Limiting the Influence of Rich Voters on Elections: The Case of Democracy Voucher in Seattle

Reza Moradi

University of Toronto

Abstract. This paper investigates the impact of public funding on elections, focusing on voters' responses to the "Democracy Voucher" policy in Seattle. The study uses empirical data to explore how public funding influences private political contributions, particularly across different incomes and political groups. The analysis reveals a decrease in private donations following the implementation of public funding, with significant variances observed across various demographic segments. The paper employs a micro-founded model to further understand these trends, particularly among high-income contributors, and examines how public funding redistributes political influence. The findings highlight the complexity of electoral financing and its implications for political equity and democracy.

Keywords: Political contribution, Elections, Public funds, Event study, Contest model

1 Introduction

In democratic countries, the imbalance between the political contributions of people with different levels of financial means has become a concern. The most recent electoral cycle, from 2021 to 2022, witnessed that a mere 0.53% of the U.S. population contributed over \$200, accounting for 75% of the overall political contributions directed towards federal candidates, political action committees (PACs), outside groups, and political parties (OpenSecrets (2023)). survey conducted by the Pew Research Center shows that 76% of respondents believe that "money has a greater influence on politics today than before," and 64% believe that the "high cost of presidential campaigns discourages good candidates (Desilver and Van Kessel (2015))." The need for a substantial sum of money to fund campaigns, along with the limited number of donors who provide the majority of these funds, raises a persistent fear that public policy could be biased towards those with the financial resources to make contributions (Overton (2011)).

One proposed solution to recalibrate political contributions is using public funds to amplify the donations of lower-income individuals (Klumpp et al. (2015) Miller (2013)). Few states and cities, such as Arizona, Maine, and New York City, have explored restoring balance in political contributions by matching campaign donations with a requisite number of supporters. To examine the effectiveness of these policies, it is crucial to extract the voters' responses regarding private contributions to political campaigns.

This study investigates voters' responses to public funding policy in elections, considering their income and political heterogeneity. Using the data from the "Democracy Voucher" program implemented in Seattle, I design an empirical event study to identify the heterogeneity between contributors from high- and low-income groups in response to distributing public funds. Although public funding can change both the amount of donation and the number of contributors, this paper focuses more on the contributors' response conditional on having an out-of-pocket donation.

The paper shows that the private political donations of individual voters decrease on average after distributing public funding in elections. Moreover, the changes in private contributions significantly differ across income and political groups. While individuals with a higher ratio of political donations and in a higher income bracket lose their margin after the policy, they decrease their private donations significantly less than other groups.

In the realm of empirical analysis, there are different studies on the effect of public funding on private donations to non-profit organizations (Borgonovi (2006), Andreoni and Payne (2011), Heutel (2014)). These studies underscore how minimal government support encourages private donations, while increased public funding tends to have a crowd-out effect. On the topic of public funding in elections, there are mixed results on the impact on private contributions (Miller (2011), Malbin et al. (2012), Dowling et al. (2012)). However, most of these studies confirm the change in the demography of political contributors after implementing a public funding policy in the elections by increasing the number of small donors.

In a recent study, Griffith and Noonen (2022) investigated the causal effect of the "Democracy Voucher" program in Seattle on some aggregate variables, i.e., donation patterns, candidate entry, and incumbency advantage. In the mentioned study, the authors find the changes in private donations statistically insignificant. This paper studies the intensive margin of political contribution to separate the effect of changes in the extensive margin on the average treatment effect, which explains the result in Griffith and Noonen (2022). This separation also helps to understand the reason behind the mixed results in the literature since the combination of both margins drives the average treatment effect. Moreover, investigating the impact of public funding across different economic and political groups opens the path to studying the changes in the political spectrum of the candidates due to the new election policy.

In microeconomics theory, the early models of political contributions were developed in the 1970s (Ben-Zion and Eytan (1974), Bental and Ben-Zion (1975), Welch (1974), Welch (1976)). These models mostly focus on the behavior of politicians to absorb campaign funding. The foundation of earlier models is based on the demand and supply of campaign funding, considering voters as passive agents and stakeholder firms as the main contributors (Bental and Ben-Zion (1981)). However, a later model investigates the result of voting games by considering different motives for the voter's political contribution (Shieh and Pan (2010)). Recently, small campaign contributions have been modeled to be motivated by the desire to impact election results, with the interaction between small and large donors notably changing key aspects of campaign finance and electoral outcomes (Bouton et al. (2018)). The model in this paper uses the framework of contest models in microeconomics theory (Tullock (2001), Cornes and Hartley (2003), Cornes and Hartley (2005), Chowdhury et al. (2013)). The main assumptions of the model are heterogeneity between different income and political groups, the implementation of political interest as a non-excludable and non-pecuniary utility component, and the assumption of voters with decreasing risk aversion by income. These assumptions help the model investigate the underlying social imbalance caused by different changes in political contribution.

I summarize the program's background in section 2. In section 3, I describe the data from the Democracy Voucher program in Seattle. In Section 4, the identification method has been introduced to conduct the empirical study. In section 5, the study results are displayed. In section 6, I propose a micro-founded model to study the response of usual contributors in high-income groups to implementing public funding in an election.

2 Background

For this research, I utilize the Seattle public funding system called "Democracy Voucher." The program was part of a ballot initiative passed in 2015 and funded by an almost $30 million property tax increase over ten years. The program authorizes the Seattle Ethics and Election Committee (SEEC) to distribute "Democracy Vouchers" to eligible Seattle residents. The program is the first to use public funding to supply election campaign finances.

In each municipal and election year, the SEEC sends out four $25 vouchers (a total of $100) to each registered voter. The program is designed to fund campaigns in three elections: the mayoral election, the city council election, and the city attorney election. The candidates' campaigns can redeem vouchers. Unused vouchers expire if they are not redeemed during the election year. After assigning the vouchers, voters can send them to the designated campaign or the SEEC to redeem.

There are also restrictions on the minimum number of contributions and signatures in the "Democracy Voucher" program. The first restriction is that participants are subject to lower cash contributions and total expenditure limits. The city attorney and at-large city council campaigns have a limit of \$375,000, by-district city council campaigns have 187,500, and mayoral campaigns have 800,000 limit funds. However, participating candidates can request release from spending limits if non-participating candidates dramatically outspend them. Besides, the City's election codes prohibit candidates from using vouchers in other ways, e.g., reimbursing contributors for their contributions.

3 Data

For this research, I utilize various datasets to identify the effect of public funding on private funds across income heterogeneity. For records related to the voucher program, the SEEC provides data on the vouchers, including those who contributed and those candidates who received these vouchers. The data for the 2017, 2019, and 2021 elections are obtained from the committee upon request.

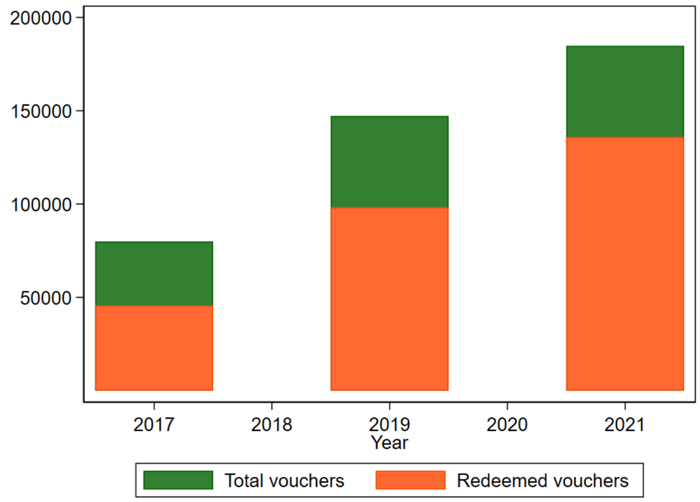
Figure 1 shows distributed and redeemed vouchers after implementing the Democracy Voucher program. The trend depicts an increasing trend in using public funding in the election from 2017 to 2021. Moreover, the share of redeemed vouchers has been almost stable through the election cycles. Ramsey et al. (2020) suggests that the increment in the usage of vouchers comes from increasing awareness of the program's dynamic and the variety of candidates. Moreover, the report argues that the share of non-redeemed vouchers has two possible explanations. First, out of 53 candidates participating in the program, 35 completed the qualifying process to receive the vouchers. Secondly, some vouchers were received after the campaign reached its limit.

To construct a reference, I use the data on the political contributions of other cities in the State of Washington. The Washington Public Disclosure Commission (PDC) gives the public access to data on financing political campaigns, lobbyist expenses, and the financial affairs of public officials and candidates for 2007 and afterward. I use two datasets from PDC containing information on contributions to each candidate's records in all Washington elections. Since the voucher datasets do not have a common key with the PDC dataset, I use a name-matching technique called "bigram[1]." described by Christen (2006), to link the tables together.

[1] Bigram is a matching method to assign a similarity score to two strings. The bigram score of s_1 and s_2 strings is:

$$sim(s_1, s_2) = \frac{1}{N-1} \sum_{i=0}^{N-1} h(i)$$

Fig. 1: Distributed and Redeemed Vouchers



Note: The data was collected from the Seattle Ethics and Election Commission. Green bars are the total number of vouchers in each election cycle. The orange part of each bar is the number of redeemed vouchers in each election cycle.

Table 1 shows the summary statistics of the political donations of individuals to candidates in local elections from 2011 to 2021. I calculate the private donation using the voucher data linked to the contribution data from PDC. A private donation is the difference between the total donations and vouchers sent to the candidates, whether they are redeemed or not, by each individual. Although private donations decrease in Seattle compared to other cities, the standard deviation is significantly larger than the drop in this table. The average total donation is larger than private donations. Many individuals contribute merely the vouchers to their desired candidates.

Most of the records in the data are donations to candidates labeled as independent or non-partisan. To assign numerical political positions to candidates without specific parties, I use out-of-data records, which are donations in Seattle contributed to all elections in local, state, and national elections but eligible elections for the DV program. First, I assign values to candidates in out-of-data records in major parties in the U.S. using the average political position estimated in Bor et al. (2023). The assigned values are described in Table 5 in the Appendix. The estimated contributor's political position is the average political position of candidates she contributes in out-of-data records. The estimated candidates' political positions in the data, the records of three eligible elections for the DV program, is the average of the estimated positions assigned to each

where N is the maximum number of characters in both strings and h_i is 1 if two characters starting from position i are the same in both strings and 0 otherwise (Akinwale and Niewiadomski (2015)).

