(p^*) = 1$. Note that, a key step in the above proof (in showing that (2) holds) is that the quantity traded at the market-clearing price, namely q^*, is greater than a half. The expected participation charges correspond to the replica economy per-agent consumer and producer surpluses, adjusted by κ; hence, (2) shows that the total surplus per agent in the replica economy is at most twice the gains from trade in the single-transaction economy.

We next provide an example in which the conclusion fails when $\text{Median}(F_B) < \text{Median}(F_S)$.

Example 1. We suppose that F_B places a mass of 0.8 on value 0, a mass of 0.1 on value 1, and is otherwise uniform on $[0,1]$. Similarly, F_S places a mass of 0.8 on value 1, a mass of 0.1 on value 0, and is otherwise uniform on $[0,1]$. (Here, the point masses are present only for expositional simplicity; the distributions can be made continuous by perturbation.)

Due to the full support of these distributions, the Green–Laffont–Holmström Theorem implies that any efficient, truthful mechanism would have to be a Groves mechanism, say with participation charges $h_B(\theta_S)$ and $h_S(\theta_B)$. No deficit then requires that $h_B(\theta_S)+h_S(\theta_B)-\mathbb{1}(\theta_B \geq \theta_S)(\theta_B - \theta_S) \geq 0$. Hence, we obtain the following inequalities

$$h_B(\theta) + h_S(\theta) \geq 0$$
$$h_B(0) + h_S(1) \geq 1$$

The *ex ante* gains from trade are $GT = \mathbb{E}_{\theta_B,\theta_S}[\mathbb{1}(\theta_B \geq \theta_S)(\theta_B - \theta_S)] = 2 \cdot 0.1^2 + \frac{0.1^2}{6} = \frac{13}{600}$. Ex ante individual rationality requires $\mathbb{E}_{\theta_B}[h_S(\theta_B)], \mathbb{E}_{\theta_S}[h_B(\theta_S)] \leq \frac{13}{600}$, which simplifies to

$$0.8h_S(0) + 0.1h_S(1) + 0.1\int_0^1 h_S(\theta)\,d\theta \leq \frac{13}{600}$$
$$0.1h_B(0) + 0.8h_B(1) + 0.1\int_0^1 h_B(\theta)\,d\theta \leq \frac{13}{600}.$$

Summing up these two inequalities, we have

$$0.1[h_B(0) + h_S(0)] + 0.8[h_B(1) + h_S(0)] + 0.1\int_0^1 [h_S(\theta) + h_B(\theta)]\,d\theta \leq \frac{13}{300}$$

From the previous inequalities from the no deficit condition, we also know that the left hand-side is at least $0.1 = \frac{30}{300} > \frac{13}{300}$, which is a contradiction. So every efficient, truthful, *ex ante* individually rational mechanism must run a deficit with these two distributions.

Hence, unlike d'Aspremont and Gérard-Varet (1979), our results rely on some hypothesis on the distributions of values. Intuitively, we need the probability of trade in the replica economy to be sufficiently large, which does not happen in general.

Note also that unlike d'Aspremont and Gérard-Varet (1979), we allow for the possibility that the mechanism can run a surplus. In our example, efficiency cannot be achieved despite this possibility. Our positive results rely on this possibility (Athey and Miller, 2007).

3. Ex-ante individually rational dissolution of partnerships

We now apply similar logic to the canonical partnership dissolution model of Cramton et al. (1987). There are N agents who are involved in a partnership. Each agent i initially owns share r_i of the partnership, with $r_i \geq 0$ and $\sum_{i=1}^{N} r_i = 1$; the initial stakes are common knowledge. Agents have independent and identically distributed values for (shares of) the partnership, which we assume are drawn from a distribution F supported on $[0, 1]$ with strictly positive density f. Let θ_i denote the type of agent i.

