Impartial Social Rankings

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Abstract

We model a situation where a set of selfish agents must rank themselves on the basis of the opinions that they hold regarding the merits of each candidate. Each agent submits her opinion in a message (a nomination, a ranking, a utility function, . . .). We call the collection of such messages a message profile and it is the input for a social ranking function that determines the final ranking. We are interested in impartial social ranking functions, that is, those where the contribution of a single agent to the social outcome can have no consequence on any social judgement between him and someone else. We obtain two types of results. On the negative side, we show some impossibility results stating the impossibility of combining impartiality with some classical properties of symmetry across agents. On the positive side, we provide two characterizations combining impartiality with weakenings of the main symmetry property across agents.

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1 Introduction

In this paper we study methods to rank a set of agents on the basis of the opinions they hold regarding the merits of each of them. In other words, we study methods to combine individual opinions when the agents themselves are involved in the final outcome. Whenever facing this type of collective decision making processes, individuals may be tempted to act strategically in order to get a social result as close as possible to their selfish interests. In order to deter such manipulations, we study how to aggregate the opinions of the agents in such a way that the contribution of a single agent to the social outcome cannot influence any social judgment between him and someone else. This property is known as Impartiality.

The Impartiality property resemblances other classical properties in Economics as strategy-proofness or implementability by weakly dominant strategies. In fact, Impartiality is equivalent to these concepts in these problems in which agents are involved in the final outcome if it is assumed that agents are selfish and cares only about themselves while being indifferent to the position of the others in the final outcome.

Our approach to analyze the Impartiality property is novel in two important aspects. First, while some important literature has analyzed how this property restricts the possibilities of aggregating the opinions when the social outcome should be the selection of a single winning candidate or a set of winning candidates, our focus is on the impact of this property on the construction of a social ranking of the candidates, a classical outcome of the aggregation procedures in social choice theory. Second, our purpose is to give a general analysis of the property and, then, we do not restrict the ways in which each agent should express her opinion, as other papers do. By the contrary, we allow each agent to express her opinion in any possible way (a nomination of an agent or a set of agents, a ranking of some or all agents, a utility function, ...). Then, we consider for each agent i an unstructured set of possible messages $\mathcal{M}_i$ such that our aggregation function, that we call a social ranking function, takes as a domain the cartesian product of the sets of possible messages of all agents, $\times_{i \in N} \mathcal{M}_i$.

The paper first tries to combine Impartiality with a very basic idea of symmetry across agents that we express here with a property we call Name Independence. This property states that if we permute the agents and their messages with a permutation, then the outcome of the social ranking function is permuted accordingly. Although this is a very basic property, we show that it is incompatible with Impartiality.[1] The reason of these impossibilities is that, meanwhile Impartiality implies that not all possible ways of ranking the agents should occur, both Name Independence and Candidate Neutrality requires that the social ranking function should have full range. These impossibility results guide our search of possibility results. We first characterize all impartial social ranking functions when there are less than four agents. These functions are called partition orderings and they

[1]Although we do not include the result in the paper, an identical impossibility holds and with the same proof combining Impartiality with another classical symmetry property, Candidate Neutrality, in which only the messages are permuted.