Table 1: Summary Statistics of the Data

	Mean	SD	Min	Max	N
Total donations	147.3	349.0	0.01	130470.0	355789
Private donations	165.7	404.8	1.0	130470.0	257436
Median Income	87776.5	26509.6	11838.9	250000	354005
Political position	0.32	0.11	0	1	355427

Note: the data is collected from the Seattle Ethics and Election Commission (SEEC), Washington Public Disclosure Commission (PDC), and U.S. Census data. Total donations are the total dollar amount of money that each individual has donated to a campaign. The private donation is the total dollar amount of donations other than vouchers. Zero values are dropped from private donations. The median income is the estimated value linked to each individual's zip code in the data from the U.S. census. The political position is the estimated political value, between zero as the most left and one as the most right candidate.

contributor in the last stage. Therefore, the "left" and "right" candidates are assigned as the bottom and top half candidates in each election and year.

I also use the median income provided by the U.S. Census of 2011. The annual data is in the Zip Code Tabulation Area levels (ZCTA). Since the SEEC and PDS datasets have zip codes for all contributors who have contributed above $25, I use Uniform Data System (UDS) mapping data to link ZCTAs to zip codes. In the Appendix, Figure 11 shows the distribution of the median income of zip codes linked to the PDC data. The median income distribution in Seattle is positively skewed compared to other cities. The difference is not unexpected since Seattle is the largest city in the state.

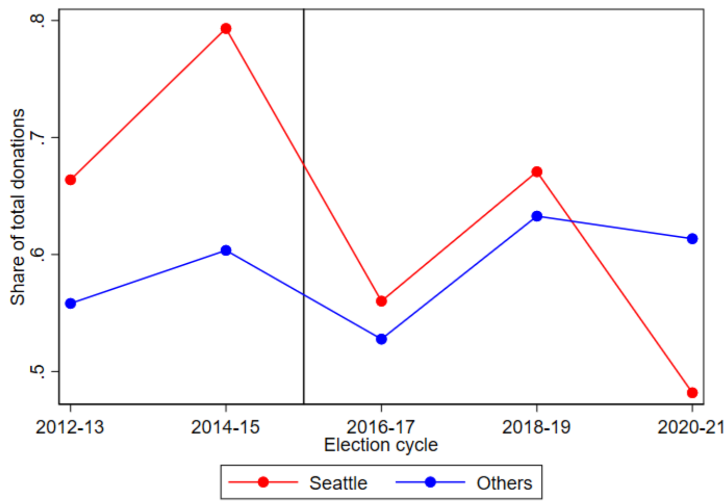
Figure 2 shows the changes in the average share of donations to the defined right candidates in elections. In other cities, the share of donations is increasing. However, the trend has been decreasing in Seattle. The drop in the share of donations is mostly after implementing the voucher policy in the 2016–17 election cycle.

Figure 3 shows the changes in the number of donors in each election cycle. The number of donors has increased since 2016. In the 2020–21 election cycle, the average total number of donors in elections in Seattle has been more than three times higher than in the 2014–15 election cycle. This is significant since, at the same time, the population increased by around 10%[2].

The gap between changes in intensive and extensive margins is evident in Figure 4. The average private donations increase by about 150% in the last cycle compared to the first one. Since the change is significantly smaller than the increment in the number of donors, it may be evidence of a decrease in the con-

[2] https://www.macrotrends.net/cities/23140/seattle/population

Fig. 2: Share of Donation to the Right-leaning Candidates



The data is collected from the Washington Public Disclosure Commission (PDC). Each point is the average share of donations to the candidates labeled as "right" across elections in a cycle.

tribution level and an increase in the number of donors simultaneously.

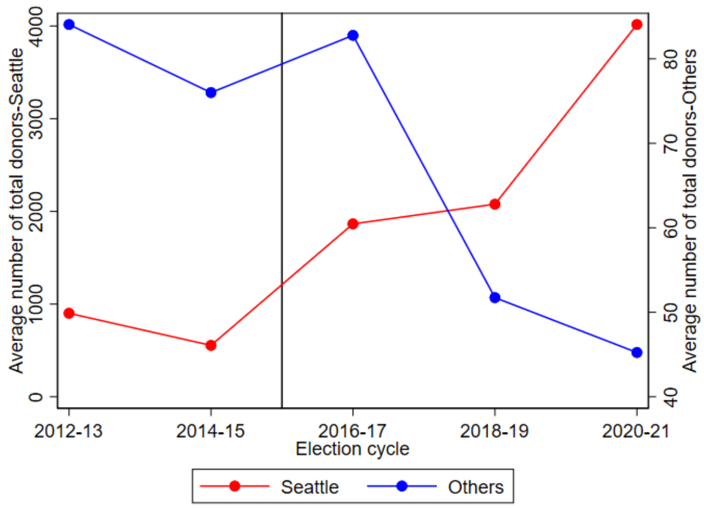
4 Identification

In this research, I implement heterogeneity in a simple event study specification to identify the different responses of groups due to the voucher program. The primary assumption is the existence of parallel trends, which requires that the expected evolution of the untreated outcome be the same inside and outside Seattle. In this case, simple Difference-in-Difference (DD) regression to identify the average treatment effect on the treated (ATT) is as follows:

$$d_{i,c,t} = \alpha_{i,c} + \lambda_t + \sum_{t=-2}^{2} \theta_t S_{i,c} \times T_t + X_{i,c,t} + \varepsilon_{i,c,t} \tag{1}$$

$d_{i,c,t}$ is the logarithm of the private donation of the contributor i to the candidate c in election cycle t, $S_{i,c}$ is a dummy variable to show whether the donor is in Seattle, and T_t is a categorical variable for each election cycle the voucher policy has been implemented, $\alpha_{i,c}$ and λ_t are the fixed effect of the election in the same jurisdiction and the time effects, respectively. $X_{i,c,t}$ are the control variables across individuals, candidates, and time. The heterogeneity of private contributions across different political positions can be estimated as follows:

Fig. 3: The Average Number of Donors in Elections



The data is collected from the Washington Public Disclosure Commission (PDC). Each point is the average number of donors across elections in each cycle.

$$d_{i,c,t} = \alpha_{i,c} + \lambda_t + \sum_{t=-2}^{2} \theta_t S_{i,c} \times T_t + \sum_{t=-2}^{2} \beta_t P_{c,t} \times S_{i,c} \times T_t + \varepsilon_{i,c,t} \quad (2)$$

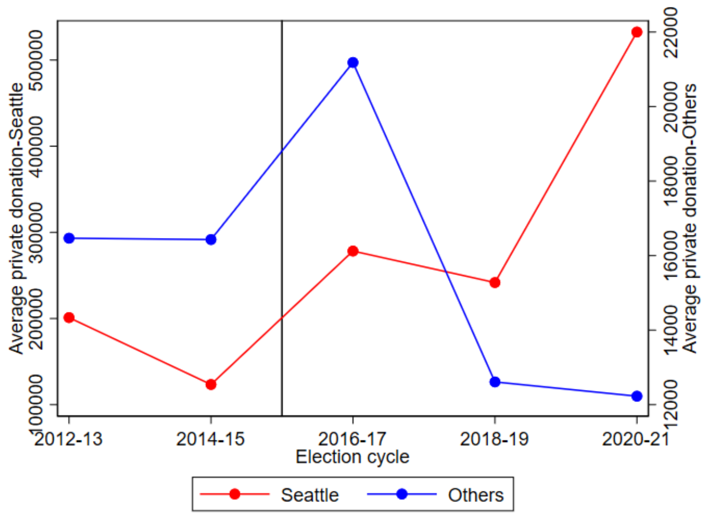
$P_{c,t}$ is a dummy variable separating candidates in two different political positions, left (0) and right (1), concerning the median political position among candidates in an election. Therefore, $\beta_{t=0}$ is the difference between the effect of voucher policy on the private donation to the "Right" candidate compared to the private donations to the "Left" candidate in the first cycle after injecting vouchers.

The main challenge is that voters respond differently to public funding in the election as their political positions and income levels differ. The primary identification assumption is that the changes in the private donation of each individual can be identified by comparing it to another unit in the same political and economic position in the control group. For instance, the change in donation of the rich voter (r) who contributes to the right politician (R) in time t is:

$$ATT_t^R = (E[d|T=1, S=1, P=1, I=1] - E[d|T=-1, S=1, P=1, I=1]) - (E[d|T=t, S=0, P=1, I=1] - E[d|T=-1, S=0, P=1, I=1]) \ \ t \geq 0 \quad (3)$$

I is a dummy variable to flag voters in higher income brackets, T is the post-treatment cycles, P is a dummy to flag "Right" politicians and S is a dummy

Fig. 4: The Average of Private Donations in Elections



The data is collected from the Washington Public Disclosure Commission (PDC). Each point is the average of donations collected in each election in each cycle.

to flag Seattle. The specification to derive the effect for contributors to each political position is as follows:

$$d^J_{i,c,t} = \alpha^J_{i,c} + \lambda^J_t + \sum_{t=-2}^{2} \theta^J_t S^J_{i,c} \times T^J_t + \sum_{t=-2}^{2} \gamma^J_t I^J_{i,t} \times S^J_{i,c} \times T^J_t + X^J_{i,c,t} + \varepsilon^J_{i,t} \quad (4)$$

$I^J_{i,t}$ is the donor's income in each election cycle for the regression on the political position $J \in \{L, R\}$. I add a control variable $X_{ic,t}$ to this regression, which changes over time and individuals. In this paper, I use the estimated number of residents in a zip code as a control variable. I also estimate the regression for both sides of defined political lines. It is also noteworthy to point out that the difference-in-difference-in-difference specification in equation 4 is fully saturated, so all combinations of the main variables $S^J_{i,c}$, T^J_t, and $I^J_{i,t}$ are controlled.