By the revelation principle, we can restrict attention to direct revelation mechanisms. A direct revelation mechanism $M = (q, t)$ specifies, for each agent i, a payment $t_i(\theta)$ and the probability $q_i(\theta) \in [0, 1]$ with which they receives the partnership, as a function of the reports of all the agents in the mechanism. The feasibility constraint is that $\sum_i q_i(\theta) = 1$ for all type profiles θ. The utility that the mechanism delivers to agent i is $\theta_i q_i(\theta) - t_i(\theta)$, while agent i's reservation utility is $\theta_i r_i$. We can therefore consider the net utility of agent i under the mechanism, which is defined by $u_i(\theta) = (q_i(\theta) - r_i)\theta_i - t_i(\theta)$. The definitions of truthfulness, efficiency, no deficit, and *ex ante* individual rationality then extend easily.

Definition 5. A mechanism (q, t) is *truthful* if for all agents i, type profiles θ, and reports $\hat{\theta}_i$:

$$q_i(\theta_i, \theta_{-i}) - t_i(\theta_i, \theta_{-i}) \geq \theta_i q_i(\hat{\theta}_i, \theta_{-i}) - t_i(\hat{\theta}_i, \theta_{-i}).$$

Definition 6. A mechanism (q, t) is *efficient* if for all agents i and type profiles θ with $\theta_i < \max_j \theta_j$, we have that $q_i(\theta) = 0$.

Definition 7. A mechanism (q, t) has *no deficit* for all type profiles θ, we have $\sum_i t_i(\theta) \geq 0$.

Definition 8. A mechanism (q, t) is *ex ante individually rational* if for all agents i, we have that $\mathbb{E}_\theta[u_i(\theta)] \geq 0$.

Our second main result shows that all partnerships can be efficiently dissolved.

Theorem 2. *For all initial ownership shares $(r_i)_{1 \leq i \leq N}$, there exists an efficient, truthful,* ex ante *individually rational mechanism that has no deficit.*

As discussed by Cramton et al. (1987), the results of Myerson and Satterthwaite (1983) imply that *ex ante* individual rationality cannot be strengthened to interim individual rationality in Theorem 2 without imposing a condition on initial ownership shares. The results of d'Aspremont and Gérard-Varet (1979) imply that Theorem 1 holds if truthfulness is relaxed to Bayesian incentive compatibility. Our contribution here is to provide a positive result for truthful mechanisms that holds for all initial ownership shares.

To prove Theorem 2, we prove a version of Theorem 1 that allows for multiple buyers, but imposes that all agents' values are drawn from the same distribution. This corresponds to the case of Theorem 2 for which there exists an agent j with $r_j = 1$.

Proposition 1. *If there exists an agent* 0 *with* $r_0 = 1$*, then there exists an efficient, truthful,* ex ante *individually rational mechanism that has no deficit.*

Using Proposition 1, the proof of Theorem 2 is straightforward.

Proof of Theorem 2 assuming Proposition 1. Let $(r_i)_{1 \leq i \leq N}$ be any initial ownership shares. Consider the following mechanism. A lottery is played to give the full property of the partnership to one of the agents, with probabilities given by the initial ownership shares $(r_i)_{1 \leq i \leq N}$. After the lottery is played, the partnership is fully owned by one of the agents. By Proposition 1, for this single-owner partnership, there exists an efficient, truthful, *ex ante* individually rational mechanism that has no deficit. The grand mechanism involving the lottery and these mechanisms is clearly efficient, truthful, and has no deficit.

From the point of view of agent i, the grand mechanism averages one *ex ante* individually rational mechanism in which they are the seller, and $N-1$ mechanisms in which they are a potential buyer. The first case provides the agent with an ex post utility of at least $(1-r_i)\mathbb{E}_{\theta_i}[\theta_i]$ and occurs with probability r_i, while the second case occurs with probability $1-r_i$ and provides the agent with a utility of at least $-r_i \mathbb{E}_{\theta_i}[\theta_i]$. Hence, the grand mechanism is *ex ante* individually rational for each agent i. □

It remains to prove Proposition 1. Intuitively, the result holds for similar reasons to Theorem 1 because the distribution of the highest value of the potential buyers first-order stochastically dominates the distribution of the value of the seller/owner. However, the detailed argument is much more involved due to the presence of multiple buyers.

