Heterogenous Firms and the Dynamics of Investment

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Abstract

We show a strong positive relationship between the magnitude of the initial response to a shock and the speed of convergence of capital in standard heterogeneous firm models of investment. This leads to a tension in matching the empirical evidence of a small response with fast convergence. To resolve this tension tension we build a model of firm investment with frictional hiring. These frictions are supported empirically by the predictability of investment by lagged hiring even when controlling for lagged investment. Quantitatively we find the labour frictions enable the model to match both the empirical response and convergence as well as standard micro investment moments.

1 INTRODUCTION

Investment models struggle to replicate both the lumpiness of investment at the microeconomic level and the sluggishness at the macroeconomic level. The classical adjustment cost formulation (Hayashi, 1982) and its variations, typically favored by macroeconomists to match aggregate data, is starkly at odds with the infrequent and large investments observed at the microeconomic level. Models of lumpy investment were designed to address these limitations and can explain key moments of the investment distribution (Caballero and Engel, 1999; Khan and Thomas, 2008). The extreme responsiveness of investment in these models, however, implies that the heterogeneity and state-dependence that distinguishes them is irrelevant for general equilibrium dynamics (House, 2014). Since an economy with lumpy investment behaves like a neoclassical economy with frictionless capital adjustment, the class of investment models consistent with the micro data appears inconsistent with the macro data.

In an attempt to match micro lumpiness without sacrificing macro sluggishness, state-of-the art investment models rely on a combination of fixed and convex capital adjustment costs. Convex capital adjustment costs dampen the responsiveness mentioned earlier, helping bring the models in line with existing evidence and breaking the general equilibrium irrelevance result (Winberry, 2021; Koby and Wolf, 2020).

This paper brings renewed attention to the dynamics of aggregate capital in investment models which is both important for the dynamics of macroeconomic models but also useful for discriminating between different models of investment. It starts by showing a close relationship between the size of the initial response and the speed of convergence dynamics back to the steady state. As the size of the initial response decreases, it takes longer for capital to converge back to the steady state level. Thus, in models that produce large responses to shocks[1], such as models with only fixed costs that match micro moments, capital converges back to the steady state level quickly. On the other hand, models with smaller responses to shocks, such as models with convex costs, capital takes a long time to converge back to the steady state level. Therefore, there is a *tension* between matching the empirically well-documented small initial response and the relatively fast speed of convergence (Curtis et al., 2021).

We then develop a model of lumpy capital adjustment with frictional hiring that helps resolve this tension. In response to a shock firm investment is dampened by the inability to flexibly adjust labour and the complementarity between capital and labour. However over time this dampening effect is mitigated as firms adjust their labour force leading to a faster convergence of capital back to the steady state level. Interestingly, our model can also generate positive autocorrelation in investment without relying on costs for adjusting the level of investment, as in so-called $\dot{I}$ models. In addition, the model generates a long tail of investment rates observed in the data, which models with convex costs fail the generate.

We use a perturbation approach to derive formulas for aggregate responses in a generalized version of

[1] Koby and Wolf (2020) show that there are strong linkages between the responses to different shocks, justifying the use of the productivity response as a general measure of the model's responsiveness

our model with frictional hiring. This allows us to shed light on how different forms of adjustment costs shape aggregate responses. Our main result here relates the time-0 extensive margin response of aggregate capital to the (weighted) second moment of investment in the cross section and an irreversibility parameter. We then build on Auclert et al. (2019) to compute impulse responses with respect to all prices in the model and study how they are shaped by different adjustment frictions.

We also examine how models with fixed costs, convex costs, and frictional hiring fit the moments of the investment distribution and the responsiveness of aggregate capital. As expected, models with only fixed capital adjustment costs can match the distributional moments but are overly responsive. Consistent with the theoretical tension, models that incorporate convex capital adjustment costs can match the initial response, but capital recovers too slowly to its steady state. For example, when we compute the dynamics triggered by permanent changes in corporate tax policy using a calibration similar to Winberry (2021), the speed of convergence contrasts sharply with the estimates in Curtis et al. (2021). Introducing frictional hiring resolves this tension, allowing the model to match both the initial response and the speed of convergence, as well as the full set of distributional moments.

Finally, we provide evidence that supports frictional hiring. Using several datasets, we find that *lagged* employment growth is a statistically significant predictor of investment. This is true even if we condition on lagged investment, the best predictor of current investment according to Eberly et al. (2012), and standard proxies for financial constraints. Moreover, hiring/firing episodes increase the likelihood of upward/downward capital adjustments in the future.

2 IMPULSE RESPONSE AND DYNAMICS

Both the impulse response and the dynamics of investment are important for macroeconomic responses. In this section we show that in standard models of capital investment the impulse response and the speed of convergence are tightly linked. This tight link creates a tension in matching empirical evidence on the response of investment to shocks. If a model matches the magnitude of the impulse response then it display excessively slow convergence back to steady state. On the other hand if a model matches the speed of convergence observed in the data the magnitude of the impulse response will be counterfactually large.

We consider models of capital investment subject to adjustment costs that have three state variables, capital $k_{i,t}$, idiosyncratic productivity e_{it} and aggregate productivity Z_t. The value function of a firm is then given by

$$V(Z, e_{i,t}, k_{i,t}) = \max_{k'} F(Z_t, e_{i,t}, k_{i,t}) + AC(Z_t, k_{i,t}, k') \tag{1}$$

$$+ \beta \mathbb{E}_{e'|e_{i,t}} \left[V(Z_t, e', k') \right] \tag{2}$$

Where $F(Z_t, e_{i,t}, k_{i,t})$ is the production function of the firm, $AC(Z_t, k_{i,t}, k')$ is the adjustment cost function and β is the discount factor.

We then make the following two assumption on the functions F and AC.

1. $F(Z_t, e, k)$ is homogeneous of degree 1 in k and Z
2. $AC(Z_t, k, k')$ is homogeneous of degree 1 in k and Z

The first of these two assumptions is generally satisfied through a redefinition of Z. For example while $F(Z_t, e, k) = Zek^{\alpha}$ does not satisfy the first condition. By defining $\tilde{Z}^{1-\alpha} = Z$ means $F(\tilde{Z}, e, k)$ satisfies the first condition for k and $\tilde{Z}$. Given a definition of productivity that satisfies the first condition, it is desirable to have the second condition satisfied as well. This ensures that when augmenting the model with growth in productivity that the adjustment frictions scale with the growth and thus remain relevant. For adjustment costs such as the cost of buying capital or convex adjustment costs (as in Winberry (2021)) this condition is satisfied by these costs being of degree 1 in k. For adjustment costs which aren't homogeneous of degree 1 in k such as fixed costs which are independent of k, scaling the fixed cost with Z can satisfies condition 2 and ensures that the fixed cost remains relevant in the case of growth.

We further assume that the economy is in the non-stochastic steady state, i.e. that Z_t is non-stochastic and constant. We will analyse responses to MIT shocks to Z and to the distribution of capital. Under these two conditions the value function can be rewritten as $V(Z, e, k) = ZV(e, \frac{k}{Z})$,for details see.

The key implication of this result is that the policy functions of the firm in $(e, \frac{k}{Z})$ space are invariant to permanent shocks to Z. Instead a permanent positive shock to Z will lead to a geometrically uniform negative shift in the distribution of firms in $\frac{k}{Z}$ space. Thus the response of aggregate investment and capital (K) to a permanent shock to Z will be observationally equivalent in $\frac{K}{Z}$ space to any other shock that induces the same geometrically uniform shift in the distribution of firms in $\frac{k}{Z}$ space. In particular this will be true for aggregate capital destruction shocks that destroy a fixed proportion of capital at every firm.

This equivalence itself is not novel and in particular has been used by Baley and Blanco (2021) to characterize the cumulative impulse response of capital to a permanent productivity shock. The novelty of our argument is in the interpretation of this equivalence. The response to the capital destruction shock which is direct shift of the distribution of capital away from the stationary distribution captures the convergence behaviour of capital back to the steady state level. While the response to the productivity shock captures the impulse response of capital to a productivity shock. This impulse response to a shock Z or related responses such as to the interest rate in capital investment models has been the focus of recent work such as Winberry (2021) and Koby and Wolf (2020). However the dynamics/convergence behaviour of these models has received less attention. Thus the equivalence implies that the impulse response of capital to a productivity shock is tightly linked to the speed of convergence of capital back to the steady state level. Models satisfying our assumptions can either have a big response to productivity shocks and a brisk convergence back of capital or a small response and slow convergence.