Regression 4 gives us the estimation of the average changes in private contribution for rich voters who donate to both right (R) and left (L) candidates:

$$\begin{aligned} A\hat{T}T^R_t &= \hat{\theta}^R_t + \hat{\gamma}^R_t \\ A\hat{T}T^L_r &= \hat{\theta}^L_t + \hat{\gamma}^L_t \end{aligned} \quad (5)$$

Therefore, the test of whether the private contribution of rich voters is significantly different between two opposite political groups is defined as:

$$\begin{cases} H_0 : \hat{\theta}_t^R + \hat{\gamma}_t^R - (\hat{\theta}_t^L + \hat{\gamma}_t^L) = 0 \\ H_1 : \hat{\theta}_t^R + \hat{\gamma}_t^R - (\hat{\theta}_t^L + \hat{\gamma}_t^L) \neq 0 \end{cases} \tag{6}$$

5 Results

Table 2: Simple Diff-in-Diff

	City Council	All
Seattle $\times$ *Time*	-0.570***	-0.565***
	(0.045)	(0.035)
Cycle	Yes	Yes
Election	Yes	Yes
Obs.	176605	257219
R-squared	0.138	0.125

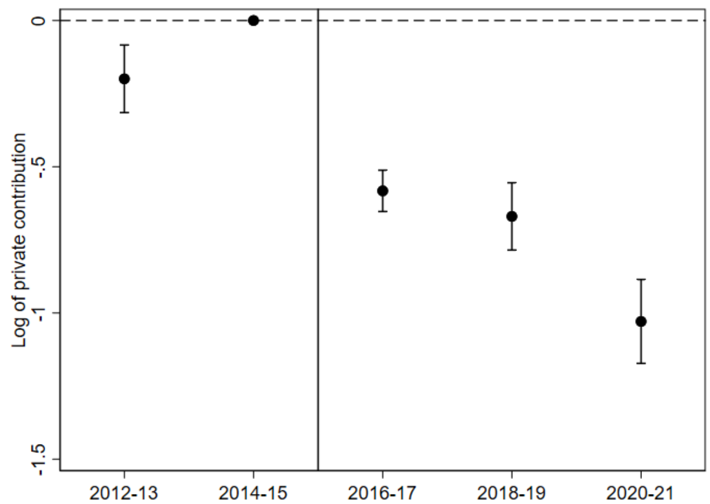
Regressions are in among individual contributors who are donating to the campaigns in elections of the cities they reside. All regressions are clustered at the city level. The combination of zip-codes and the position in the election and the time effects of each election cycle are controlled.
* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Table 2 shows the simple difference-in-difference for city council elections and all elections with voucher policy in Seattle. The table shows that private donations, defined as the positive value of non-voucher donations, have decreased on average since introducing the democracy voucher program. Moreover, the regression result with city council election records is not significantly different from the result of considering all eligible elections, including mayoral and city attorney elections. Figure 5 shows the event study of the treatment in each election cycle. The private donations drop significantly after implementing the voucher policy, marked by the vertical line.

Table 3 shows the regression results related to the equation 4. In each regression, the separating border for the income flag is displayed on top of the column. The coefficient of income variables increases as we compare smaller groups at the top of the income distribution to others in the regression. The first two columns show that the difference between the private contributions above and below the 50th and 80th percentile of the median income distribution is statistically insignificant. However, from the last two columns, it is clear that the contribution of the voters in the higher income percentiles drops significantly compared to others.

Figure 6 shows the heterogeneity of private contributions across candidates on both sides of the political spectrum in elections. Although the contributions to both political positions are not significantly different before the treatment, the private contribution to candidates in "right" political positions is higher than candidates in "left" two cycles after implementing the voucher policy. Therefore,

Fig. 5: Changes in Private Contribution in Each Election Cycle



Each point is the estimated coefficient of the event study in equation 1 and the confidence interval. The baseline is set on the cycle before implementing the DV program, which is the 2014-15 cycle.

Table 3: Diff-in-Diff with Income Heterogeneity

	Median-p50	Median-p80	Median-p90	Median-p99
Seattle × Time	-0.569***	-0.579***	-0.582***	-0.575***
	(0.033)	(0.037)	(0.036)	(0.035)
Income × Seattle × Time	0.017	0.038	0.099***	0.115***
	(0.017)	(0.025)	(0.031)	(0.041)
Cycle	Yes	Yes	Yes	Yes
Election	Yes	Yes	Yes	Yes
Obs.	257219	257219	257219	257219
R-squared	0.127	0.128	0.129	0.129

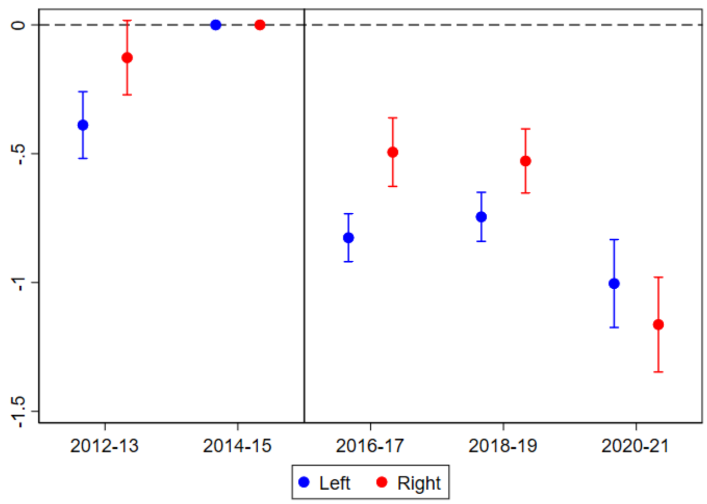
Regressions are in among individual contributors who are donating to the campaigns in elections of the cities they reside. All regressions are clustered at the city level. The combination of zip-codes and the position in the election and the time effects of each election cycle are controlled.

* $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$

private contribution changes can vary across income and political gaps.

Table 4 shows the result of the equation 4. The pattern of regressions is consistent across different income lines. Private donations have decreased for all groups in income and political dimensions. However, the drop in private contributions is significantly smaller for rich voters who prefer to donate to relatively "Right" politicians.

Fig. 6: Changes in Private Contribution to Both Defined Political Positions



Each point is the estimated coefficient of the event study in equation 2 and the confidence interval. The baseline is set on the cycle before implementing the DV program, which is the 2014-15 cycle.

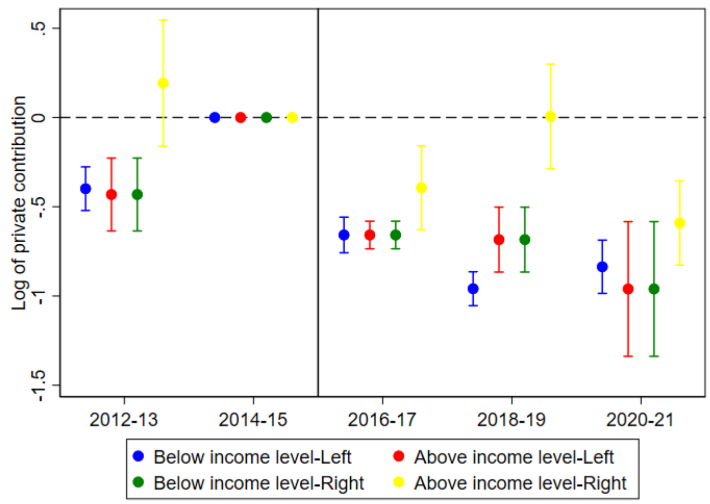
Table 4: Diff-in-Diff with Income and Political Heterogeneity

	Pct. 90		Pct. 99	
	Left	Right	Left	Right
Seattle $\times$ *Time*	-0.449***	-0.502***	-0.485***	-0.484***
	(0.043)	(0.052)	(0.041)	(0.051)
Income $\times$ *Seattle* $\times$ *Time*	-0.109***	0.150***	0.042	0.188***
	(0.041)	(0.055)	(0.076)	(0.043)
Cycle	Yes	Yes	Yes	Yes
Election	Yes	Yes	Yes	Yes
Population	Yes	Yes	Yes	Yes
Obs.	112920	143001	112920	143001
R-squared	0.184	0.123	0.185	0.124

Regressions are in among individual contributors who are donating to the campaigns in elections of the cities they reside. All regressions are clustered at the city level. The combination of zip-codes and the position in the election and the time effects of each election cycle are controlled.
* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Figure 7 depicts the same results across election cycles while assuming the income line is on the 99th percentile. The drop in the donation of contributors above the 99th income percentile who contribute to the right candidate is significantly less than the drop in other groups' donations in the 2018-19 cycle. Although the same difference exists in the 2016-17 cycle, it is not statistically significant in %5. The effect does not seem to be present in 2020-21. Moreover, the figures' confidence intervals overlapped in the first election cycle, showing no statistically significant gap between these groups before the voucher policy, which assures the parallel trend assumption. The results considering other income lines, such as 80th, 90th, and 95th percentiles, are consistent and available in the Appendix.

Fig. 7: Changes in Private Contribution for Different Groups Across Election Cycles (Income Level: 99th Pct.)



Each point is the estimated coefficient of the event study in equation 4 and the confidence interval. The baseline is set on the cycle before implementing the DV program, which is the 2014–15 cycle.

One concern over the credibility of the results is the problem of the few treated clusters. Olden and Møen (2022) discusses the advantage of the DDD compared to DD specifications regarding the over-rejection problem. However, since the correction methods for inference are mostly developed for DD specifications, I use the randomization inference based on t-statistics discussed in **?** to infer the second regression in Table 2. In the Appendix section, Figure 15 shows that the difference-in-difference estimation is also rejected using the correction method.

Another concern over the results comes from the changes in the combination of donors. Since there is some attrition of the old donors and some new donors after the treatment, some may argue that comparing the two groups threatens identification. In the Appendix, the table 6 shows the results of the DDD specification, considering the 90th percentile as the income line, for those donors who have records in the data before the voucher policy. Although the difference between above and below income levels in each political group is not statistically significant, the results are consistent with Table 4 since private donations decrease less for the right-leaning voters above the income level.

The limits on the program's campaigns can be another cause for concern. It is possible that the donors will not send their private donations and vouchers to the campaign anymore after the announcement that the limit has been reached. From 140 campaigns in all of the elections in Seattle after the voucher policy, 34 of them have reached their limit of fund-raising. However, the substantial gap between the recorded total donation in the data and the limit for all of these campaigns suggests that there were probably lifting the limits upon request. In the Appendix, Table **??** shows the results of DDD specification, considering the 90th percentile as the income line, for those records related to the committees that did not reach their limits. The results are consistent with the findings in Table 4.

6 Theoretical Framework

There are two political candidates, L and R, announcing their policies as $y_L = \frac{1}{2} - \frac{\Delta}{2}$ and $y_R = \frac{1}{2} + \frac{\Delta}{2}$. In this model, I assume that candidates fix their positions before the election and do not change them through the election process. Moreover, voters are heterogeneous in two dimensions: **ideal political position**, $x_i \in X = \{x_0, x_1\}$, $x_0 = 0$, $x_1 = 1$, and **wealth**, $w_i \in W = \{w_p, w_r\}$, $w_p < w_r$. There are $n_{0,p} > 1$ voters with w_p and x_0, and $n_{1,p} > 1$ voters with w_p and $x_1 = 1$. Moreover, there are $n_{0,r} > 1$ voters with w_r and x_0, and $n_{1,r} > 1$ voters with w_r and $x_1 = 1$. To simplify, I assume that $n = n_{0,p} + n_{1,p} + n_{0,r} + n_{1,r}$ and the share of each group in the population is defined as $s_{x,w} = \frac{n_{x,w}}{n}$.