We once again construct a Groves (1973) mechanism. The seller's participation charge is a piecewise linear function of the highest value of the buyers. The key complication is that the buyer's participation charges must depend on the values of other buyers in addition to the value of the seller. The participation charge of buyer i as a piecewise linear function of the larger of the value of the seller and the second-highest value of buyers other than i.

Proof of Proposition 1. Since the distribution F has strictly positive density, ties occur with probability 0 and we ignore them thereafter. We use the following notation. For $Z \subset \{0, \dots, N-1\}$, $\theta_{-Z} = (\theta_i)_{i \notin Z}$. For any vector of random variables θ, $\theta^{(1)}$ (resp $\theta^{(2)}$) denotes the first (resp. second) order statistic of this family. With a slight abuse of notation, for example, $\theta_{-0}^{(1)} = \max_{1 \leq j \leq N-1} \theta_j$.

We propose the following Groves' scheme (q, t), where $q_i(\theta) = \mathbb{1}(\theta_i \geq \max_{j \neq i} \theta_j)$ and

$$t_0(\theta) = -\theta_{-0}^{(1)}(1 - q_0(\theta)) + \max(\theta_{-0}^{(1)} - p^*, 0)$$

$$t_i(\theta) = \theta_{-i}^{(1)} q_i(\theta) + \frac{1}{N-1} \max(p^* - \max(\theta_{-\{0,i\}}^{(2)}, \theta_0), 0) \qquad \text{for } 1 \leq i \leq N-1.$$

The utilities derived by the agents when they report truthfully are $u_0(\theta) = \theta_0(q_0(\theta) - 1) - t_0(\theta)$ for the seller, and $u_i(\theta) = q_i(\theta)\theta_i - t_i(\theta)$ for a representative buyer. All along, we refer to the first part of the payments of agents as their VCG payment, while we refer to the second part in which p^* intervenes as the participation charges. We for now do not give the exact definition of p^* but return to this question later in the proof.

For the seller, the whole payment is very similar to the bilateral trade case. The VCG payment is null if they do not sell, while they receive the buyer's value $\theta_{-0}^{(1)}$ if they sell. In both cases, the participation charge is a function of the highest potential buyers' value $\theta_{-0}^{(1)}$. The buyers who do not end up buying the good have a VCG payment of zero, while if one buyer ends up buying the good, they has a VCG payment of the second highest value (which might be that of the seller or another buyer's value). Contrary to the bilateral trade case, the participation charge is not simply a function of the seller's value.

We examine the four desired properties of our mechanism in turn. First, the mechanism is a Groves' scheme, so efficiency and dominant-strategy IC are guaranteed.

Second, we verify that the mechanism has no deficit by investigating two cases:

(1) if the seller keeps the good ($q_0(\theta) = 1$), then

$$\sum_{j=0}^{N-1} t_j(\theta) = \max(\theta_{-0}^{(1)} - p^*, 0) + \max(p^* - \theta_0, 0) \geq 0.$$

(2) if the seller sells the good ($q_0(\theta) = 0$) and there exists an agent $i \in \{1, \ldots, N-1\}$ such that $q_i(\theta) = 1$, then we have that

$$\begin{aligned}\sum_{j=0}^{N-1} t_j(\theta) &= -\theta_{-0}^{(1)} + \max(\theta_{-0}^{(1)} - p^*, 0) + \theta_{-i}^{(1)} \\ &\quad + \frac{1}{N-1} \max(p^* - \max(\theta_{-\{0,i\}}^{(2)}, \theta_0), 0) + \frac{1}{N-1} \sum_{j \neq 0,i} \max(p^* - \max(\theta_{-\{0,j\}}^{(2)}, \theta_0), 0).\end{aligned}$$

It follows that

$$\sum_{j=0}^{N-1} t_j(\theta) \geq -\theta_i + \max(\theta_i - p^*, 0) + \theta_{-i}^{(1)} + \max(p^* - \theta_{-i}^{(1)}, 0) \geq 0,$$

where we use the facts that $\forall j \notin \{0, i\}$, $\theta_{-\{0,j\}}^{(2)} \leq \theta_{-i}^{(1)}$, that $\theta_{-0}^{(1)} = \theta_i \geq \theta_{-i}^{(1)}$, and that $\theta_{-\{0,i\}}^{(2)} \leq \theta_{-i}^{(1)}$

We now turn to proving that our mechanism is *ex ante* individually rational in aggregate—i.e., that $\mathbb{E}_\theta \left[u_0(\theta) + \sum_{i=1}^{N-1} u_i(\theta) \right] \geq 0$. As in the bilateral case, we can then use lump sum transfers between the agents to ensure *ex ante* individual rationality for each agent.