If the empirical evidence aligns with either big and brisk or small and slow then models of the class we consider can match the data. However if there is a disconnect between the impulse response and the speed

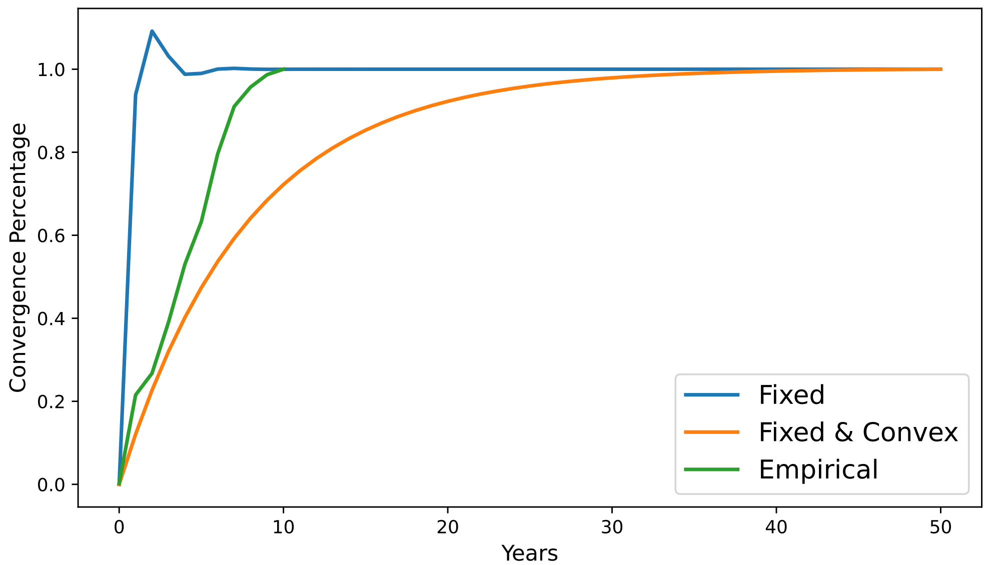


Figure 1: Impulse Response of Capital to Cost of Capital Changes

of convergence then models of the class we consider will struggle to match the data. Curtis et al. (2021) estimate the response of investment to changes in depreciation timing in the US. We plot their results as well as the predictions of a model with fixed costs based on Khan and Thomas (2008) and a model with fixed and convex adjustment costs based on Koby and Wolf (2020) in figure 1. The model with fixed costs has a counterfactually large on impact response as has been noted by previous work. Consistent with the theoretical result it also features a fast convergence to the new steady state level of capital. The model with fixed and convex adjustment costs matches the on impact response as it was calibrated to do. However the convergence back to the steady state level of capital is very slow being 75% of the way back to the steady state level after 10 years and still only 93% after 20 years. This is inconsistent with the empirical evidence in which the convergence is much faster and is complete after 10 years.

A particularly important feature of the empirical dynamics is the linear growth of capital followed by a sharp slowdown in the growth rate. This is in contrast to the exponential decay in the speed of convergence in the model with fixed and convex adjustment costs which leads to the drawn out dynamics. To understand how a model with labor frictions can resolve this tension and match this feature of the empirical dynamics it is useful to consider the micro level policy functions that determine the aggregate response to either of the two shocks we have considered.

2.1 The Role of the Expected Change in Capital

To understand the micro level behaviour driving aggregate behaviour write the law of motion $\Delta K_t = K_{t+1} - K_t$ for the capital in the model as a function of micro policy functions and the distribution of firms.

$$\Delta K_t = \int \Delta k(k_t) f(k_t) dk_t \tag{3}$$

where $\Delta k(k_t)$ is the expected net change in capital for a firm with capital k_t and $f(k_t)$ is the steady state distribution. In steady state $\Delta K_t = 0$.

Now suppose the economy is hit by a capital destruction shock that hits all firms, though as discussed above this could also be interpreted as an aggregate productivity shock. The net investment response will thus be given by the derivative of ΔK_t with respect to γ evaluated at $\gamma = 0$.

$$\frac{\partial \Delta K_t}{\partial \gamma} = -\int \frac{\partial \Delta k((1-\gamma)k_t)}{d\gamma} f(k_t) dk_t \tag{4}$$

$$= -\int \frac{\partial \Delta k(k_t)}{d\log(k_t)} f(k_t) dk_t \tag{5}$$

$$\tag{6}$$

The responsiveness of net investment will thus be governed by the slope of the expected change in capital as a function of the capital stock $\frac{\partial \Delta k(k_t)}{d\log(k_t)}$. When this slope is steep the shifting of the distribution of firms will induce a large change individual firms' expected change in capital and thus the aggregate response will be large. Furthermore due to the steep slope the rate of convergence of capital will also be fast as on average firms will quickly replace the destroyed capital.

Key to this is that as both shocks can be thought of as shifts to the distribution, the slope of the expected change in capital as a function of the capital stock is invariant to the shock. Thus both responses are governed by the same object and are thus constrained to be linked.

2.2 Labour Adjustment Costs can Resolve the Tension

Now we discuss how a model with labor frictions can resolve this equivalence. When there are convex hiring costs, labor becomes an endogenous state variable. In this model a permanent productivity shock can be thought of as a shock to the distribution of both capital and labor. The net change in capital can then be written.

$$\Delta K_t = \int\int \Delta k(k_t, l_t) f(k_t, l_t) dk_t dl_t \tag{7}$$

And the net investment response to the shock will be given by the derivative of ΔK_t with respect to γ.

$$\frac{\partial \Delta K_t}{\partial \gamma} = -\int\int \Delta \frac{\partial k(k_t(1-\gamma), l_t(1-\gamma))}{d\gamma} f(k_t, l_t) dk_t dl_t \tag{8}$$

So now the the expected change in capital as a function of the capital stock is not invariant to the shock. If the shock is negative then due to the shifting down of the labor stock for all firms the expected change in capital will shift to the left. So the on impact response will be determined by the slope as well as by how much the expected change in capital curve shifts in response to the shock.

To study the local speed of convergence an important factor is if labor or capital converge faster. If labor converges faster then the local speed of convergence of capital will be determined by the slope of the expected capital change function when labor is at the steady state level. If capital converges faster then the local speed of convergence of capital will be dominated by the slope of the expected capital function when labor is shifted away from steady state. Assuming that labor converges faster then the speed local speed of convergence will be determined by the slope of the expected capital change function around the steady state distribution. This is the same object as in the model with only capital as an endogenous state variable.

This then implies that the model with labor frictions can resolve the tension between the impulse response and the speed of convergence by varing the degree to which the expected change in capital as a function of the capital stock shifts as labor shifts. Thus adding frictions in hiring allows models of capital investment to match both the short and medium run responses observed in the data.

3 MEASUREMENT

In the previous section we provided a theoretical justification for introducting labour adjustment costs in order to separate shock responsiveness from convergence speed. In this section study empirically if adding frictions in adjusting labour is a modelling choice supported by direct empirical evidence.

3.1 FIRM-LEVEL REGRESSIONS

The empirical literature studying firm investment has robustly documented that lagged investment is a significant predictor of current investment. This holds even when controlling for anticipated investment returns such as Tobin's Q and proxies for financial constraints such as the cash to assets ratio. For example, Eberly et al. (2012) argue that in terms of explanatory power, lagged investment is the best predictor of current investment. These results are suggestive of frictions in the adjustment of capital. In a frictionless enviroment firms would instantaneously adjust their capital stock to the optimal level requiring no additional response in the future.

In this subsection we study the predictive power of lagged employment growth on investment. To test for the importance of hiring frictions for investment decisions in this section we study how predictive hiring is of investment. To do this we start with the standard specification from the lagged investment literature

$$i_{it} = \alpha + \beta i_{it-1} + \gamma X_{it} + \epsilon_{it} \tag{9}$$

Where i_{it} is the investment of firm i at time t as a proportion of the capital stock, X_{it} is a vector of control variables including Tobin's Q and the cash to assets ratio. To construct investment we use the perpetual inventory method to calculate the capital stock and define investment as the change in the capital stock.

We also deflate capital to convert investment to real terms. We then augment this specification with lagged employment growth e_{it-1}.

$$i_{it} = \alpha + \beta i_{it-1} + \gamma X_{it} + \delta e_{it-1} + \epsilon_{it} \tag{10}$$

We estimate this regression on two data sets, first on data from Compustat which contains data from required filings by publically traded firms in the United States. As a consequence of using data on employment we are restricted to annual data rather than quarterly data. This makes are results not directly comparable to the literature which typically uses quarterly data.

The second data set is from Chilean plants. This is a panel data set of manufacturing plants in Chile containing information on investment, employment and other variables at the annual frequency. The two main differences between this dataset and Compustat are that it is at the plant level rather than the firm level and that it is more representative of the universe of firms than the Compustat data which is biased towards larger firms.

The results for the Compustat regressions are contained in Table 1. Consistent with the previous literature we find lagged investment is a statistically significant predictor of future investment. However, we also find that lagged employment growth is a significant predictor of future investment. In column 3 our main specification the coefficients on lagged investment and employment growth are similar in magnitude, suggesting that the stock of workers is an important factor in investment decisions. Furthermore, the magnitude of the coefficient on employment growth is more robust to the inclusion of firm fixed effects in columns 4 and 6. In these columns, the coefficient of lagged investment is over 50% smaller than in the baseline specification, while the coefficient on employment growth is less than 30% smaller.

These results are consistent that the evolution of employment and therefore the frictions faced in hiring are important for investment decisions. The existence of such a strong relationship is particularly suprising given that the data is at the annual frequency. Firm's decisions to invest are dependent on hiring that could have taken place up to 2 years prior.

3.2 Inaction and hiring

The next approach we take to study the relationship between employment and investment is to look at the probability of inaction conditional on lagged hiring. The existence of inaction was a major motivation for (s,S) models of investment. We look at how the probability of inaction changes with lagged employment growth.