Each voter receives a voucher of value v, which can be used only for political contributions. Voter i chooses her level of consumption c_i and political donation to candidate L, d_i^l, and to candidate R, d_i^r, to optimize her expected utility:

$$\begin{aligned} \max_{c_i, d_i^L, d_i^R} \quad & \pi_i^L[-\gamma_i(x_i - y_L)^2 + \ln c_i] + \pi_i^R[-\gamma_i(x_i - y_R)^2 + \ln c_i] \\ s.t. \quad & c_i \le w_i - \max\{0, d_i - v\} \\ & d_i = d_i^L + d_i^R \end{aligned} \tag{7}$$

where π_i^L and π_i^R are the winning probabilities of candidates L and R, respectively. γ_i is the sensitivity of the individual to change in p policy, described as the marginal utility of political polarization. To simplify the model, I define the ratio $\xi = \frac{\gamma_r}{\gamma_p}$.

The timing of the game can be formalized as follows:

1. Candidates announce their policies.
2. Each voter chooses their level of consumption and political contribution to the candidates.

3. The winner of the election is determined through a Tullock contest:

$$\pi_R = \frac{\sum_{i=1}^n d_i^R}{\sum_{i=1}^n d_i^R + \sum_{i=1}^n d_i^L} \quad \pi_L = \frac{\sum_{i=1}^n d_i^L}{\sum_{i=1}^n d_i^R + \sum_{i=1}^n d_i^L} \tag{8}$$

Lemma 1. *There is no equilibrium that for an individual i, $d_i^L > 0$ and $d_i^R > 0$.*

From Lemma 1, it is evident that voters with x_0 merely donate to candidate L, and voters with x_1 merely donate to candidate R. The assumption in the model is justified because the incentive for donating to both candidates is mostly related to changing the candidates' political positions in favor of the donor. Since the candidates' political positions are fixed, the result of Lemma 1 can be acceptable. For the rest of the paper, I denote d_i^0 as d_i^L and d_i^1 as d_i^R.

Lemma 2. *There is no equilibrium that individuals i and j with the same wealth w and political position x have different donations.*

Lemma 3. *There is no equilibrium such that either $\pi_R = 0$ or $\pi_R = 1$.*

Lemma 2 demonstrates that heterogeneity exists between groups rather than within them. Consequently, equilibrium is achieved when contributors within a particular political and economic bracket donate at similar levels. Additionally, Lemma 3 corroborates the notion that at least one group of voters donates to their preferred candidates at each extreme of the political spectrum.

To move further, all variables are divided by w_r to simplify the results and intuitions. Therefore:

$$\theta = \frac{w_p}{w_r}, \tau = \frac{v}{w_r}, \delta_r^R = \frac{d_r^R - v}{w_r}, \delta_r^L = \frac{d_r^L - v}{w_r}, \delta_p^R = \frac{d_p^R - v}{w_r}, \delta_p^L = \frac{d_p^L - v}{w_r} \tag{9}$$

where δ_i^C is the normalized private contribution of the donor i to candidate C. The following proposition opens the path to conducting comparative statics analysis on the model:

Proposition 1. *There exists an upper bound $\bar{\tau}$ on the normalized level of the voucher and a lower bound $\underline{\gamma}_r$ on the marginal utility of political polarization such that, for all $\tau \leq \bar{\tau}$ and all $\gamma_r \geq \underline{\gamma}_r$, there is a unique equilibrium in which all agents contribute in equilibrium with:*

$$\begin{aligned} \pi_R = \frac{(s_{1,r} + s_{1,p}\xi)\delta_r^R + (s_{1,p}(\theta - \xi) + s_1\tau)}{(s_{1,r} + s_{1,p}\xi)\delta_r^R + (s_{0,r} + s_{0,p}\xi)\delta_r^L + (s_p(\theta - \xi) + \tau)}, \\ \delta_r^R = 1 - \frac{n(s_{1,r} + s_{1,p}\theta + s_1\tau)}{\phi_r + n(s_{1,r} + s_{1,p}\xi)}, \\ \delta_r^L = 1 - \frac{n(s_{0,r} + s_{0,p}\theta + s_0\tau)}{\phi_r + n(s_{0,r} + s_{0,p}\xi)} \end{aligned} \tag{10}$$

where $s_p = s_{0,p} + s_{1,p}$ and $\phi_r = \pi_R(1 - \pi_R)\gamma_r\Delta$.

The term ϕ_r is derived by combining the risk of changing the election result and the marginal benefit of political polarization in the model. This term can be interpreted as the expected utility cost of losing the election. The donation will increase for each rich individual in the political spectrum when the expected utility cost of losing the election increases. This is also evident in proposition 1.

The existence of an equilibrium with positive donations for all individuals in the proposition 1 is a crucial result of the model since it makes analyzing the changes in the donation of the voters possible. Before going through the comparative statics part, the model should be set to the status quo in the empirical section. The next lemma is to set a condition that $\pi_r > \frac{1}{2}$:

Lemma 4. *In the equilibrium described in proposition 1 and $\tau = 0$, if $(s_{1,r} - s_{0,r}) + (s_{1,p} - s_{0,p})\theta > 0$, then there exist a cutoff $\hat{\gamma}_r$ such that for all $\gamma_r > \hat{\gamma}_r$, the equilibrium winning probability of candidate R is $\pi_R > \frac{1}{2}$.*

Lemma 4 is intuitive since it asserts that the concentration of more wealth on one political side can lead to a higher probability of winning the election. However, it does not necessarily mean that the wealth gap between the two sides should favor the same political side at every income level. For instance, if $s_{0,p} < s_{1,p}$, the same condition holds if $\theta < \frac{s_{1,r} - s_{0,r}}{s_{0,p} - s_{1,p}}$. Therefore, if the concentration of wealth is higher in one political group among poor voters, their winning probability is less than the other if the inequality gap is large enough.

Lemma 4 sufficiently establishes the basis for the forthcoming proposition, which holds significant relevance to the empirical findings:

Proposition 2. *In the equilibrium described in proposition 1, if $\pi_R > \frac{1}{2}$ and $s_1 < \frac{1}{2}$, then there exists a cutoff $\underline{n}$ such that for all $n > \underline{n}$, $\left.\frac{\partial \delta_r^J}{\partial \tau}\right|_{\tau=0} < 0$.*

Proposition 2 states that if the number of voters is large enough, the people's private donations to the election's winning side will decrease after implementing the voucher policy, whether the policy favors their agenda or not. The result comes from the fact that the response to the voucher policy comes from three channels: the substitution for private donations, the shock to the balance of the election due to the changes in the winning probability, and the response to the changes in the donations of other groups. A higher number of voters decreases other effects on the changes in donations rather than the substitution effect, which is negative for all voters.

Moreover, the following proposition helps to compare the changes in the donations of rich voters:

Proposition 3. *In the equilibrium described in proposition 1, if $\pi_R > \frac{1}{2}$ and $s_1 < \frac{1}{2}$, then there exist cutoff points $\tilde{n}$ and $\tilde{\gamma}_r$ such that for all $n > \tilde{n}$, $\left.\frac{\partial \delta_r^J}{\partial \tau}\right|_{\tau=0} < 0$ and for all $\gamma_r > \tilde{\gamma}_r$:*

$$\left.\frac{\partial \delta_r^R}{\partial \tau}\right|_{\tau=0} - \left.\frac{\partial \delta_r^L}{\partial \tau}\right|_{\tau=0} > 0$$

Proposition 3 shows that a range of parameters γ_r and n exist such that the rich donor who contributes to the right-leaning candidate resists the voucher policy by decreasing her donation less than the other group. Therefore, the voucher policy's substitution effect is weaker if one political side's financial means do not match their population in a majority-rule democracy.

A simulation of the model shows the result more clearly. First, I set the parameters of the model as follows:

$$n = 1000, s_{0,r} = 0.02, s_{1,r} = 0.08, s_{0,p} = 0.52, s_{1,p} = 0.38$$
$$\gamma_r = 100, \xi = 0.11, \theta = 0.1, \Delta = 0.5$$

In this case, the total wealth of the right-leaning voters is significantly higher than the other side, although they are a minority. Figure 8 shows the changes in private donations of rich voters on both political sides (δ_r^R and δ^L) concerning small increments in the share of vouchers. Since the number of donors is significant, both variables decrease.

Fig. 8: Share of Donations In Income

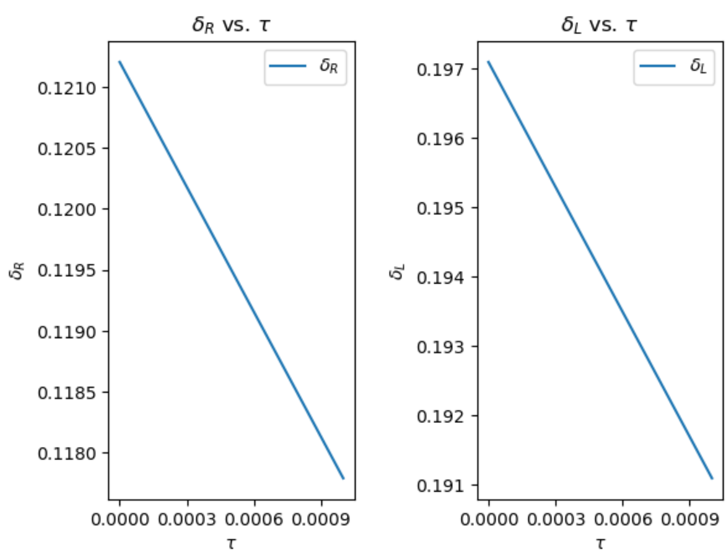


Figure **??** depicts the winning probability of the right-leaning candidate. First, the winning probability is above $\frac{1}{2}$ when $\tau = 0$, which is consistent with Lemma 4. Secondly, the winning probability decreases after implementing the voucher policy, which is due to creating more balance regarding the donations of both sides.