Step 1. Decomposing the expected sum of utilities. Note that

$$(q_0(\theta) - 1)\theta_0 + \theta_{-0}^{(1)}(1 - q_0(\theta)) + \sum_{i=1}^{N-1} q_i(\theta) \left(\theta_i - \theta_{-i}^{(1)} \right)$$

can be rewritten by observing the following facts:

(1) if the seller sells the good ($q_0(\theta) = 0$), which happens with probability $\frac{N-1}{N}$ independently of the values of the order statistics $\theta^{(k)}$, we have

$$(q_0(\theta) - 1)\theta_0 + \theta_{-0}^{(1)}(1 - q_0(\theta)) + \sum_{i=1}^{N-1} q_i(\theta) \left(\theta_i - \theta_{-i}^{(1)}\right) = -\theta_0 + \theta^{(1)} + \theta^{(1)} - \theta^{(2)}.$$

(2) if the seller keeps the good ($q_0(\theta) = 0$), which happens with probability $\frac{1}{N}$ independently of the order statistics $\theta^{(k)}$, we have

$$(q_0(\theta) - 1)\theta_0 + \theta_{-0}^{(1)}(1 - q_0(\theta)) + \sum_{i=1}^{N-1} q_i(\theta) \left(\theta_i - \theta_{-i}^{(1)}\right) = -\theta_0 + \theta^{(1)} = 0.$$

Hence, we can rewrite

$$\mathbb{E}_\theta \left[(q_0(\theta) - 1)\theta_0 + \theta_{-0}^{(1)}(1 - q_0(\theta)) + \sum_{i=1}^{N-1} q_i(\theta) \left(\theta_i - \theta_{-i}^{(1)}\right)\right] = \mathbb{E}_\theta \left[\theta^{(1)} - \theta_0\right] + \frac{N-1}{N}\mathbb{E}_\theta \left[\theta^{(1)} - \theta^{(2)}\right]$$

That is, we find the usual decomposition of the sum of utilities

$$\mathbb{E}_\theta \left[u_0(\theta) + \sum_{i=1}^{N-1} u_i(\theta)\right] = GT + VCG_{def} - TPC$$

where $GT = \mathbb{E}_\theta \left[\theta^{(1)} - \theta_0\right]$ are the finite-market expected gains from trade, $VCG_{def} = \frac{N-1}{N}\mathbb{E}_\theta \left[\theta^{(1)} - \theta^{(2)}\right]$ is the expected VCG deficit, and

$$TPC = E_\theta \left[\max(\theta_{-0}^{(1)} - p^*, 0) + \sum_{i=1}^{N-1} \frac{1}{N-1} \max(p^* - \max(\theta_{-\{0,i\}}^{(2)}, \theta_0), 0)\right]$$

is the expected total participation charges in the Groves' scheme.

Step 2. Using quantiles and Renyi's representation. We start by working in the quantile space. For each type θ_i, let $U_i = F(\theta_i)$, so that we have a family of $(U_0, \ldots, U_{N-1})$ of independently and identically distributed random variables following the uniform distribution on the unit interval. We remind the reader of the following two facts. First, fixing $Z \subset \{0, \ldots, N-1\}$, $\theta_{-Z}^{(1)}$ has the same distribution as $F^{-1}(U_{-Z}^{(1)})$. Second, from the Renyi representation of order statistics, we have that $U^{(k)} \sim Beta(n - k + 1, k)$. Hence, for all quantiles $u \in [0, 1]$, we have that

$$\begin{aligned} f_{U_{-i}^{(1)}}(u) &= (N-1)u^{N-2} \\ f_{U_{-i}^{(2)}}(u) &= (N-1)(N-2)u^{N-3}(1-u) \\ &= (N-1)(N-2)u^{N-3} - (N-1)(N-2)u^{N-2} \end{aligned}$$