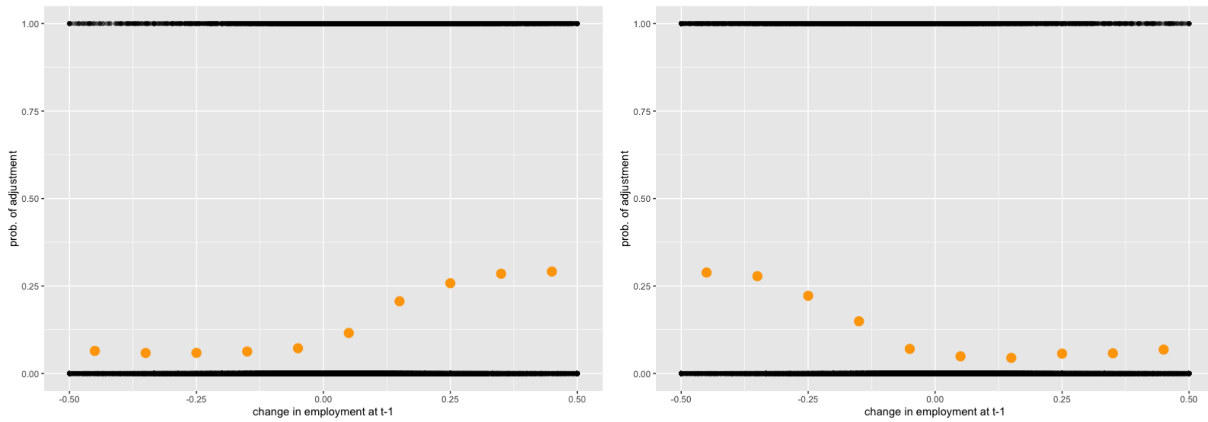
We define inaction as when the investment rate, i.e. investment as a percentage of the capital stock, is below 20% in absolute terms. This definition is in line with previous literature which have studied inaction in the context of investment.

In line with our findings in Section 3.1, Figure 2 shows that lagged changes in employment predict

Dependent Variable:	Lagged Investment					
Model:	(1)	(2)	(3)	(4)	(5)	(6)
Variables						
Lagged Investment	0.1759***	0.1695***	0.1710***	0.0640***	0.1608***	0.0432***
	(0.0049)	(0.0049)	(0.0049)	(0.0050)	(0.0051)	(0.0053)
Lagged Hiring	0.1566***	0.1446***	0.1448***	0.1189***	0.1322***	0.1008***
	(0.0043)	(0.0044)	(0.0044)	(0.0046)	(0.0046)	(0.0048)
Lagged Q		0.0135***	0.0126***	0.0202***	0.0125***	0.0180***
		(0.0013)	(0.0013)	(0.0022)	(0.0014)	(0.0021)
Lagged Cash to assets			0.0491***	0.2214***	0.1083***	0.2807***
			(0.0080)	(0.0120)	(0.0089)	(0.0135)
Fixed-effects						
Firm				Yes		Yes
Year - Industry					Yes	Yes
Fit statistics						
Observations	132,729	132,729	132,729	132,729	132,729	132,729
R^2	0.08580	0.09819	0.09883	0.24135	0.24523	0.37415
Within R^2				0.05551	0.08771	0.04612

*Signif. Codes: ***: 0.01, **: 0.05, *: 0.1*

Table 1: Predicting investment



(a) Prob. of adjusting up and lagged emp change

(b) Prob. of adjusting down and lagged emp change

Figure 2: Adjustment probabilities and lagged changes in employment with Compustat data

the likelihood of adjustment at the firm level. When the lagged hiring rate is below zero the probability of adjusting up is close to zero. On the other hand as the lagged hiring rate increases above zero the probability of adjusting up increases.

This relationship also holds in reverse for downward adjustments in the Compustat data. When the lagged hiring rate is above zero the probability of adjusting down is close to zero.

So firms empirically make larg adjustments in their capital stack after they have adjusted their stock of employment. This is consistent with the idea that capital and labour are complements in production and that there are frictions in hiring. Firms wait until they have built up a large enough stock of workers before they invest in capital in order to make effective use of the capital.

4 MODEL

Consider a discrete time economy with a continuum of firms who choose sequences of capital $\{k_t\}$ and labour $\{l_t\}$ to maximise the net present value of profits. They take as given the interest rate r and the wage w and face adjustment costs for both capital and labour, solving[2]

$$\max_{\{k_t,l_t\}_{t=1}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t \left[e_t F(k_t, l_t) - wl_t - AC(k_t, k_{t+1}, l_t, l_{t+1})\right]. \tag{11}$$

Idiosyncratic productivity $\{e_t\}$ follows an AR(1) with normal innovations. Capital depreciates at rate δ_k and there is labour attrition proportionate to δ_l. The production function F is CES with elasticity of substitution $\rho > 0$ and returns to scale parameter $\alpha \in (0, 1)$. We allow for various forms of adjustment costs, so we let

[2]We do not distinguish between employees and hours, i.e. we are assuming that hours are fixed.

the adjustment cost function AC be given by

$$
\begin{aligned}
AC(k_t, k_{t+1}, l_t, l_{t+1}) = &\underbrace{w\xi}_{fixed\ cost}\ \mathbb{1}_{\{k_{t+1} \neq (1-\delta_k)k_t\}} \\
&+ p \underbrace{\left[1 - (1-\gamma)\mathbb{1}_{\{k_{t+1} \leq (1-\delta_k)k_t\}}\right]}_{partial\ irreversibility} (k_{t+1} - (1-\delta_k)k) \\
&+ \frac{\chi}{2} \underbrace{\left(\frac{k_{t+1} - (1-\delta_k)k_t}{k_t}\right)^2 k_t}_{variable\ cost\ capital} + \frac{\phi}{2} \underbrace{\left(\frac{l_{t+1} - (1-\delta_l)l_t}{l_t}\right)^2 l_t}_{variable\ cost\ labour}.
\end{aligned}
$$

The first adjustment cost is a fixed cost of capital adjustment which must be paid whenever tomorrow's capital stock is different from that which would be implied by depreciation. We assume that the fixed adjustment cost ξ is drawn at the start of each period from distribution G with mean μ. This adjustment cost captures the fact that output may be lost whenever new capital is installed.[3] The presence of this term is motivated by the literature on lumpy investment that has found this cost useful to replicate the long tail of firm investment observed in the data. As Caballero (1999) points out, this pattern of adjustment requires more than proportional adjustment costs. There has to be some benefit from incurring large adjustments and the simplest recipe is to introduce fixed costs. The mechanism by which it does so is that firms who receive a sequence of high fixed cost draws may choose to delay their investment till they receive a low draw at which point they will have a large gap between their capital and their target level of capital. The randomness plays a dual role: it helps computations by smoothing the problem and allows the model to generate more realistic investment distributions.

The second term represents partial irreversibility of capital and is motivated by evidence on the firm specificity of capital. To capture that specificity we allow a firm to buy capital at price p but assume it only receives γp whenever it decides to sell. The lost wedge $1-\gamma$ represents the share of the value of the capital that is firm specific. Irreversibility makes disinvesting very unattractive due to the low return on selling capital. Thus, when making a positive investment firms will reduce the size of it so as to lessen the loss if future shocks would leave them above their target capital. So irreversibility leads to a reduction in large investments and infrequent downward adjustments in capital.[4]

The third term captures variable costs that arise when adjusting the capital stock. This quadratic form was recently used by Winberry (2021) in order to generate investment to interest rate semi-elasticities that were closer to empirical evidence. The convex variable costs reduce the investment to interest rate semi-elasticity as they limit the size of investment any individual firm wants to make in any one period. Firms are instead incentivised to spread out their capital response to a shock over more periods in order to reduce the average cost per unit of investment. This solves the problem encountered in pure *Ss* models such as

[3] This may be because labour input has to be diverted to install new machines or because it is difficult to incorporate new machines into the production process.

[4] This is consistent with what we observe in the data- downward adjustments in capital are a lot less likely than upward adjustments. See Eslava et al. (2010)

Khan and Thomas (2008) where firms who receive a small fixed cost draw make very large investments. At the same time, in the calibration used by Winberry (2021) the investment distribution is no longer skewed and there is no downward capital adjustment whenever there is some degree of irreversibility ($\gamma < 1$).

We depart from most of the literature on firm investment by introducing variable costs for changes in employment, the last term in the expression for AC. This quadratic form is somewhat standard in the dynamic labour demand literature and is a stand in for output loss that arises due to inexperience of new workers or severance payments that accrue whenever the firm decides to lay off workers. Although it is somewhat restrictive, it is consistent with labour hoarding, procyclical labour productivity and imperfect substitution between incumbent workers and new hires.[5] We also find that the empirical distribution of hiring/firing rates is roughly symmetric, making the use of this quadratic form somewhat less problematic.

The standard approach in the investment literature is to treat labour as a frictionless factor that can be *maximized out* of the production function. Incorporating this into the model makes employment decisions forward looking and slow-moving. In addition, it draws out the response of the marginal productivity of capital to productivity shocks, making investment less responsive. As we show below, this allows the model to better match the autocorrelation structure of investment. In addition, because labour is now forward looking the model can generate relationships like the ones we document in the empirical section.

To show this more formally, we now turn to our characterization of the firm's problem. We will use our recursive formulation of the problem to derive some formulas to understand the factors that determine the price-elasticity of investment.