Fig. 9: The winning probability of right-leaning candidate

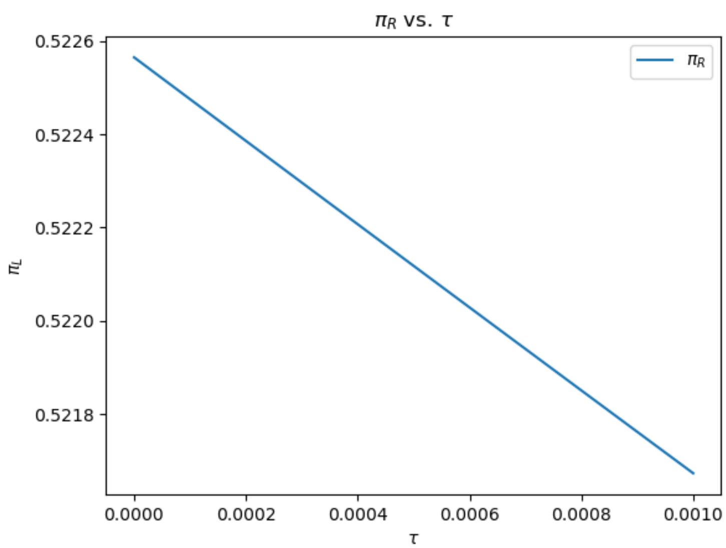
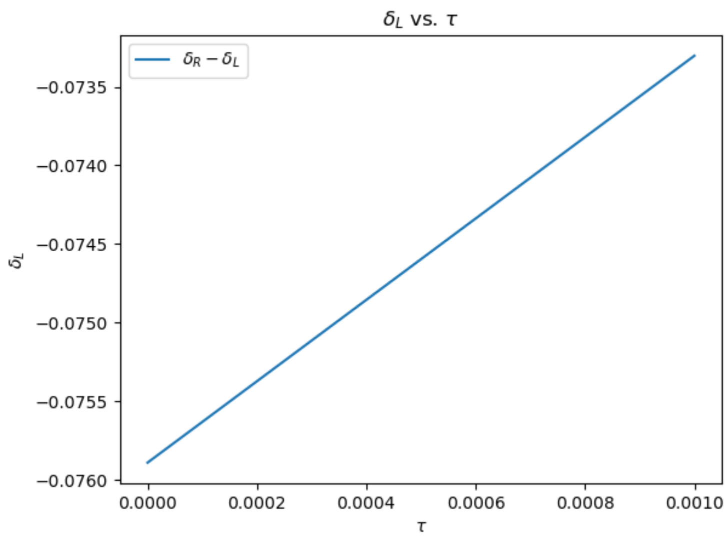


Figure 10 shows the difference in private donations of two groups of rich voters, $\delta_r^R - \delta_r^L$, concerning the changes in the share of vouchers in their wealth. Since in the equilibrium $\delta_r^R < \delta_r^L$, the increasing value shows that the gap between the two variables is decreasing. The result is consistent with the outcome of proposition 3.

Fig. 10: Difference of Private Donations $\delta_r^R - \delta_r^L$



7 Conclusion

This study illustrates that public funding in elections, exemplified by Seattle's "Democracy Voucher" policy, significantly alters the landscape of private political contributions. The decrease in private contributions following the policy implementation suggests a shift in political engagement dynamics. Notably, high-income groups show a less pronounced reduction in private donations than other groups, indicating a nuanced impact of public funding across different income brackets.

This research contributes to understanding how public funding can reshape political donations, potentially leveling the playing field in political campaigns. Future implications include the need for ongoing assessment of such policies to ensure they effectively democratize political participation without unintended consequences. The study's findings are vital for policymakers, indicating the importance of considering income and political heterogeneity when designing and implementing public funding schemes in elections.

References

Akinwale, A. and A. Niewiadomski (2015). Efficient similarity measures for texts matching. *Journal of Applied Computer Science 23*(1), 7–28.

Andreoni, J. and A. A. Payne (2011). Is crowding out due entirely to fundraising? evidence from a panel of charities. *Journal of public Economics 95*(5-6), 334–343.

Ben-Zion, U. and Z. Eytan (1974). On money, votes, and policy in a democratic society. *Public Choice 17*(1), 1–10.

Bental, B. and U. Ben-Zion (1975). Political contribution and policy: Some extensions. *Public Choice*, 1–12.

Bental, B. and U. Ben-Zion (1981). A simple model of political contributions. *Public Finance Quarterly 9*(2), 143–157.

Bor, D., B. S. Lee, and E. J. Oughton (2023). Quantifying polarization across political groups on key policy issues using sentiment analysis. *arXiv preprint arXiv:2302.07775*.

Borgonovi, F. (2006). Do public grants to american theatres crowd-out private donations? *Public choice*, 429–451.

Bouton, L., M. Castanheira, and A. Drazen (2018). A theory of small campaign contributions. Technical report, National Bureau of Economic Research.

Chowdhury, S. M., D. Lee, and R. M. Sheremeta (2013). Top guns may not fire: Best-shot group contests with group-specific public good prizes. *Journal of Economic Behavior & Organization 92*, 94–103.

Christen, P. (2006). A comparison of personal name matching: Techniques and practical issues. In *Sixth IEEE International Conference on Data Mining-Workshops (ICDMW'06)*, pp. 290–294. IEEE.

Cornes, R. and R. Hartley (2003). Risk aversion, heterogeneity and contests. *Public Choice 117*(1-2), 1–25.

Cornes, R. and R. Hartley (2005). Asymmetric contests with general technologies. *Economic theory 26*, 923–946.

Desilver, D. and P. Van Kessel (2015). As more money flows into campaigns, americans worry about its influence.

Dowling, C. M., R. D. Enos, A. Fowler, and C. Panagopoulos (2012). Does public financing chill political speech? exploiting a court injunction as a natural experiment. *Election Law Journal 11*(3), 302–315.

Griffith, A. and T. Noonen (2022). The effects of public campaign funding: Evidence from seattle's democracy voucher program. *Journal of Public Economics 211*, 104676.

Heutel, G. (2014). Crowding out and crowding in of private donations and government grants. *Public Finance Review 42*(2), 143–175.

Klumpp, T., H. M. Mialon, and M. A. Williams (2015). Leveling the playing field? the role of public campaign funding in elections. *American Law and Economics Review 17*(2), 361–408.

Malbin, M. J., P. W. Brusoe, and B. Glavin (2012). Small donors, big democracy: New york city's matching funds as a model for the nation and states. *Election Law Journal 11*(1), 3–20.

Miller, M. G. (2011). After the gao report: what do we know about public election funding? *Election Law Journal 10*(3), 273–290.

Miller, M. G. (2013). *Subsidizing democracy: How public funding changes elections and how it can work in the future.* Cornell University Press.

Olden, A. and J. Møen (2022). The triple difference estimator. *The Econometrics Journal 25*(3), 531–553.

OpenSecrets (2023). Election overview. Accessed on September 01, 2023.

Overton, S. (2011). Matching political contributions. *Minn. L. Rev. 96*, 1694.

Ramsey, K., B. Anderson, J. Tesch, and J. Major (2020). Seattle democracy voucher program: 2019 election cycle evaluation. Technical report, BERK Consulting.

Shieh, S. and W.-H. Pan (2010). Individual campaign contributions in a downsian model: expressive and instrumental motives. *Public Choice 145*, 405–416.

Tullock, G. (2001). Efficient rent seeking. *Efficient rent-seeking: Chronicle of an intellectual quagmire*, 3–16.

Welch, W. P. (1974). The economics of campaign funds. *Public Choice 20*(1), 83–97.

Welch, W. P. (1976). A rejoinder to silberman. *Public Choice (pre-1986) 25*, 75.

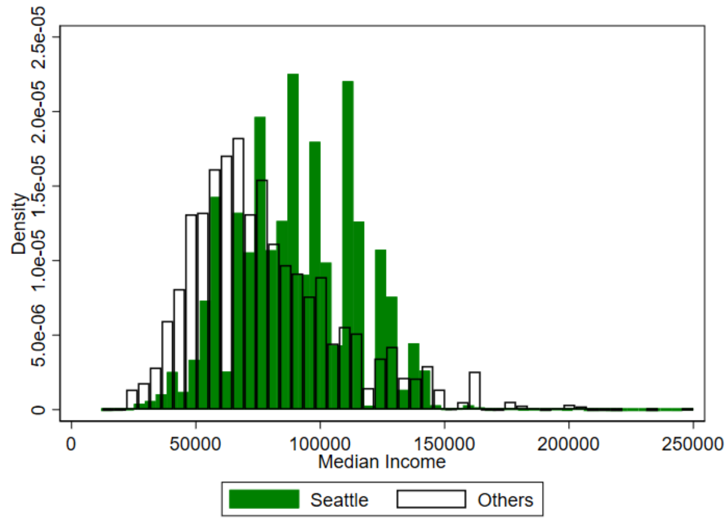
8 Appendix

Table 5: Tabulation of Political Parties

	Assigned values	Out of Data Obs.	Pct.	In Data Obs.	Pct.	Total Obs.	Pct.
American Heritage	1	119	0.01	13	0.00	132	0.01
Constitution	1	21	0.00	0	0.00	21	0.00
Democratic	0.28	354406	26.63	18561	5.22	372967	22.11
Green	0	197	0.01	0	0.00	197	0.01
Independent	-	4398	0.33	2984	0.84	7382	0.44
Libertarian	1	1781	0.13	11	0.00	1792	0.11
Non Partisan	-	61516	4.62	82135	23.09	143651	8.52
None	-	27775	2.09	87416	24.57	115191	6.83
Other	-	8083	0.61	27103	7.62	35186	2.09
Republican	0.71	254411	19.12	570	0.16	254981	15.12
Workers World	0	3	0.00	50	0.01	53	0.00
.	-	618163	46.45	136946	38.49	755109	44.77
Total	-	1330873	100.00	355789	100.00	1686662	100.00

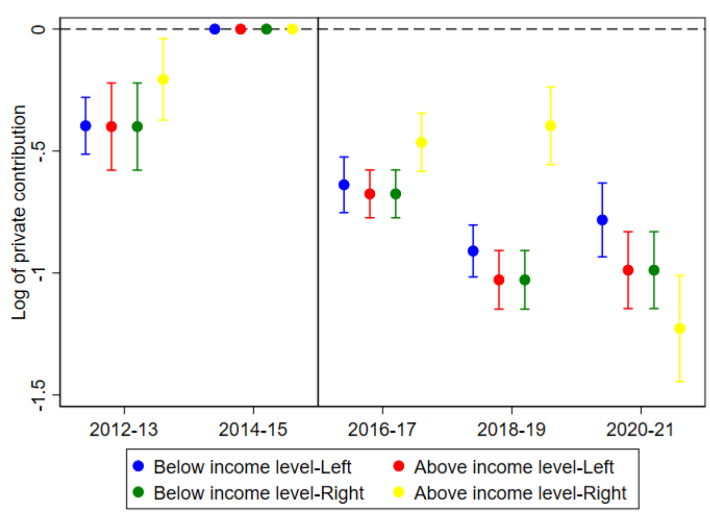
The data is collected from the Washington Public Disclosure Commission (PDC). Columns labeled "In Data" are the records of donations in eligible elections for the Democracy Voucher program, i.e., city council elections, city attorney elections, and mayoral elections. Columns labeled "Out of Data" are records of donations in other elections at local, state, and national levels. The first column is the assigned value of political position to campaigns that received donations in "Out of Data" records from the estimated values in Bor et al. (2023).