Step 3. Calculating the expected gains from trade. We calculate the three terms GT, VCG_{def}, TPC in turn. We start with the expected gains from trade.

$$\begin{aligned} GT &= \mathbb{E}_\theta \left[\theta^{(1)} - \theta_0\right] \\ &= \int_0^1 (Nx^{N-1} - 1)F^{-1}(x)\,dx \\ &= \int_0^1 (x - x^N)\frac{dF^{-1}(x)}{dx}\,dx \end{aligned}$$

where the first inequality is by definition, the second is by using the quantiles U_0 and $U^{(1)}$ and the Renyi representation for $U^{(1)}$, and the third is obtained by integration by parts.

Step 4. Calculating the expected VCG deficit. Using similar techniques, we have

$$\begin{aligned} VCG_{def} &= \frac{N-1}{N}\int_0^1 (Nx^{N-1} - N(N-1)x^{N-2}(1-x))F^{-1}(x)\,dx \\ &= (N-1)\int_0^1 (Nx^{N-1} - (N-1)x^{N-2})F^{-1}(x)\,dx \\ &= (N-1)\int_0^1 (x^{N-1} - x^N)\frac{dF^{-1}(x)}{dx}\,dx \end{aligned}$$

Step 5. Calculating the total expected participation charges. We must first find the distribution of $\max(U^{(2)}_{-\{0,i\}}, U_0)$ for an arbitrary $i \in \{1, \dots, N-1\}$. We partition an event

$$\left\{\max(U^{(2)}_{-\{0,i\}}, U_0) \le x\right\} = \left\{U^{(1)}_{-i} \le x, U_0 = U^{(1)}_{-i}\right\} \cup \left\{U^{(2)}_{-i} \le x, U_0 \ne U^{(1)}_{-i}\right\}.$$

Indeed, if U_0 is the highest realization among all realizations except U_i, then we have that $\max(U^{(2)}_{-\{0,i\}}, U_0) = U^{(1)}_{-i}$. If not, we have essentially two cases, either $U_0 = U^{(2)}_{-i}$, in which case $\max(U^{(2)}_{-\{0,i\}}, U_0) = U_0 = U^{(2)}_{-i}$; or $U_0 < U^{(2)}_{-i}$, in which case $\max(U^{(2)}_{-\{0,i\}}, U_0) = U^{(2)}_{-i}$. That is, when $U_0 \ne U^{(1)}_{-i}$, we always have $\max(U^{(2)}_{-\{0,i\}}, U_0) = U^{(2)}_{-i}$.

Denoting by $Z_i = \max(U^{(2)}_{-\{0,i\}}, U_0)$, we have Z_i has the same distribution as a mixture of $\frac{1}{N-1}U^{(1)}_{-i}$ and $\frac{N-2}{N-1}U^{(2)}_{-i}$. By definition of mixtures, for all $z \in [0,1]$, we then have

$$\begin{aligned} f_{Z_i}(z) &= z^{N-2} + (N-2)^2 z^{N-3} - (N-2)^2 z^{N-2} \\ &= (N-2)^2 z^{N-3} - (N-1)(N-3)z^{N-2} \end{aligned}$$

Hence, the cumulative distribution function of Z_i is given by

$$F_{Z_i}(z) = (N-2)z^{N-2} - (N-3)z^{N-1}$$

We let p^* be such that $(N-2)F(p^*)^{N-2}-(N-3)F(p^*)^{N-1}+F(p^*)^{N-1} = (N-2)F(p^*)^{N-2} - (N-4)F(p^*)^{N-1} = 1$. The $N-4$ term comes from the fact that, similarly to the bilateral trade case, we determine the market-clearing price by $F_S(p^*)+F_B(p^*) = 1$. Here, the buyer's cumulative distribution function is that of the first-order statistic among the $N-1$ potential buyers, so it is $F(\theta)^{N-1}$.