4.1 Recursive characterization

Let V denote the firm's value function. Given the state (e, k, l) and the realization of the fixed cost ξ, V tells the value of the maximized objective function (11). From the dynamic programming principle, V solves the following Bellman equation

$$V(e,k,l;\xi) = \max_{l',k'} \left\{ eF(k,l) - wl - AC(k,k',l,l';\xi) + \frac{1}{1+r}\mathbb{E}_{e'|e}\left[V^0(e',k',l')\right] \right\}, \tag{12}$$

where V^0 denotes the ex-ante value function (i.e. prior to the realization of the fixed cost)

$$V^0(e,k,l) \equiv \int V(e,k,l;\xi)G(d\xi). \tag{13}$$

We break up the maximization into two steps. First, given some value for productivity and employment today, we fix k' and find the optimal choice of employment going into next period. This defines an intermediate value function in terms of next period's capital stock and today's productivity and employment

[5] See Hamermesh (1993) and Mercan et al. (2021). This particular formulation makes the marginal cost of adjustment independent of the scale of the firm. Moreover, it implicitly assumes that adjustment costs depend on net rather than gross changes in employment. An alternative would be to let it depend on the change in the level of the input rather than the percentage change. At the end of the day, what cost structure is more realistic is an empirical question that this paper does not seek to address.

$$\tilde{V}(e,k',l) \equiv \max_{l'} \left\{ \frac{1}{1+r} \mathbb{E}_{e'|e} \left[V^0(e',k',l') \right] - \frac{\phi}{2} \left(\frac{l' - (1-\delta_l)l}{l} \right)^2 l \right\}. \tag{14}$$

For future reference, we denote by $\tilde{\mathcal{L}}$ the employment policy that solves the problem associated with the intermediate value function $\tilde{V}$. Once we know the firm's optimal choice of capital going into next period, we can back out the firm's optimal employment policy by evaluating $\tilde{\mathcal{L}}$ at the optimal value of k'. It is convenient to work with $\tilde{V}$ because it makes the optimization problem for k' one-dimensional. To see this, note that given $\tilde{V}$ we can rewrite (12) as

$$\begin{aligned} V(e,k,l;\xi) = \max_{k'} \Bigg\{ & eF(k,l) - wl - w\xi \mathbb{1}_{\{k' \neq (1-\delta_k)k\}} - \frac{\chi}{2} \left(\frac{k' - (1-\delta_k)k}{k} \right)^2 k \\ & - p \left[1 - (1-\gamma) \mathbb{1}_{\{k' \leq (1-\delta_k)k\}} \right] \left[k' - (1-\delta_k)k \right] + \tilde{V}(e,k',l) \Bigg\}. \end{aligned}$$

Going through the maximization over k', we arrive at

$$V(e,k,l;\xi) = eF(k,l) - wl + \max \left\{ V^{a,u}(e,k,l) - w\xi,\ V^{a,d}(e,k,l) - w\xi,\ \tilde{V}(e,(1-\delta_k)k,l) \right\} \tag{15}$$

with

$$V^{a,u}(e,k,l) \equiv \max_{k' \geq (1-\delta_k)k} \left\{ \tilde{V}(e,k',l) - \frac{\chi}{2} \left(\frac{k' - (1-\delta_k)k}{k} \right)^2 k - p \left[k' - (1-\delta_k)k \right] \right\}$$

and

$$V^{a,d}(e,k,l) \equiv \max_{k' \leq (1-\delta_k)k} \left\{ \tilde{V}(e,k',l) - \frac{\chi}{2} \left(\frac{k' - (1-\delta_k)k}{k} \right)^2 k - \gamma p \left[k' - (1-\delta_k)k \right] \right\}.$$

NO IRREVERSIBILITY, $\gamma = 1$

To understand the structure of the solution, let us first consider the case without irreversibility. In the absence of the price wedge, the Bellman equation becomes

$$V(e,k,l;\xi) = eF(k,l) - wl + \max \left\{ V^a(e,k,l) - w\xi,\ \tilde{V}(e,(1-\delta_k)k,l) \right\} \tag{16}$$

with

$$V^a(e,k,l) \equiv \max_{k'} \left\{ \tilde{V}(e,k',l) - \frac{\chi}{2} \left(\frac{k' - (1-\delta_k)k}{k} \right)^2 k - p \left[k' - (1-\delta_k)k \right] \right\}$$

From this, it is easy to see that the firm's decision on the extensive margin can be summarized by a cutoff rule according to which the firm adjusts its capital stock whenever

$$\xi \leq \frac{V^a(e,k,l) - \tilde{V}(e,(1-\delta_k)k,l)}{w} \equiv \hat{\xi}(e,k,l). \tag{17}$$

Combining this adjustment threshold with the cdf G, we obtain the state-dependent generalized hazard function of the model $\Lambda(e,k,l)$ that captures the adjustment probability given the state today:

$$\Lambda(e,k,l) \equiv G\left(\hat{\xi}(e,k,l)\right). \tag{18}$$

On the intensive margin, the firm chooses next period's capital stock in order to maximize the intermediate value function $\tilde{V}$ net of adjustment costs. For future reference, we denote by $\mathcal{K}$ the firm's optimal choice of capital conditional on adjustment

$$\mathcal{K}(e,k,l) \equiv \arg\max_{k'} \left\{ \tilde{V}(e,k',l) - \frac{\chi}{2}\left(\frac{k'-(1-\delta_k)k}{k}\right)^2 k - p\left[k'-(1-\delta_k)k\right] \right\}.$$

Integrating equation (16) with respect to the fixed cost ξ, using the definition of V^0 in (13) and the form of the investment policy described above, we arrive at an expression for the ex-ante value function without irreversibility:

$$\begin{aligned} V^0(e,k,l) = eF(k,l) - wl + \Lambda(e,k,l)\left(V^a(e,k,l) - w\mathbb{E}\left[\xi|\xi \le \hat{\xi}(e,k,l)\right]\right) \\ + (1-\Lambda(e,k,l))\tilde{V}(e,(1-\delta_k)k,l) \end{aligned} \tag{19}$$

This shows that the ex-ante continuation value of the firm is a weighted average of the value from adjusting both capital and labour optimally and the value from adjusting labour optimally and letting the capital stock depreciate. We now use this expression to derive necessary conditions for a firm's optimal choice of capital conditional on adjustment and the optimal employment policy. We will then use these optimality conditions to study the effects of a small exogenous change in the price of investment goods.

Optimal employment decision, $\gamma = 1$

From (19), the envelope condition for labour is

$$\frac{\partial V^0(e,k,l)}{\partial l} = eF_l(k,l) - w + \frac{\phi}{2}\left[\frac{\mathbb{E}_\xi\left[\mathcal{L}(e,k,l;\xi)^2\right] - (1-\delta_l)^2 l^2}{l^2}\right], \tag{20}$$

with the *ex-post employment policy* $\mathcal{L}(\cdot;\xi)$ defined as

$$\mathcal{L}(e,k,l;\xi) \equiv \begin{cases} \tilde{\mathcal{L}}(e,(1-\delta_k)k,l), & \text{if } \xi > \hat{\xi}(e,k,l) \\ \tilde{\mathcal{L}}(e,\mathcal{K}(e,k,l),l), & \text{if } \xi \le \hat{\xi}(e,k,l) \end{cases}.$$

In the absence of adjustment costs for labour, the marginal benefit is simply given by the marginal product. When $\phi > 0$, the marginal value of hiring an additional worker to the firm takes into account the effect on adjustment costs. Given this, the first order condition for employment is

$$\phi\left(\frac{l'-(1-\delta_l)l}{l}\right) = \frac{1}{1+r}\Bigg(\mathbb{E}_{e'|e}\left[e'\right]F_l(k',l') - w \tag{21}$$

$$+ \frac{\phi}{2}\left[\frac{\mathbb{E}_{e'|e,\xi}\left[\mathcal{L}(e',k',l';\xi)^2\right] - (1-\delta_l)^2 l'^2}{l'^2}\right]\Bigg). \tag{22}$$

When the firm chooses not to adjust the capital stock today, $k' = (1-\delta_k)k$ and the condition implicitly defines $\tilde{\mathcal{L}}(e,(1-\delta_k)k,l)$. When the firm chooses to adjust, $k' = \mathcal{K}(e,k,l)$ and (21) implicitly defines $\tilde{\mathcal{L}}(e,\mathcal{K}(e,k,l),l)$.

OPTIMAL INVESTMENT DECISION, $\gamma = 1$

The envelope condition for capital is

$$\begin{aligned}\frac{\partial V^0(e,k,l)}{\partial k} =& eF_k(k,l) + (1-\delta_k)\Bigg\{\Lambda(e,k,l)\ p \\ &+ (1-\Lambda(e,k,l))\frac{1}{1+r}\mathbb{E}\left[\frac{\partial V^0(e',(1-\delta_k)k,\tilde{\mathcal{L}}(e,(1-\delta)k,l))}{\partial k}\right]\Bigg\}\end{aligned}$$

Note the fact that the envelope theorem does not apply when the firm chooses to let the capital stock depreciate and there is a direct effect of changing capital stock in those states of the world. Bringing back the explicit dependence on time, we can iterate this equation to derive the *sequence-space representation* for the marginal value of capital

$$\frac{\partial V^0(e_t,k_t,l_t)}{\partial k} = \mathbb{E}_t \sum_{j=0}^{\infty}\left(\frac{1-\delta_k}{1+r}\right)^j \left(\prod_{s=0}^{j-1}\left[1-G(\hat{\xi}_{t+s})\right]\right)\Big(e_{t+j}F_{k,t+j} + G(\hat{\xi}_{t+j})p(1-\delta_k)\Big) \tag{23}$$

Conditional on adjustment, the firm's choice optimal choice of capital is pinned down by the following optimality condition

$$p = \frac{1}{1+r}\times \mathbb{E}_t \sum_{j=0}^{\infty}\left(\frac{1-\delta_k}{1+r}\right)^j \left(\prod_{s=0}^{j-1}\left[1-G(\hat{\xi}_{t+1+s})\right]\right)\Big(e_{t+1+j}F_{k,t+1+j} + G(\hat{\xi}_{t+1+j})p(1-\delta_k)\Big)$$