Fig. 11: Distribution of Median Income Linked to Zip Codes in PDC Data



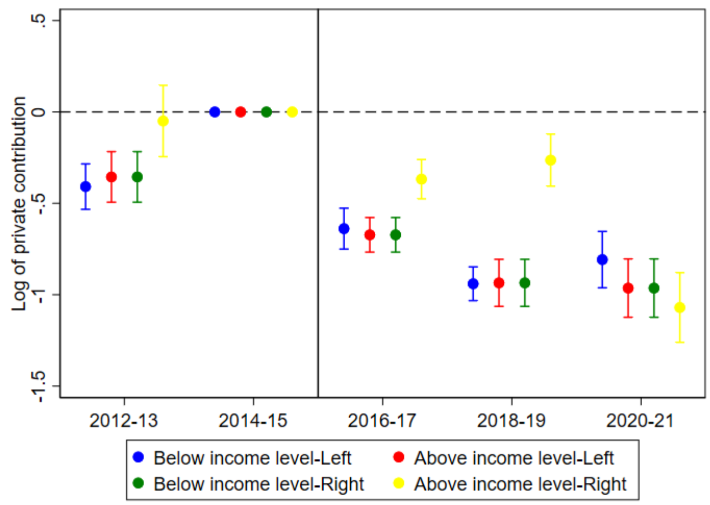
The data is collected from the Washington Public Disclosure Commission (PDC) and the U.S. Census.

Fig. 12: Changes in Private Contribution for Different Groups Across Election Cycles (Income Level: 80th Pct.)



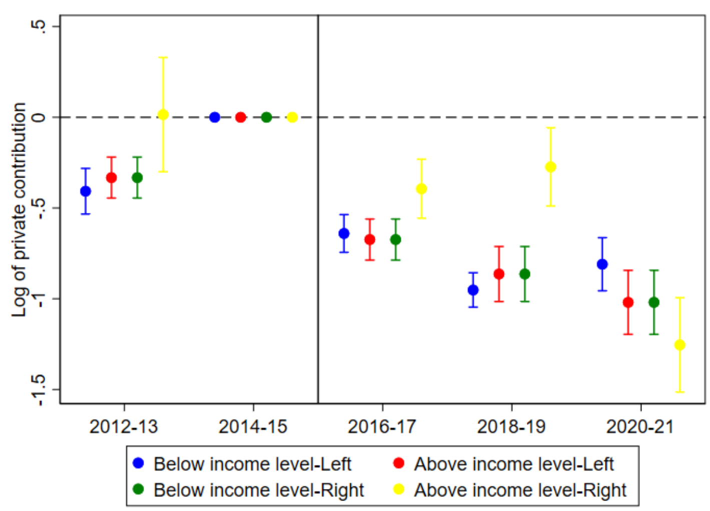
Each point is the estimated coefficient of the event study in equation 4 and the confidence interval. The baseline is set on the cycle before implementing the DV program, which is the 2014–15 cycle.

Fig. 13: Changes in Private Contribution for Different Groups Across Election Cycles (Income Level: 90th Pct.)



Each point is the estimated coefficient of the event study in equation 4 and the confidence interval. The baseline is set on the cycle before implementing the DV program, which is the 2014–15 cycle.

Fig. 14: Changes in Private Contribution for Different Groups Across Election Cycles (Income Level: 95th Pct.)



Each point is the estimated coefficient of the event study in equation 4 and the confidence interval. The baseline is set on the cycle before implementing the DV program, which is the 2014–15 cycle.

Fig. 15: Randomization Inference based on t-statistics

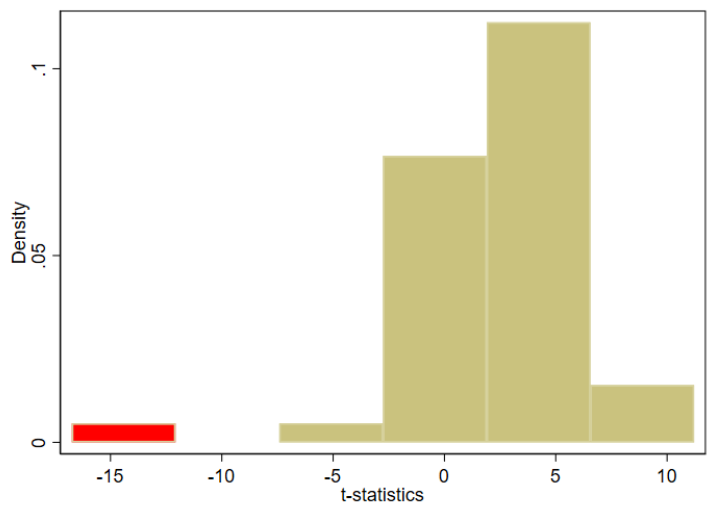


Table 6: Diff-in-Diff with Income Heterogeneity By 90th Percentile

	Left	Right
Seattle × *Time*	-0.392***	-0.197***
	(0.032)	(0.057)
Income × *Seattle* × *Time*	0.100*	0.066
	(0.059)	(0.077)
Cycle	Yes	Yes
Election	Yes	Yes
Population	Yes	Yes
Log(income)	Yes	Yes
Obs.	24556	16845
R-squared	0.137	0.127

Regressions are in among indvidual contributors who are donating to the campaigns in elections of the cities they reside. All regressions are clustered in city level. The combination of zip-codes and the position in election and the times effects of each election cycle are controlled.

* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Lemma 1:

Proof. Each individual i should be either $L \succeq_i R$ or $R \succeq_i L$ in ex-post. Suppose that $d_i^L > 0$ and $d_i^R > 0$ are equilibrium donations of individual i, and WLOG suppose that $L \succeq_i R$. If individual i decreases a small portion of the donation to R, which is ϵ, a d adds it to the donation to L, the consumption does not change. However, the ex-ante utility has improved since:

$$\pi_L(d_i^L, d_i^R, d_{-i}^L, d_{-i}^R)[-\gamma_i(x_i - y_L)^2] + \pi_R(d_i^L, d_i^R, d_{-i}^L, d_{-i}^R)[-\gamma_i(x_i - y_R)^2] <$$
$$\pi_L(d_i^L + \epsilon, d_i^R - \epsilon, d_{-i}^L, d_{-i}^R)[-\gamma_i(x_i - y_L)^2] + \pi_R(d_i^L + \epsilon, d_i^R - \epsilon, d_{-i}^L, d_{-i}^R)[-\gamma_i(x_i - y_R)^2]$$

Table 7: Diff-in-Diff with Income and Political Heterogeneity

	Left	Right
Seattle $\times$ *Time*	-0.769***	-0.827***
	(0.044)	(0.085)
Income $\times$ *Seattle* $\times$ *Time*	0.077*	0.307***
	(0.046)	(0.068)
Cycle	Yes	Yes
Election	Yes	Yes
Population	Yes	Yes
Log(income)	Yes	Yes
Obs.	92631	60335
R-squared	0.157	0.149

Regressions are in among indvidual contributors who are donating to the campaigns in elections of the cities they reside. All regressions are clustered in city level. The combination of zip-codes and the position in election and the times effects of each election cycle are controlled.
* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Therefore, there is no equilibrium in which $d_i^L > 0$ and $d_i^R > 0$.■

Lemma 2:

Proof. The first-order condition of the equation 7 is:

$$-\frac{\partial U_i(y_j, x_i, c_i)}{\partial c_i} + \frac{\partial \pi_j}{\partial d_i^j}[U_i(y_j, x_i, c_i) - U_i(y_{-j}, x_i, c_i)] \tag{11}$$

Suppose that i and j exists that $w_i = w_j$ and $x_i = x_j$ and $d_i^C \neq d_j^C$. Since the marginal benefit of donation is the same for voters in the same group, then the marginal utility of the consumption should be the same :

$$\frac{1}{w_i - d_i^C} = \frac{1}{w_j - d_j^C}$$

Because $w_i = w_j$, so $d_i^C = d_j^C$ which contradicts with initial assumption. The case can be proved if it is assumed that one of the donations is equal to zero by using the same logic.■

Lemma 3:

Proof. Without loss of generality, I suppose that $d_i^L = 0$ for all individuals with $x_i = 0$. If all individuals with $x_j = 1$ have $d_j^R = 0$, then each individual with $x_j = 1$ has the incentive to donate a small amount. However, if there is an individual j that $d_j^R > 0$, then the individual has an incentive to decrease her donation. Since the probability of winning R does not change with a small drop in the donation of the voter j, the equilibrium is not incentive-compatible.■

Proposition 1:

Proof. Based on lemma 1, 2, and 3, there is at least one group on each side of the political spectrum that contributes to their desired candidate. If all of the individuals have a positive private donation, the response of each will be:

$$-\frac{\partial U_i(y_j, x_i, c_i)}{\partial c_i} + \frac{\partial \pi_j}{\partial d_i^j}[U_i(y_j, x_i, c_i) - U_i(y_{-j}, x_i, c_i)] = 0 \implies$$

For example, individuals whose political positions are $x = 1$ respond through the following equation:

$$-\frac{1}{w_r + v - d_r^R} + \frac{\pi_R(1-\pi_R)}{n(s_{1,r}d_r^R + s_{1,p}d_p^R)}\gamma_r\Delta = 0 \tag{12}$$

$$-\frac{1}{w_p + v - d_p^R} + \frac{\pi_R(1-\pi_R)}{n(s_{1,r}d_r^R + s_{1,p}d_p^R)}\gamma_p\Delta = 0 \tag{13}$$