We calculate the expected participation charges of the seller and of a representative buyer separately. For the seller, we have that

$$\begin{aligned}
\mathbb{E}_{\theta_0}[\max(\theta_{-0}^{(1)} - p^*, 0)] &= \mathbb{E}_{U_0}[\max(F^{-1}(U_{-0}^{(1)}) - p^*, 0)] \\
&= \int_{F(p^*)}^{1} (F^{-1}(x) - p^*)(N-1)x^{N-2}\, dx \\
&= (1-p^*) - \int_{F(p^*)}^{1} \frac{dF^{-1}(x)}{dx} x^{N-1}\, dx \\
&= \int_{F(p^*)}^{1} \frac{dF^{-1}(x)}{dx}(1 - x^{N-1})\, dx
\end{aligned}$$

where the fourth inequality is obtained by integration by parts, and, for a representative buyer $i \in \{1, \dots, N-1\}$,

$$\begin{aligned}
&\mathbb{E}_{\theta}\left[\frac{1}{N-1}\max(p^* - \max(\theta_{-\{0,i\}}^{(2)}, \theta_0), 0)\right] \\
&= \frac{1}{N-1}\int_0^{F(p^*)} (p^* - F^{-1}(z))[(N-2)^2 z^{N-3} - (N-1)(N-3)z^{N-2}]\, dz \\
&= \frac{1}{N-1}\int_0^{F(p^*)} (p^* - F^{-1}(z))[(N-2)^2 z^{N-3} - (N-1)(N-3)z^{N-2}]\, dz \\
&= \frac{1}{N-1}\int_0^{F(p^*)} [(N-2)z^{N-2} - (N-3)z^{N-1}]\frac{dF^{-1}(z)}{dz}\, dz
\end{aligned}$$

Hence, the total participation charge can be written as

$$\begin{aligned}
&\mathbb{E}_{\theta}\left[\max(\theta_{-0}^{(1)} - p^*, 0) + \sum_{i=1}^{N-1}\frac{1}{N-1}\max(p^* - \max(\theta_{-\{0,i\}}^{(2)}, \theta_0), 0)\right] \\
&= \int_0^{F(p^*)} [(N-2)y^{N-2} - (N-3)y^{N-1}]\frac{dF^{-1}(y)}{dy}\, dy + \int_{F(p^*)}^{1} \frac{dF^{-1}(y)}{dy}(1 - y^{N-1})\, dy
\end{aligned}$$

Step 6. Rewriting aggregate ex ante individual rationality. We rewrite the condition we want to investigate using the results from the previous steps. Indeed, the property that

$\mathbb{E}_\theta\left[u_0(\theta) + \sum_{i=1}^{N-1} u_i(\theta)\right] \geq 0$ is equivalent to

$$\begin{aligned}
&\int_0^1 (x - x^N)\frac{dF^{-1}(x)}{dx}\,dx + (N-1)\int_0^1 (x^{N-1} - x^N)\frac{dF^{-1}(x)}{dx}\,dx \\
&= \int_0^1 (x + (N-1)x^{N-1} - Nx^N)\frac{dF^{-1}(x)}{dx}\,dx \\
&\geq \int_0^{F(p^*)} [(N-2)x^{N-2} - (N-3)x^{N-1}]\frac{dF^{-1}(x)}{dx}\,dx + \int_{F(p^*)}^1 \frac{dF^{-1}(x)}{dx}(1 - x^{N-1})\,dx
\end{aligned}$$

We have defined $F(p^*)$ to be the solution to the equation $(N-2)x^{N-2} - (N-3)x^{N-1} = 1 - x^{N-1}$, that is when the two integrands from the above integrals meet. The first integral has an increasing integrand $(N-2)x^{N-2} - (N-3)x^{N-1}$ on $[0,1]$, while the second integrand $1 - x^{N-1}$ is decreasing. Hence, we can rewrite the above inequality as

$$\begin{aligned}
&\int_0^1 (x + (N-1)x^{N-1} - Nx^N)\frac{dF^{-1}(x)}{dx}\,dx \\
\geq &\int_0^1 \min\{(N-2)x^{N-2} - (N-3)x^{N-1}, 1 - x^{N-1}\}\frac{dF^{-1}(x)}{dx}\,dx
\end{aligned}$$