We can also write the envelope condition for capital as

$$\frac{\partial V^0(e,k,l)}{\partial k} = eF_k(k,l) + (1-\delta_k)\mathbb{E}_{\xi}\left[Q(e,k,l;\xi)\right]$$

where $Q(\cdot;\xi)$ denotes *ex-post marginal value of capital*

$$Q(e,k,l;\xi) \equiv \begin{cases} \frac{1}{1+r}\mathbb{E}_{e'|e}\left[\frac{\partial V^0\left(e',(1-\delta_k)k,\tilde{\mathcal{L}}(e,(1-\delta_k)k,l)\right)}{\partial k}\right], & \text{if } \xi > \hat{\xi}(e,k,l) \\ p, & \text{if } \xi \le \hat{\xi}(e,k,l)\end{cases}. \tag{24}$$

So an alternative way to write the FOC for capital is

$$p = \frac{1}{1+r}\Big(\mathbb{E}_{e'|e}\left[e'\right]F_k(k',l') + (1-\delta_k)\mathbb{E}_{e'|e,\ \xi}\left[Q(e',k',l';\xi)\right]\Big) \tag{25}$$

PARTIAL IRREVERSIBILITY, $\gamma < 1$

When there is a wedge between the price of buying and the price of selling capital, the firm's investment policy changes. We now need to keep track of whether the firm is adjusting up or down.

Following the same logic as in the case without irreversibility, the firm's extensive margin decision can be summarized using two cutoff rules: adjust up whenever

$$\xi \le \frac{V^{a,u}(e,k,l) - \tilde{V}(e,(1-\delta_k)k,l)}{w} \equiv \hat{\xi}^u(e,k,l) \tag{26}$$

and adjust down whenever

$$\xi \leq \frac{V^{a,d}(e,k,l) - \tilde{V}(e,(1-\delta_k)k,l)}{w} \equiv \hat{\xi}^d(e,k,l). \tag{27}$$

Combining these thresholds with the cdf G, we obtain a pair of state-dependent generalized hazard functions which we denote by Λ^u and Λ^d, respectively

$$\Lambda^u(e,k,l) \equiv G(\hat{\xi}^u(e,k,l)) \tag{28}$$

$$\Lambda^d(e,k,l) \equiv G(\hat{\xi}^d(e,k,l)) \tag{29}$$

On the intensive margin, the firm's optimal choice of capital when adjusting up maximizes the intermediate value function $\tilde{V}$ net of adjustment costs that apply whenever the firm chooses to increase capital. We denote by $\mathcal{K}^u$ the firm's optimal choice of capital conditional on upward adjustment

$$\mathcal{K}^u(e,k,l) \equiv \arg \max_{k' \geq (1-\delta_k)k} \left\{ \tilde{V}(e,k',l) - \frac{\chi}{2}\left(\frac{k'-(1-\delta_k)k}{k}\right)^2 k - p\left[k'-(1-\delta_k)k\right] \right\}.$$

Similarly, we let $\mathcal{K}^d$ denote the firm's optimal choice of capital conditional on downward adjustment

$$\mathcal{K}^d(e,k,l) \equiv \arg \max_{k' \leq (1-\delta_k)k} \left\{ \tilde{V}(e,k',l) - \frac{\chi}{2}\left(\frac{k'-(1-\delta_k)k}{k}\right)^2 k - p\,\gamma\left[k'-(1-\delta_k)k\right] \right\}.$$

Using the thresholds (26) and (27), we can integrate (15) to arrive at the following expression for the ex-ante value function

$$\begin{aligned} V^0(e,k,l) = {} & eF(k,l) - wl + \Lambda^u(e,k,l)\left(V^{a,u}(e,k,l) - w\mathbb{E}\left[\xi|\xi \leq \hat{\xi}^u(e,k,l)\right]\right) \\ & + \Lambda^d(e,k,l)\left(V^{a,d}(e,k,l) - w\mathbb{E}\left[\xi|\xi \leq \hat{\xi}^d(e,k,l)\right]\right) \\ & + \left(1-\Lambda^u(e,k,l)-\Lambda^d(e,k,l)\right)\tilde{V}(e,(1-\delta_k)k,l) \end{aligned} \tag{30}$$

This expression relies on the fact that $\hat{\xi}^u(e,k,l) > 0 \implies \hat{\xi}^d(e,k,l) = 0$ and viceversa. This is because the firm is either to the left or to the right of it's target capital stock and thus never faces a non-zero probability of adjusting up and adjusting down within the same period (see Appendix . . . for a proof).

Optimal employment decision, $\gamma < 1$

The first order condition for employment is as in the case without irreversibility but with a slightly modified ex-post employment policy. This modification takes into account the fact that the target capital stock when adjusting up differs from the target when adjusting down. with the *ex-post employment policy* $\mathcal{L}(\cdot;\xi)$ now defined as

$$\mathcal{L}(e,k,l;\xi) \equiv \begin{cases} \tilde{\mathcal{L}}(e,(1-\delta_k)k,l), & \text{if } \xi > \max\left\{\hat{\xi}^u(e,k,l), \hat{\xi}^d(e,k,l)\right\} \\ \tilde{\mathcal{L}}(e,\mathcal{K}^u(e,k,l),l), & \text{if } \xi \leq \hat{\xi}^u(e,k,l) \\ \tilde{\mathcal{L}}(e,\mathcal{K}^d(e,k,l),l), & \text{if } \xi \leq \hat{\xi}^d(e,k,l) \end{cases}.$$

Optimal investment decision, $\gamma < 1$

We can also write the same first order condition for capital (25) as long as we modify the definition of $Q(\cdot;\xi)$ to take into account the difference in the ex-post marginal value of capital due to the difference between the price of buying and selling capital

$$Q^{irrev}(e,k,l;\xi) \equiv \begin{cases} \frac{1}{1+r}\mathbb{E}_{e'|e}\left[\frac{\partial V^0(e',(1-\delta_k)k,\mathcal{L}(e,k,l))}{\partial k}\right], & \text{if } \xi > \max\left\{\hat{\xi}^u(e,k,l),\hat{\xi}^d(e,k,l)\right\} \\ p, & \text{if } \xi \leq \hat{\xi}^u(e,k,l) \\ \gamma\, p, & \text{if } \xi \leq \hat{\xi}^d(e,k,l) \end{cases}. \tag{31}$$

5 Aggregate Responses

The goal in this section is to understand what drives the response of firm aggregates to small changes in prices across different variants of the model laid out in Section 4. We first use a perturbation approach to characterize the time-0 elasticity of capital with respect to the price of investment goods.

5.1 Time-0 Responses

Our approach relies on equations (17), (18),(21) and (25) to compute the first order effect of a small change in the price of investment goods dp on firm aggregates. We denote by $d\mathcal{L}(x)$ the first-order response of the employment policy of a firm whose current state is given by $x = (e,k,l)$. Similarly, we let $d\mathcal{K}(x)$ denote the first-order response of the investment policy on the intensive margin. When there is no irreversibility, we let $d\Lambda(x)$ denote the first-order effect on the extensive margin. Whenever $\gamma < 1$, we use $d\Lambda^u(x)$ and $d\Lambda^d(x)$ for the first-order response of the pair of extensive margin hazards.[6]

Extensive margin, $\gamma = 1$

Let us begin with the price-elasticity of aggregate capital in the absence of irreversibility. The first-order time-0 response of aggregate capital $K \equiv \int \mathcal{K}(x;\xi)dD(x;\xi)$ is given by

$$dK = \int \underbrace{d\Lambda(x)\ \left[\mathcal{K}(x) - (1-\delta_k)k\right]}_{\text{extensive margin}} dD(x) + \int \underbrace{\Lambda(x)\, d\mathcal{K}(x)}_{\text{intensive margin}}\, dD(x) \tag{32}$$

A perturbation of (17) and (18) combined with the envelope theorem, establishes the following result on the extensive margin:

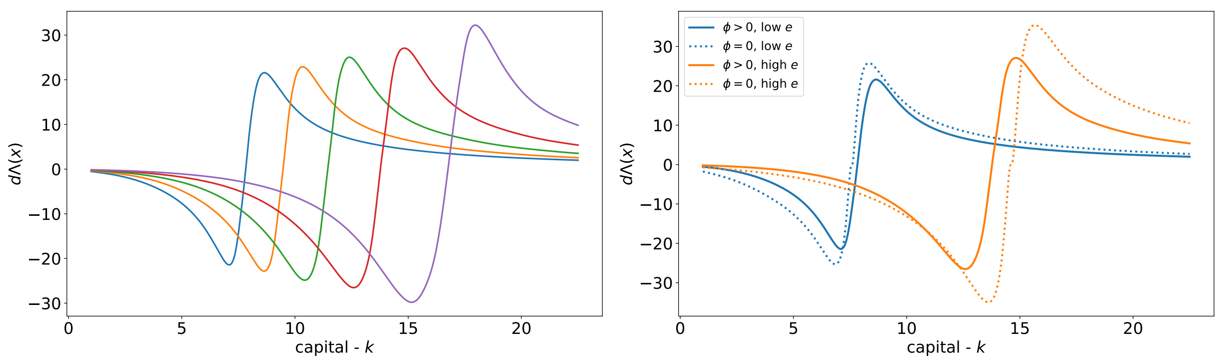
Proposition 1 *The first-order effect on the extensive margin $d\Lambda(x)$ is given by*

$$d\Lambda(x) = -\hat{g}(x) \times \frac{dp}{w} \times \left[\mathcal{K}(x) - (1-\delta_k)k\right],$$

where $\hat{g}(x) \equiv g(\hat{\xi}(x))$. Therefore, the first-order time-0 response of aggregate capital due to the extensive margin is

$$dK^{ext} = -\frac{dp}{w} \times \int \hat{g}(x)\left[\mathcal{K}(x) - (1-\delta_k)k\right]^2 dD(x).$$

[6] Formally, these are the Gateaux differentials of the optimal policies.