Using these equations and the normalization described in 9, then:

$$\delta_p^R = \frac{d_p^R - v}{w_r} = \theta - \xi(1 - \delta_r^R) \tag{14}$$

Therefore, the share of donation of an individual with w_r and $x = 1$ can be obtained using equations in 9, equation 12, and equation 14:

$$\delta_r^R = 1 - \frac{n(s_{1,r} + s_{1,p}\theta + s_1\tau)}{\phi_r + n(s_{1,r} + s_{1,p}\xi)} \tag{15}$$

where $\phi_r = \pi_R(1 - \pi_r)\gamma_r\Delta$. Using the same logic, the share of donation of an individual with w_r and $x = 0$ is:

$$\delta_r^L = 1 - \frac{n(s_{0,r} + s_{0,p}\theta + s_0\tau)}{\phi_r + n(s_{0,r} + s_{0,p}\xi)} \tag{16}$$

Since both equations are related to an endogenous variable, which is π_R through ϕ_r, it is necessary to have the probability of winning candidate R in our system of equations:

$$\pi_R = \frac{(s_{1,r} + s_{1,p}\xi)\delta_r^R + s_{1,p}(\theta - \xi) + s_1\tau}{(s_{1,r} + s_{1,p}\xi)\delta_r^R + (s_{0,r} + s_{0,p}\xi)\delta_r^L + s_p(\theta - \xi) + \tau} \tag{17}$$

Based on equation 14 and its counterpart for δ_p^L, and also equations 15 and 16, conditions that the equilibrium has a positive value of private donation for each donor are as follows:

$$\delta_r^R > \max\left\{0, 1 - \frac{\theta}{\xi}\right\}, \;\; \delta_r^L > \max\left\{0, 1 - \frac{\theta}{\xi}\right\} \tag{18}$$

If $\xi < \theta$, then the conditions hold if:

$$\phi_r > n \max\left\{(s_{1,p}(\theta - \xi) + s_1\tau), (s_{0,p}(\theta - \xi) + s_0\tau)\right\}$$

If the above inequality holds for the possible infimum of ϕ_r, it also holds for other values. Because ϕ_r is concave with a maximum at $\pi_R = \frac{1}{2}$, then:

$$\inf \phi_r = \min \{\inf \{\pi_R\}(1 - \inf \{\pi_R\}), \sup \{\pi_R\}(1 - \sup \{\pi_R\})\}\gamma_r \Delta \tag{19}$$

Since in equation 17, π_R is increasing in δ_r^R and decreasing in δ_r^L, therefore:

$$\inf_{\delta_r^R, \delta_r^L} \pi_R = \frac{s_{1,p}(\theta - \xi) + s_1 \tau}{s_{0,r} + s_{0,p}\xi + s_p(\theta - \xi) + \tau} \tag{20}$$

$$\sup_{\delta_r^R, \delta_r^L} \pi_R = \frac{s_{1,r} + s_{1,p}\theta + s_1 \tau}{s_{1,r} + s_{1,p}\xi + s_p(\theta - \xi) + \tau} \tag{21}$$

Therefore, if $\xi < \theta$, each individual in the equilibrium donates a positive value if:

$$\gamma_r > n \frac{\max \{(s_{1,p}(\theta - \xi) + s_1 \tau), (s_{0,p}(\theta - \xi) + s_0 \tau)\}}{\min \{\inf \{\pi_R\}(1 - \inf \{\pi_R\}), \sup \{\pi_R\}(1 - \sup \{\pi_R\})\}\Delta} \tag{22}$$

If $\xi \geq \theta$, then:

$$\phi_r > n \max \left\{ \left(s_{1,r} \left(\frac{\xi}{\theta} - 1 \right) + \frac{\xi}{\theta} s_1 \tau \right), \left(s_{0,r} \left(\frac{\xi}{\theta} - 1 \right) + \frac{\xi}{\theta} s_0 \tau \right) \right\}$$

$$\inf_{\delta_r^R, \delta_r^L} \pi_R = \frac{s_{1,r} \left(1 - \frac{\theta}{\xi} \right) + s_1 \tau}{s_{1,r} \left(1 - \frac{\theta}{\xi} \right) + s_{0,r} + s_{0,p}\theta + \tau} \tag{23}$$

$$\sup_{\delta_r^R, \delta_r^L} \pi_R = \frac{s_{1,r} + s_{1,p}\theta + s_1 \tau}{s_{1,r} + s_{0,r} \left(1 - \frac{\theta}{\xi} \right) + s_{1,p}\theta + \tau} \tag{24}$$

Therefore, if $\xi \geq \theta$, each individual in the equilibrium donates a positive value if:

$$\gamma_r > n \frac{\max \left\{ \left(s_{1,r} \left(\frac{\xi}{\theta} - 1 \right) + \frac{\xi}{\theta} s_1 \tau \right), \left(s_{0,r} \left(\frac{\xi}{\theta} - 1 \right) + \frac{\xi}{\theta} s_0 \tau \right) \right\}}{\min \{\inf \{\pi_R\}(1 - \inf \{\pi_R\}), \sup \{\pi_R\}(1 - \sup \{\pi_R\})\}\Delta} \tag{25}$$

From equations 15 and 16, it is also evident that the conditions hold for any $\gamma_r > 0$ if $\tau = 0$ and $\theta = \xi$. As a result, the model has a positive equilibrium if $\gamma_r > \bar{\gamma}$ where $\bar{\gamma}$ is the maximum of the right-hand sides of the inequalities in 22 and 25.

The system of nonlinear equations 15, 16, and 17 are defined in $[0,1]^3 \rightarrow [0,1]^3$. So, the metric space of $([0,1]^3, ||.||)$, in which $||.||$ is the matrix norm, is a complete metric space. The nonlinear system is a contraction mapping if the norm of the Jacobian matrix of the system has a norm of less than 1:

$$g(\delta_r^R, \delta_r^L, \pi_R) = \begin{bmatrix} 1 - \frac{n(s_{1,r}+s_{1,p}\theta+s_1\tau)}{\phi_r+n(s_{1,r}+s_{1,p}\xi)} \\ 1 - \frac{n(s_{0,r}+s_{0,p}\theta+s_0\tau)}{\phi_r+n(s_{0,r}+s_{0,p}\xi)} \\ \frac{(s_{1,r}+s_{1,p}\xi)\delta_r^R+s_{1,p}(\theta-\xi)+s_1\tau}{(s_{1,r}+s_{1,p}\xi)\delta_r^R+(s_{0,r}+s_{0,p}\xi)\delta_r^L+s_p(\theta-\xi)+\tau} \end{bmatrix} \tag{26}$$

$$D^g = \begin{bmatrix} 0 & 0 & \frac{n(1-2\pi_R)(1-\delta_r^R)\gamma_r\Delta}{\phi_r+n(s_{1,r}+s_{1,p}\xi)} \\ 0 & 0 & \frac{n(1-2\pi_R)(1-\delta_r^L)\gamma_r\Delta}{\phi_r+n(s_{0,r}+s_{0,p}\xi)} \\ \frac{(s_{1,r}+s_{1,p}\xi)(1-\pi_R)}{(s_{1,r}+s_{1,p}\xi)\delta_r^R+(s_{0,r}+s_{0,p}\xi)\delta_r^L+s_p(\theta-\xi)+\tau} & \frac{-(s_{0,r}+s_{0,p}\xi)\pi_R}{(s_{1,r}+s_{1,p}\xi)\delta_r^R+(s_{0,r}+s_{0,p}\xi)\delta_r^L+s_p(\theta-\xi)+\tau} & 0 \end{bmatrix} \tag{27}$$

For each element of the derivative matrix D^g, it is evident that:

$$|D^g_{i,j}| < 1$$

Considering the ∞-norm, the absolute value of the sum of all rows in D^g is less than 1. So:

$$||D^g|| < 1$$

The Banach fixed point theorem secures the existence and uniqueness of equilibrium.■

Lemma 4:

Proof. From equation 17:

$$\frac{(s_{1,r} + s_{1,p}\xi)\delta_r^R + s_{1,p}(\theta - \xi)}{(s_{1,r} + s_{1,p}\xi)\delta_r^R + (s_{0,r} + s_{0,p}\xi)\delta_r^L + s_p(\theta - \xi)} > \frac{1}{2} \implies$$
$$(s_{1,r} + s_{1,p}\xi)\delta_r^R - (s_{0,r} + s_{0,p}\xi)\delta_r^L + (s_{1,p} - s_{0,p})(\theta - \xi) > 0$$

Replacing δ_r^R and δ_r^L:

$$(s_{1,r}+s_{1,p}\xi)\left(1 - \frac{n(s_{1,r} + s_{1,p}\theta)}{\phi_r + n(s_{1,r} + s_{1,p}\xi)}\right) - (s_{0,r}+s_{0,p}\xi)\left(1 - \frac{n(s_{0,r} + s_{0,p}\theta)}{\phi_r + n(s_{0,r} + s_{0,p}\xi)}\right) +$$
$$(s_{1,p}-s_{0,p})(\theta-\xi) > 0 \implies \phi_r\left(\frac{n(s_{1,r} + s_{1,p}\theta)}{\phi_r + n(s_{1,r} + s_{1,p}\xi)} - \frac{n(s_{0,r} + s_{0,p}\theta)}{\phi_r + n(s_{0,r} + s_{0,p}\xi)}\right) > 0$$
$$\implies [(s_{1,r} - s_{0,r}) + (s_{1,p} - s_{0,p})\theta]\phi_r + n(s_{1,r}s_{0,p} - s_{0,r}s_{1,p})(\xi - \theta) > 0$$

If $s_{1,r} + s_{1,p}\theta > s_{0,r} + s_{0,p}\theta$, then if $\theta = \xi$ all positive values of $\gamma_r > 0$ hold the inequality. If $\theta \neq \xi$:

$$\gamma_r > \max\left\{0, \frac{n(s_{1,r}s_{0,p} - s_{0,r}s_{1,p})(\theta - \xi)}{\min\{\inf\{\pi_R\}(1 - \inf\{\pi_R\}), \sup\{\pi_R\}(1 - \sup\{\pi_R\})\}\Delta}\right\}$$

where the fact that denominator is determined through equations 20 and 21 or equations 23 or 24 depends on whether $\xi < \theta$ or $\xi > \theta$, respectively.■

Proposition 2

Proof. The system of nonlinear equations in proposition 1 is reshaped as follows:

$$\begin{aligned} f_1 &:= 1 - \frac{n(s_{1,r} + s_{1,p}\theta + s_1\tau)}{\phi_r + n(s_{1,r} + s_{1,p}\xi)} - \delta_r^R = 0 \\ f_2 &:= 1 - \frac{n(s_{0,r} + s_{0,p}\theta + s_0\tau)}{\phi_r + n(s_{0,r} + s_{0,p}\xi)} - \delta_r^L = 0 \\ f_3 &:= \frac{(s_{1,r} + s_{1,p}\xi)\delta_r^R + s_{1,p}(\theta - \xi) + s_1\tau}{(s_{1,r} + s_{1,p}\xi)\delta_r^R + (s_{0,r} + s_{0,p}\xi)\delta_r^L + s_p(\theta - \xi) + \tau} - \pi_R = 0 \end{aligned} \tag{28}$$