Step 7. Some final calculus and algebra to show that this inequality holds. A sufficient condition for the aggregate ex-ante IR condition to hold is that for all $x \in [0,1]$

$$x + (N-1)x^{N-1} - Nx^N \geq \min\{(N-2)x^{N-2} - (N-3)x^{N-1}, 1 - x^{N-1}\}$$

In particular, it suffices to show that the left-hand side is greater or equal than a convex combination of the two terms on the right-hand side. We take as weights $1 - x^{N-1}$ for the left-hand side, and x^{N-1} for the right-hand side, which gives us

$$\begin{aligned}
&(N-2)x^{N-2} - (N-3)x^{N-1} - (N-2)x^{2N-3} + (N-3)x^{2N-2} + x^{N-1} - x^{2N-2} \\
&\qquad = (N-2)x^{N-2} - (N-4)x^{N-1} - (N-2)x^{2N-3} + (N-4)x^{2N-2}.
\end{aligned}$$

Hence, it suffices to show that for all $x \in [0,1]$,

$$x + (N-1)x^{N-1} - Nx^N \geq (N-2)x^{N-2} - (N-4)x^{N-1} - (N-2)x^{2N-3} + (N-4)x^{2N-2}$$

This is equivalent to

$$x + (2N-5)x^{N-1} + (N-2)x^{2N-3} \geq (N-2)x^{N-2} + Nx^N(N-4)x^{2N-2}$$

By arithmetic-geometric mean comparisons, we have $\frac{1}{N-2}x + \frac{N-3}{N-2}x^{N-1} \geq x^{N-2}$, so that $x + (N-3)x^{N-1} \geq (N-2)x^{N-2}$. Hence, all is left is to show that

$$(N-2)x^{N-1} + (N-2)x^{2N-3} \geq Nx^N + (N-4)x^{2N-2}$$

Since this clearly holds for $x = 0$, we can divide through by x^{N-1}, and we need only show that $(N-2)+(N-2)x^{N-2} \geq Nx+(N-4)x^{N-1}$. We prove the latter using calculus. Define

$$g(x) = (N-2) + (N-2)x^{N-2} - Nx - (N-4)x^{N-1}$$

We have $g(1) = 0$ and $g'(x) = -N + (N-2)^2x^{N-3} - (N-1)(N-4)x^{N-2}$. If we can show that $g'(x) \leq 0$, then we are done. We do it using again an arithmetic-geometric mean comparison.

$$\frac{N}{(N-2)^2} + \frac{(N-1)(N-4)}{(N-2)^2}x^{N-2} \geq x^{\frac{(N-1)(N-4)}{N-2}}$$

and $(N-1)(N-4) \leq (N-2)(N-3)$, so that $x^{\frac{(N-1)(N-4)}{N-2}} \geq x^{N-3}$, since $x \in [0,1]$. We conclude that $(N-2)^2x^{N-3} \leq (N-1)(N-4)x^{N-2} + N$, so that the desired inequality $g'(x) \leq 0$ holds. □

4. Conclusion

This paper investigated truthful implementation under an *ex ante* individual rationality constraint. We obtained positive results on the achievability of full efficiency in a canonical bilateral trade model (under a hypothesis on distributions) and a canonical partnership dissolution model. This contrasts with the Myerson and Satterthwaite (1983) impossibility results under an interim individual rationality constraint, and sharpens the d'Aspremont and Gérard-Varet (1979) results which do not provide truthful mechanisms.

Our analysis highlights the importance of relaxing the interim individual rationality property to *ex ante* individual rationality in order to achieve efficiency. The latter condition is economically natural in settings in which agents can commit to participating in the mechanism in advance of learning what is needed to determine their values (e.g., before a partnership makes investments). In future work, we plan to investigate truthful implementation under *ex ante* individual rationality constraints in other mechanism design problems.

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