(a) $d\Lambda(x)$ along the k dimension

(b) $d\Lambda(x)$ with $\phi = 0$ and $\phi > 0$

Figure 3: Extensive margin responses without irreversibility

This shows that the $\hat{g}$-weighted second moment of investment in the cross section is a sufficient statistic for the contemporaneous first-order effect due to the extensive margin. Much of the prior literature has used a uniform distribution leading $\hat{g}(x)$ to be constant. A uniform distribution estimated to generate the standard deviation of investment will imply that there are many firms on the margin of adjusting or not given their shock who could be induced to invest by a shock Thus leading to an extensive margin response much larger than has been estimated empirically. Alternative fixed cost distributions with support on the positive reals such as the log normal can help the model jointly match both the steady state distribution of investment and the response. Distributions such as the log normal will generate large investments from a sequence of high draws same as with the uniform, i.e. $\prod_{t=1}^{T}(1 - G(\text{Value of adjusting t periods after last adjustment}))$ is large enough. The difference is that for the log normal,$g(\text{Value of adjusting t periods after last adjustment})$ can be very small for t's that imply a large investment

Corollary 1 *The first-order time 0 response of aggregate labour due to the extensive margin is*

$$dL^{ext} = -\frac{dp}{w} \times \int \hat{g}(x) \left[\mathcal{K}(x) - (1-\delta_k)k\right] \left[\mathcal{L}^a(x) - \mathcal{L}^n(x)\right] dD(x).$$

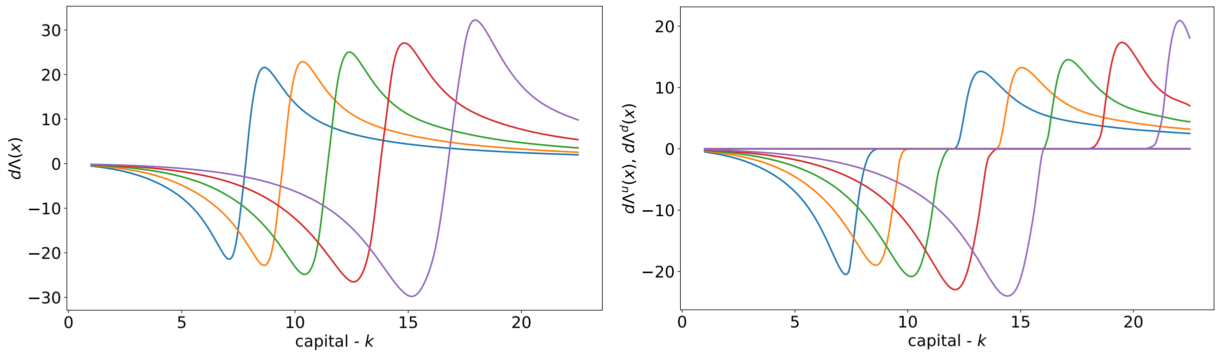
where $\mathcal{L}^a(x) \equiv \tilde{\mathcal{L}}(e, \mathcal{K}(x), l)$ *and* $\mathcal{L}^n(x) \equiv \tilde{\mathcal{L}}(e, (1-\delta_k)k, l)$. □

EXTENSIVE MARGIN, $\gamma < 1$

With partial irreversibility, the expression for the time-0 extensive margin response of aggregate capital becomes

$$dK^{ext} = \int d\Lambda^u(x) \left[\mathcal{K}^u(x) - (1-\delta_k)k\right] dD(x) + \int d\Lambda^d(x) \left[\mathcal{K}^d(x) - (1-\delta_k)k\right] dD(x) \tag{33}$$

Using an argument similar to the one behind Proposition 1, we can establish the following result regarding the extensive margin response in the presence of irreversibility:



(a) $d\Lambda(x)$ along the k dimension

(b) $d\Lambda^u(x)$ and $d\Lambda^d(x)$ along the k dimension

Figure 4: Extensive margin responses with and without irreversibility

Proposition 2 *With irreversibility, the first-order effect on the extensive margin pair Λ^u and Λ^d is given by*

$$d\Lambda^u(x) = -\hat{g}^u(x) \times \frac{dp}{w} \times [\mathcal{K}^u(x) - (1-\delta_k)k]$$
$$d\Lambda^d(x) = -\hat{g}^d(x) \times \gamma\frac{dp}{w} \times \left[\mathcal{K}^d(x) - (1-\delta_k)k\right], \qquad \square$$

where $\hat{g}^u(x) \equiv g(\hat{\xi}^u(x))$ and $\hat{g}^d(x) \equiv g(\hat{\xi}^d(x))$. Therefore, the first-order response of aggregate capital due to the extensive margin is

$$dK^{ext} = -\frac{dp}{w} \times \left(\int \hat{g}^u(x)\,[\mathcal{K}^u(x) - (1-\delta_k)k]^2\,dD(x) + \gamma\int \hat{g}^d(x)\left[\mathcal{K}^d(x) - (1-\delta_k)k\right]^2 dD(x)\right).$$

With irreversibility, the sufficient statistics for the first-order effect on the extensive margin are the $\hat{g}$-weighted second moments of investment.[7]

Corollary 2 *With irreversibility, the first-order time 0 response of aggregate labour due to the extensive margin is*

$$dL^{ext} = -\frac{dp}{w} \times \left\{\int \hat{g}^u(x)\,[\mathcal{K}^u(x) - (1-\delta_k)k]\,[\mathcal{L}^{a,u}(x) - \mathcal{L}^n(x)]\,dD(x)\right.$$
$$\left. + \gamma\int \hat{g}^d(x)\left[\mathcal{K}^d(x) - (1-\delta_k)k\right]\left[\mathcal{L}^{a,d}(x) - \mathcal{L}^n(x)\right] dD(x)\right\}.$$

where $\mathcal{L}^{a,u}(x) \equiv \tilde{\mathcal{L}}(e, \mathcal{K}^u(x), l)$, $\mathcal{L}^{a,d}(x) \equiv \tilde{\mathcal{L}}(e, \mathcal{K}^d(x), l)$ and $\mathcal{L}^n(x) \equiv \tilde{\mathcal{L}}(e, (1-\delta_k)k, l)$. $\square$

INTENSIVE MARGIN, $\gamma = 1$

We now turn to the characterization of the intensive margin response, which is somewhat more involved due to the interdependence of capital and labour. From (25), the perturbed FOC for capital is

$$p + \mu dp = \frac{1}{1+r}\left(\bar{e}(x)F_k(\mathcal{K} + \mu d\mathcal{K}, \mathcal{L} + \mu d\mathcal{L}) + (1-\delta_k)\left\{\mathbb{E}_{x'|x}\left[Q^0(e', \mathcal{K} + \mu d\mathcal{K}, \mathcal{L} + \mu d\mathcal{L})\right]\right\}\right)$$

[7] One also needs to take a stand on γ but maybe dK^{ext} could be the calibration target for γ?.We could target the intensive and extensive margin shares of the response and show how both K&T and Winberry miss the intensive margin. Can you hit them both with the things we have now?

with $\bar{e}(x) \equiv \mathbb{E}_{e'|x}[e']$ and $Q^0(x) \equiv \mathbb{E}_{\xi}[Q(x;\xi)]$. Taking a first-order Taylor expansion of this equation around the steady state and letting $\mu \to 0$, we find

$$d\mathcal{K}(x) = \frac{(1+r)\,dp - \left\{\bar{e}(x)F_{kl}(x) + (1-\delta_k)\mathcal{E}^{Q_l^0}(x)\right\} d\mathcal{L}(x)}{\bar{e}(x)F_{kk}(x) + (1-\delta_k)\mathcal{E}^{Q_k^0}(x)} \tag{34}$$

where $\mathcal{E}^{Q_l^0}(x) \equiv \mathbb{E}_{x'|x}\left[\frac{\partial}{\partial l}Q^0(x')\right]$ and $\mathcal{E}^{Q_k^0}(x) \equiv \mathbb{E}_{x'|x}\left[\frac{\partial}{\partial k}Q^0(x')\right]$ are the one-period ahead expected sensitivity of ex-ante marginal value of capital. To get some intuition, let us begin with the case where labour is fully flexible. In this case, the ex-post marginal value of capital Q does not depend on the firm's employment policy and the equation becomes

$$d\mathcal{K}(x) = \frac{(1+r)\,dp - \bar{e}(x)F_{kl}(x)d\mathcal{L}(x)}{\bar{e}(x)F_{kk}(x)\, + (1-\delta_k)\mathcal{E}^{Q_k^0}(x)}$$

where the change in Q^0 due to a small variation in the capital stock is given by

$$\frac{\partial}{\partial k}Q^0(x) = (1-\delta_k)\Bigg\{\hat{g}(x)\left(p - \frac{1}{1+r}\mathbb{E}_{e'|e}\left[\frac{\partial V^0\left(e',(1-\delta_k)k\right)}{\partial k}\right]\right)^2 + \left(\frac{1-\Lambda(x)}{1+r}\right)\mathbb{E}_{e'|e}\left[\frac{\partial^2 V^0\left(e',(1-\delta_k)k\right)}{\partial k^2}\right]\Bigg\}$$