The implicit function theorem suggests that:

$$\begin{bmatrix} \frac{\partial f_1}{\partial \delta_r^R} & \frac{\partial f_1}{\partial \delta_r^L} & \frac{\partial f_1}{\partial \pi R} \\ \frac{\partial f_2}{\partial \delta_r^R} & \frac{\partial f_2}{\partial \delta_r^L} & \frac{\partial f_2}{\partial \pi R} \\ \frac{\partial f_3}{\partial \delta_r^R} & \frac{\partial f_3}{\partial \delta_r^L} & \frac{\partial f_3}{\partial \pi R} \end{bmatrix} \times \begin{bmatrix} \frac{\partial \delta_r^R}{\partial \tau} \\ \frac{\partial \delta_r^L}{\partial \tau} \\ \frac{\partial \pi_R}{\partial \tau} \end{bmatrix} = - \begin{bmatrix} \frac{\partial f_1}{\partial \tau} \\ \frac{\partial f_2}{\partial \tau} \\ \frac{\partial f_3}{\partial \tau} \end{bmatrix} \tag{29}$$

Based on Cramer's rule:

$$\frac{\partial \delta_r^R}{\partial \tau} = \frac{\begin{vmatrix} -\frac{\partial f_1}{\partial \tau} & \frac{\partial f_1}{\partial \delta_r^L} & \frac{\partial f_1}{\partial \pi R} \\ -\frac{\partial f_2}{\partial \tau} & \frac{\partial f_2}{\partial \delta_r^L} & \frac{\partial f_2}{\partial \pi R} \\ -\frac{\partial f_3}{\partial \tau} & \frac{\partial f_3}{\partial \delta_r^L} & \frac{\partial f_3}{\partial \pi R} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f_1}{\partial \delta_r^R} & \frac{\partial f_1}{\partial \delta_r^L} & \frac{\partial f_1}{\partial \pi R} \\ \frac{\partial f_2}{\partial \delta_r^R} & \frac{\partial f_2}{\partial \delta_r^L} & \frac{\partial f_2}{\partial \pi R} \\ \frac{\partial f_3}{\partial \delta_r^R} & \frac{\partial f_3}{\partial \delta_r^L} & \frac{\partial f_3}{\partial \pi R} \end{vmatrix}} \tag{30}$$

The determinant of the Jacobian matrix in the denominator is:

$$\begin{vmatrix} -1 & 0 & \frac{(1-2\pi_R)\gamma_r\Delta(1-\delta_r^R)}{\phi_r+n(s_{1,r}+s_{1,p}\xi)} \\ 0 & -1 & \frac{(1-2\pi_R)\gamma_r\Delta(1-\delta_r^L)}{\phi_r+n(s_{0,r}+s_{0,p}\xi)} \\ \frac{(s_{1,r}+s_{1,p}\xi)(1-\pi_R)}{(s_{1,r}+s_{1,p}\xi)\delta_r^R+(s_{0,r}+s_{0,p}\xi)\delta_r^L+s_p(\theta-\xi)} & \frac{-(s_{0,r}+s_{0,p}\xi)\pi_R}{(s_{1,r}+s_{1,p}\xi)\delta_r^R+(s_{0,r}+s_{0,p}\xi)\delta_r^L+s_p(\theta-\xi)} & -1 \end{vmatrix} =$$

$$\left(\frac{(s_{1,r} + s_{1,p}\xi)(1 - \pi_R)}{(s_{1,r} + s_{1,p}\xi)\delta_r^R + (s_{0,r} + s_{0,p}\xi)\delta_r^L + s_p(\theta - \xi)} \times \frac{(1 - 2\pi_R)\gamma_r\Delta(1 - \delta_r^R)}{\phi_r + n(s_{1,r} + s_{1,p}\xi)} \right) -$$
$$\left(\frac{(s_{0,r} + s_{0,p}\xi)\pi_R}{(s_{1,r} + s_{1,p}\xi)\delta_r^R + (s_{0,r} + s_{0,p}\xi)\delta_r^L + s_p(\theta - \xi)} \times \frac{(1 - 2\pi_R)\gamma_r\Delta(1 - \delta_r^L)}{\phi_r + n(s_{0,r} + s_{0,p}\xi)} \right) - 1 < 0$$

The nominator is as follows:

$$
\begin{vmatrix}
\frac{ns_1}{\phi_r+n(s_{1,r}+s_{1,p}\xi)} & 0 & \frac{(1-2\pi_R)\gamma_r\Delta(1-\delta_r^R)}{\phi_r+n(s_{1,r}+s_{1,p}\xi)} \\
\frac{ns_0}{\phi_r+n(s_{0,r}+s_{0,p}\xi)} & -1 & \frac{(1-2\pi_R)\gamma_r\Delta(1-\delta_r^L)}{\phi_r+n(s_{0,r}+s_{0,p}\xi)} \\
\frac{\pi_R-s_1}{(s_{1,r}+s_{1,p}\xi)\delta_r^R+(s_{0,r}+s_{0,p}\xi)\delta_r^L+s_p(\theta-\xi)} & \frac{-(s_{0,r}+s_{0,p}\xi)\pi_R}{(s_{1,r}+s_{1,p}\xi)\delta_r^R+(s_{0,r}+s_{0,p}\xi)\delta_r^L+s_p(\theta-\xi)} & -1
\end{vmatrix} =
$$

$$
\begin{aligned}
&\underbrace{\left(\frac{\pi_R-s_1}{(s_{1,r}+s_{1,p}\xi)\delta_r^R+(s_{0,r}+s_{0,p}\xi)\delta_r^L+s_p(\theta-\xi)}\times\frac{(1-2\pi_R)\gamma_r\Delta(1-\delta_r^R)}{\phi_r+n(s_{1,r}+s_{1,p}\xi)}\right)}_{\text{negative}}+ \\
&\underbrace{\frac{(s_{0,r}+s_{0,p}\xi)\pi_R}{(s_{1,r}+s_{1,p}\xi)\delta_r^R+(s_{0,r}+s_{0,p}\xi)\delta_r^L+s_p(\theta-\xi)}\left(\frac{ns_1}{\phi_r+n(s_{1,r}+s_{1,p}\xi)}\times\frac{(1-2\pi_R)\gamma_r\Delta(1-\delta_r^L)}{\phi_r+n(s_{0,r}+s_{0,p}\xi)}\right.}_{\text{negative}}- \\
&\underbrace{\left.\frac{ns_0}{\phi_r+n(s_{0,r}+s_{0,p}\xi)}\times\frac{(1-2\pi_R)\gamma_r\Delta(1-\delta_r^R)}{\phi_r+n(s_{1,r}+s_{1,p}\xi)}\right)}+\underbrace{\frac{ns_1}{\phi_r+n(s_{1,r}+s_{1,p}\xi)}}_{\textit{positive}}
\end{aligned} \tag{31}
$$

The first term is negative based on the assumptions. It is also straightforward to show that if $\pi_R > \frac{1}{2}$ and $\tau = 0$, then $dr^L > d_r^R$. Therefore, the second term is also negative. However, the third term is positive. If n increases, the first and second terms go to zero, but the last term goes to a positive value. As a result, $\bar{n}$ exists such that for all $n > \bar{n}$ the determinant is positive, which means that:

$$
\exists \bar{n} \in \mathbb{N}, \forall n > \bar{n}: \quad \frac{\partial \delta_r^R}{\partial \tau} < 0
$$

Using the same path, it is straightforward to show the same result for the δ_r^L.■

Proposition 3

Proof. The equation 31 can be written for δ_r^L as well:

$$
\begin{aligned}
&\left(\frac{\pi_R-s_1}{(s_{1,r}+s_{1,p}\xi)\delta_r^R+(s_{0,r}+s_{0,p}\xi)\delta_r^L+s_p(\theta-\xi)}\times\frac{(1-2\pi_R)\gamma_r\Delta(1-\delta_r^L)}{\phi_r+n(s_{0,r}+s_{0,p}\xi)}\right)+ \\
&\frac{(s_{1,r}+s_{1,p}\xi)(1-\pi_R)}{(s_{1,r}+s_{1,p}\xi)\delta_r^R+(s_{0,r}+s_{0,p}\xi)\delta_r^L+s_p(\theta-\xi)}\left(\frac{ns_1}{\phi_r+n(s_{1,r}+s_{1,p}\xi)}\times\frac{(1-2\pi_R)\gamma_r\Delta(1-\delta_r^L)}{\phi_r+n(s_{0,r}+s_{0,p}\xi)}-\right. \\
&\left.\frac{ns_0}{\phi_r+n(s_{0,r}+s_{0,p}\xi)}\times\frac{(1-2\pi_R)\gamma_r\Delta(1-\delta_r^R)}{\phi_r+n(s_{1,r}+s_{1,p}\xi)}\right)+\frac{ns_0}{\phi_r+n(s_{0,r}+s_{0,p}\xi)}
\end{aligned} \tag{32}
$$

Therefore, based on the proof of the proposition 2:

$$
\frac{\partial \delta_r^R}{\partial \tau} = -\frac{ns_1}{\phi_r+n(s_{1,r}+s_{1,p}\xi)} + O(n^{-2})
$$

$$
\frac{\partial \delta_r^L}{\partial \tau} = -\frac{ns_0}{\phi_r+n(s_{0,r}+s_{0,p}\xi)} + O(n^{-2})
$$

Therefore:

$$\frac{\partial \delta_r^R}{\partial \tau} - \frac{\partial \delta_r^L}{\partial \tau} = n \frac{(s_0 - s_1)\phi_r + n(s_{1,r}s_{0,p} - s_{0,r}s_{1,p})(1-\xi)}{(\phi_r + n(s_{1,r} + s_{1,p}\xi))(\phi_r + n(s_{0,r} + s_{0,p}\xi))} + O(n^{-2})$$

Using the multiplier of π_r and the degree of the remaining part of the equation, it is straightforward to show that if $s_0 > s_1$, $\tilde{\gamma}_r$ and $\tilde{n}$ exist, which for all $\gamma_r > \tilde{\gamma}_r$ and $n > \tilde{n}$:

$$\frac{\partial \delta_r^R}{\partial \tau} - \frac{\partial \delta_r^L}{\partial \tau} > 0 \blacksquare$$