After perturbing the FOC for labour when $\phi = 0$, we find

$$d\mathcal{L}(x) = -\frac{F_{lk}(x)}{F_{ll}(x)} \times d\mathcal{K}(x)$$

Substituting this into (34) and solving for $d\mathcal{K}(x)$ we can establish the following result for the intensive margin responses:

Proposition 3 *When $\phi = 0$, the intensive margin response $d\mathcal{K}(x)$ is given by*

$$d\mathcal{K}(x) = \frac{(1+r)\,dp}{\bar{e}(x)\left[F_{kk}(x) - \frac{F_{lk}(x)^2}{F_{ll}(x)}\right] + (1-\delta_k)\mathcal{E}^{Q_k^0}(x)}$$

Therefore, the first-order time-0 response of aggregate capital due to the intensive margin is

$$dK^{int} = (1+r)dp \times \int \frac{\Lambda(x)}{\bar{e}(x)\left[F_{kk}(x) - \frac{F_{lk}(x)^2}{F_{ll}(x)}\right] + (1-\delta_k)\mathcal{E}^{Q_k^0}(x)} dD(x)$$

Let us now consider the case where $\phi > 0$. The perturbed FOC for labour when the firm chooses not to adjust its capital stock is

$$(1+r)\phi\left(\frac{\mathcal{L} + \mu\hat{\mathcal{L}} - (1-\delta_l)l}{l}\right) = \bar{e}(x)F_l\left((1-\delta_k)k, \mathcal{L} + \mu\hat{\mathcal{L}}\right) - w + \frac{\phi}{2}\left[\frac{\mathbb{E}_{x'|x}\left[\mathcal{L}_2^0\left(e',(1-\delta_k)k,\mathcal{L}+\mu\hat{\mathcal{L}}\right)\right] - (1-\delta_l)^2\left(\mathcal{L}+\mu\hat{\mathcal{L}}\right)^2}{\left(\mathcal{L}+\mu\hat{\mathcal{L}}\right)^2}\right] \tag{35}$$

where $\mathcal{L}_2^0(x) \equiv \mathbb{E}_\xi\left[\mathcal{L}(x;\xi)^2\right]$. This is the equation that applies for the employment policy of all those firms whose realization of the fixed cost $\xi > \hat{\xi}(e,k,l)$.[8] It is easy to see that the first order response of employment for the firms who choose not to adjust their capital stock when the shock hits is identically equal to zero. When the firm decides to adjust its capital stock, the perturbed FOC becomes

$$\begin{aligned}(1+r)\phi\left(\frac{\mathcal{L}+\mu\hat{\mathcal{L}}-(1-\delta_l)l}{l}\right) &= \bar{e}(x)F_l\left(\mathcal{K}+\mu\hat{\mathcal{K}},\mathcal{L}+\mu\hat{\mathcal{L}}\right)-w \\ &\quad+\frac{\phi}{2}\left[\frac{\mathbb{E}_{x'|x}\left[\mathcal{L}_2^0\left(e',\mathcal{K}+\mu\hat{\mathcal{K}},\mathcal{L}+\mu\hat{\mathcal{L}}\right)\right]-(1-\delta_l)^2\left(\mathcal{L}+\mu\hat{\mathcal{L}}\right)^2}{\left(\mathcal{L}+\mu\hat{\mathcal{L}}\right)^2}\right]\end{aligned} \tag{36}$$

Taking a first-order Taylor expansion of this equation and letting $\mu \to 0$, we find

$$\begin{aligned}&\left\{\left(\frac{(1+r)\phi}{l}-\bar{e}(x)F_{ll}(x)\right)\mathcal{L}(x)-\frac{\phi}{2}\left(\mathbb{E}_{x'|x}\left[\frac{\frac{\partial}{\partial l}\mathcal{L}_2^0(x')}{\mathcal{L}(x)}\right]-2\,\mathbb{E}_{x'|x}\left[\frac{\mathcal{L}_2^0(x')}{\mathcal{L}(x)^2}\right]\right)\right\}d\mathcal{L}(x) \\ &=\left\{\bar{e}(x)F_{lk}(x)\mathcal{L}(x)+\frac{\phi}{2}\mathbb{E}_{x'|x}\left[\frac{\frac{\partial}{\partial k}\mathcal{L}_2^0(x')}{\mathcal{L}(x)}\right]\right\}d\mathcal{K}(x)\end{aligned} \tag{37}$$

Defining $\mathcal{E}^{\mathcal{L}_{2,l}^0}(x) \equiv \mathbb{E}_{x'|x}\left[\frac{\frac{\partial}{\partial l}\mathcal{L}_2^0(x')}{\mathcal{L}(x)}\right]$, $\mathcal{E}^{\mathcal{L}_{2,k}^0}(x) \equiv \mathbb{E}_{x'|x}\left[\frac{\frac{\partial}{\partial k}\mathcal{L}_2^0(x')}{\mathcal{L}(x)}\right]$, and $\mathcal{E}^{\mathcal{L}_2^0}(x) \equiv \mathbb{E}_{x'|x}\left[\frac{\mathcal{L}_2^0(x')}{\mathcal{L}(x)^2}\right]$ and solving for $d\mathcal{L}(x)$,

$$d\mathcal{L}(x)=\frac{\left\{\bar{e}(x)F_{lk}(x)\mathcal{L}(x)+\frac{\phi}{2}\mathcal{E}^{\mathcal{L}_{2,k}^0}(x)\right\}d\mathcal{K}(x)}{\left(\frac{(1+r)\phi}{l}-\bar{e}(x)F_{ll}(x)\right)\mathcal{L}(x)-\frac{\phi}{2}\left(\mathcal{E}^{\mathcal{L}_{2,l}^0}(x)-2\,\mathcal{E}^{\mathcal{L}_2^0}(x)\right)} \tag{38}$$

Proposition 4 *When $\phi > 0$, the first-order responses $d\mathcal{K}(x)$ and $d\mathcal{L}(x)$ due to a small one-time change in the price of investment goods dp are given by the solution to the following system*

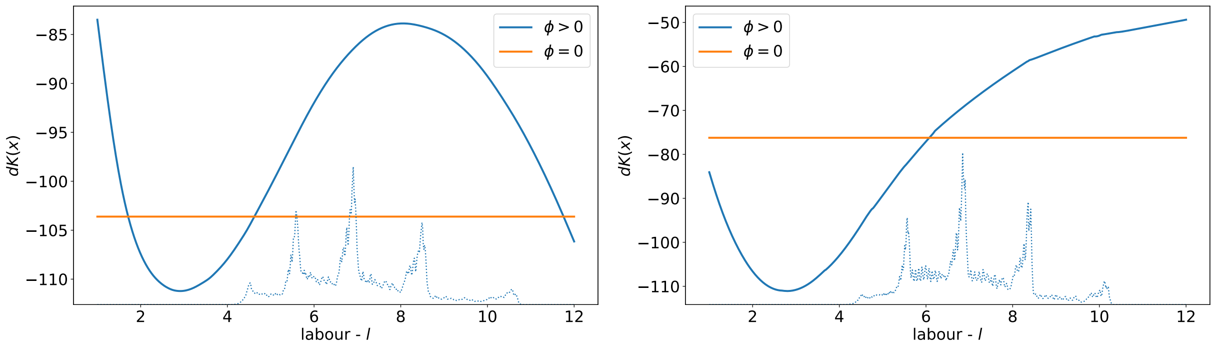
$$d\mathcal{K}(x)=\frac{(1+r)\,dp-\left\{\bar{e}(x)F_{kl}(x)+(1-\delta_k)\mathcal{E}^{Q_l^0}(x)\right\}d\mathcal{L}(x)}{\bar{e}(x)F_{kk}(x)+(1-\delta_k)\mathcal{E}^{Q_k^0}(x)} \tag{39}$$

$$d\mathcal{L}(x)=\frac{\left\{\bar{e}(x)F_{lk}(x)\mathcal{L}(x)+\frac{\phi}{2}\mathcal{E}^{\mathcal{L}_{2,k}^0}(x)\right\}d\mathcal{K}(x)}{\left(\frac{(1+r)\phi}{l}-\bar{e}(x)F_{ll}(x)\right)\mathcal{L}(x)-\frac{\phi}{2}\left(\mathcal{E}^{\mathcal{L}_{2,l}^0}(x)-2\,\mathcal{E}^{\mathcal{L}_2^0}(x)\right)} \tag{40}$$

□

The primary reason for the rise and then dip in the absolute value of the intensive margin response is the responsiveness of the marginal value of capital to additional capital first decreases and then increases. The initial decrease the responsiveness of the marginal value of capital to capital is driven by the fall in the effective DRS in capital as labour increases. Due to the quadratic adjustment costs on labour, when adjusting capital at a low level of labour, labour tomorrow will also be low and so the firm faces a high degree of DRS in capital causing it to not change its investment choice much in response to a change in capital price. But as labour rises this DRS lessens making firms for responsive. The later increase in the responsiveness of the marginal value of capital to capital is driven by the change in the derivative of the continuation value of capital in capital. If a firm is likely to adjust tomorrow additional capital will just

[8] Note that we hold fixed tomorrow's employment policy because we are only considering a one-time shock that has no direct effect on future policies.



(a) $d\mathcal{K}(x)$ without irreversibility

(b) $d\mathcal{K}(x)$ with irreversibility

Figure 5: Intensive margin responses with and without irreversibility

increase or decrease the amount bought or sold tomorrow, thus the marginal value is constant. However if the firm is not likely to adjust then it will hold onto the additional value going forward so it the marginal value changes by the expected second derivative of the value function with respect to capital. Therefore as the probability of adjusting tomorrow falls the responsiveness of the marginal value of capital increases decreasing the intensive margin response.

6 Model Comparison

6.1 Targeting Moments

While we will discuss the relative impulse responses of the different models, the tension we highlighted earlier relates to the calibrating the responses of models of only capital adjustment. The key calibration targets will be the responsivness of firm investment and capital which Koby and Wolf (2020) argue are the correct targets to discipline the general equilbrium effects in heterogenous firm models.

We chose targets for responsivness from Zwick and Mahon (2017) and Curtis et al. (2021). These two papers study the same shocks to the cost of investment which came about due to changes in tax depreciaton schedules. There are, however, two major differences, first Zwick and Mahon (2017) estimates an elasticity while Curtis et al. (2021) remain agnostic about the size of the shocks in their reduced form estimation.Thus we will use the semi-elasticity of aggregate investment to capital cost changes of 7.2 from Zwick and Mahon (2017) as our target for the short run responsivness of investment. Secondly Curtis et al. (2021) estimate responses up to 10 years while Zwick and Mahon (2017) focusses on the shorter run. These longer run responses has been of less focus in the previous literature on firm investment subject to investment frictions. Curtis et al. (2021) find a permanent increase in investment rates and capital in their figure 2. Given the agnosticism about shock magnitudes we can only look at the shape of the responses. The investment response is noisy though peaks 5 years after the shock While the capital stock monotonically increases, though flattens out by 7 years after the shock. Since the capital stock estimates are smoother, we take a conservative target of 10 years post a permanent shock for the capital distribution to return to steady state.

The second set of moments that we target are the micro-level moments typical in this literature. The exact moments we use are taken from Zwick and Mahon (2017). We target the mean and standard deviation of annual firm investment rates. We also target the percentage of investments that are positive.

6.2 Model Calibration

We then move on to calibrating the three models, the fixed cost model, the convex cost model and the labour adjustment model. In all of these calibrations we hold a number of parameters constant. These include the returns to scale of the production function (α) and the parameters of the productivity process. The parameters of the productivity process in particular are not well identified by the moments we are using here Clementi and Palazzo (2015). We choose values of 0.8 for the returns to scale and 0.9, 0.053 for the persistence and variance of the idiosyncratic productivity terms.

To start we calibrate the fixed cost model to the micro level moments in the spirit of Khan and Thomas (2008). As expected we find a good fit of the model moments. We estimate μ, σ of the lognormal distribution to be -0.76 and 2.0 respectively. Additionally as expected this calibrated model fails to match the responsivness measured in the data as is well known in the data.

We then calibrate the convex cost model to the micro level moments as well as the one year response of investment to costs. Looking at the 10 year horizon, however, the aggregate capital level has not even reached 90% of the steady state level. This highlights numerically the tension between the initial responsivness and the speed of convergence. Adjusting the parameters to speed up the convergence of capital causes the responsivness to increase. Thus this model cannot match the full dynamics of investment observed in the data

Finally we show that the model with labour adjustment costs can match the full set of moments. By allowing the convex cost on labour and the elasticity of substitution between capital and labour to vary, the model moments closely match the moments in the data. To acheive this the model requires a relatively low elasticity of substitution of 0.45. This is consistent however with the evidence from Curtis et al. (2021) who break up labour into production labour and non production labour and find that production labour is complementary to capital while non production labour is substitutable.

6.3 Comparison of Model Responses

In 6 we plot the response of capital in each model to a permanent interest rate shock. The response is scaled by both the final new level of capital and the size of the shock so it represents the semi-elasticity to the interest rate.

The major features of the calibration are clear. In the fixed cost model capital converges very quickly with a large response in the first period. Additionally, the slow response of the convex cost model is illustrated. The level of capital only visably converges 20 years after the shock occured, far slower than observed in the data. On the other hand the model with labour adjustment costs while having the same initial investment response converges in a manner more consistent with the data.

7 Conclusion

In this paper we demonstrate that models of capital adjustment that have capital has the only endogenous state variable are unable to match both the short and medium runs responses observed in the data. We then introduce a model of dynamic labour demand into a lumpy investment model to show that the dynamic interdependence of capital and labour can help rationalize the empirical evidence on the relationship between employment and investment. To justify this modelling choice we provide evidence that lagged employment growth is a statistically significant predictor of future investment even conditional on lagged investment. We also show when labour is not a flexible factor, the elasticity of substitution between labour and capital is a key parameter in determining the magnitude of the investment response to interest rate shocks.

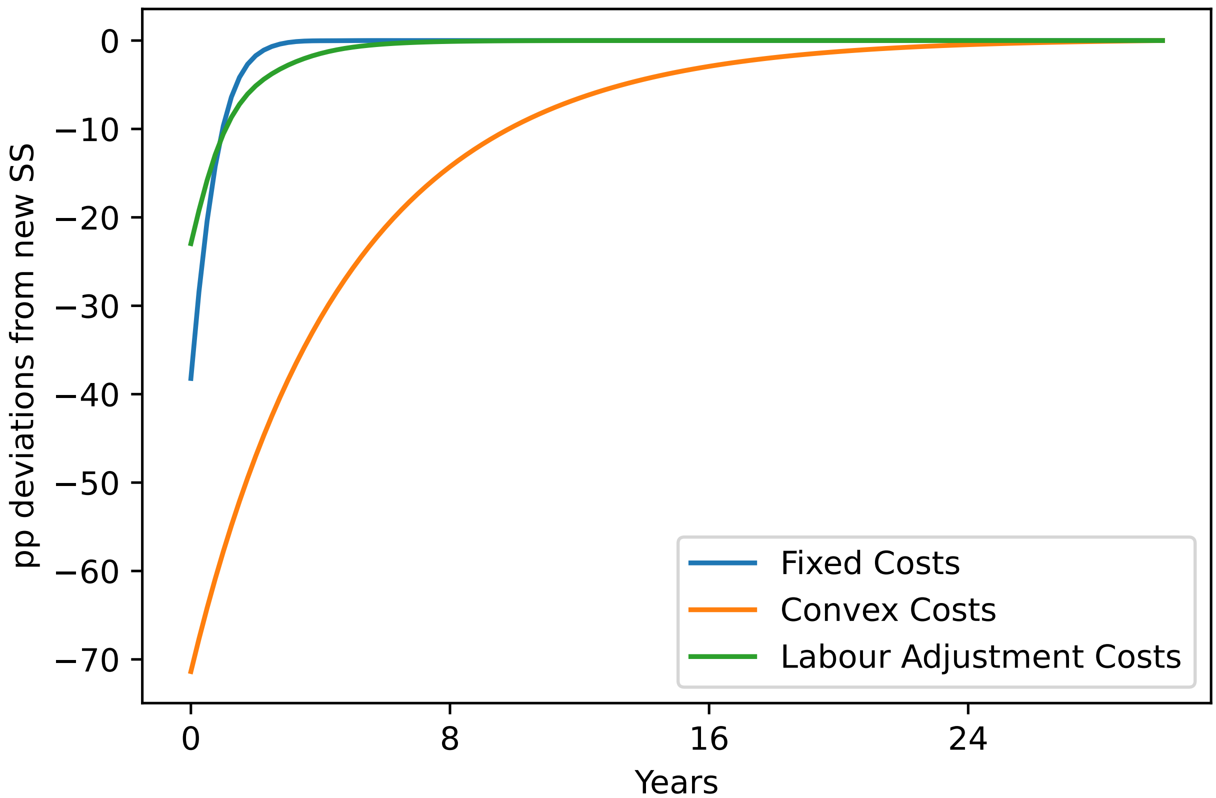


Figure 6: Response of capital in fixed cost, convex cost and labour adjustment models to a permanent interest rate shock

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A CONTINUOUS TIME MODEL

In the continuous time version of the model, the firm solves a combined stochastic control and impulse control problem. When there is no adjustment, capital evolves as

$$dk_t = -\delta k_t dt + \sigma dB_t.$$

With impulse Δk, the capital stock jumpts from k to $k' = k + \Delta k$ but the firm has to pay a cost $\mathcal{G}$ that can be given by

$$\mathcal{G}(\Delta k) = F + p \max\{\Delta k, 0\} + p(1-\gamma) \min\{\Delta k, 0\}$$

As in the discrete time model, we assume that the firm faces convex and symmetric adjustment costs in labour. It can hire workers at rate n_t but must pay a cost $\mathcal{C}$ that is allowed to depend on current employment at the firm. A fraction ρ of workers leave the firm each period, so the evolution of labour is governed by

$$dl_t = [n_t - \rho l_t]\, dt$$

The program of the firm can be written as

$$v(k_0, l_0) = \sup_{\{\Delta k_t, n_t\}_{t \geq 0}} \mathbb{E}_0 \int_0^\infty e^{-rt} \left[eF(k_t, l_t) - wl_t - AC(\Delta k_t, n_t, k_t, l_t) \right] dt$$

subject to the evolution of capital and labour above and with

$$AC(\Delta k_t, n_t) := \mathcal{G}(\Delta k_t, k_t) + \mathcal{C}(n_t, l_t)$$