have the following structure. First, they fix a partition of the agents ex-ante $\{N_1, \ldots, N_s\}$ such that, independently of the opinions of the agents, the agents of N_1 are always ranked above the rest of agents, the agents of N_2 are always ranked above the rest of agents except those of N_1, and so on. Secondly, the ranking of the agents of the same set of the partition is defined depending on the messages of the agents outside this set. After that, we focus on the general case in which there could be more than four agents. Then, given that Impartiality restricts some binary comparisons ex-ante, we propose to apply the idea of Name Independence only to the permutations that change the role of the agents that are not fixed by the social ranking function when Impartiality is imposed. This is what the property of Weak Name Independence exactly does. We provide a characterization of all the social ranking functions that satisfy both Impartiality and Weak Name Independence. This family, called impartial partition orderings, is a superfamily of the partition orderings because the ranking of the agents of each set of the partition is not necessarily determined in this superfamily only by the messages of the agents outside the set and can incorporate also the messages of the agents inside the set if they do it in an impartial way. The family of impartial partition orderings incorporate Impartiality as a property in its own definition: it says that, fixing the messages of the agents outside a set of the partition, the ranking of the agents inside the set should be done aggregating the messages of the agents inside the set in an impartial way. Then, it should be of interest to define a subfamily with a definition in a closed form. Then, we propose a stronger property than Weak Name Independence, Consistent Weak Name Independence, and we obtain a bigger family of impartial social ranking functions that we call sequential partition orderings. These functions coincide with the partition orderings in the fact that they also initially partition the set of agents in such a way that all agents belonging to N_p are always ranked above any agent of N_q for all $q > p$. They also coincide with the impartial partition orderings in that here the ordering of the agents situated in the same set N_p of the initial partition does not necessarily only depend on the messages of the agents outside that set. By the contrary, the messages of the agents outside N_p only partitions that set into small sets, $(N_{(p,1)}, \ldots, N_{(p,t)})$, such that the agents of $N_{(p,i)}$ are going to be ranked above the agents of $N_{(p,j)}$ with $j > i$, but the order of the agents of the same set $N_{(p,i)}$ are not decided by only the messages of the agents outside N_p, but also by the messages of the agents of other subsets of N_p different from $N_{(p,i)}$. Then, this process is replicated iteratively until all subsets are singletons and a complete ranking has been reached.

Finally, we also present the impossibility of combining in a non-constant function the property of Impartiality with the idea of Anonymity. Meanwhile the classical Anonymity property requires that a permutation of the agents should not change the outcome of the function, our Weak Anonimity property used for the impossibility requires this idea only for the outcomes of the agents that are not changed by the permutation.

Related Literature

Several papers introduce the idea of impartiality, which applies to cases where voters are also candidates to be elected or, as in our case, to be ranked. Holzman and Moulin (2013) describe a situation where a group of peers must choose one of them to receive a prize in a way such that one's own message does not influence whether or not he wins the prize. Their main result states that for any impartial rule, two desirable properties cannot be satisfied at the same time: that the winner always gets at least one nomination, and that an agent nominated by everyone else always wins. This impossibility is in line with our result in Theorem 1 stating that there is no impartial social ranking function satisfying some symmetry condition.[2] The form of the social ranking functions characterized in our Proposition 1 for $n \leq 3$ concurs with Holzman and Moulin's (2013) main example of an impartial rule, *Partition Methods*.[3]

Following Holzman and Moulin (2013), Tamura and Ohseto (2013) consider nomination correspondences that determine the subset of agents to get a prize on the basis of each agent's nomination of a single winner. They show that there exists a nomination correspondence that satisfies impartiality, positive unanimity, and negative unanimity, and that any impartial nomination correspondence that satisfies anonymous ballots is not necessarily constant, but it violates positive unanimity. We would also like to mention de Clippel at al. (2008) who first introduced the notion of impartiality. They study how could partners divide a cash surplus and they characterize a family of rules in which an agent's part is derived from aggregate reports on the share that other agents propose for him.

Another recent literature studies impartiality within lottery based nominations. For instance, Mackenzie (2015) studies impartial methods that assign to each agent his probability to win the prize. In his paper the author characterizes two classes of impartial rules that satisfy, using our terminology, anonymity and neutrality, respectively.

The study of robustness to manipulation of the aggregation rules, which map profiles of individual opinions or preferences into social orderings of the alternatives, has not been developed to a great extend.[4] It is clear that some commonly used rules like Borda are susceptible to misrepresentations. The problem is that there is an absence of a uniformly accepted notion defining precisely robustness of aggregation rules to preference misrepresentations. Bossert and Sprumont (2014) adapt the classic notion of strategy-proofness in the framework of aggregation rules. They call a rule strategy-proof if misreporting one's

[2]A crucial fact is stated in Lemma 1 that implies that under impartiality not all social rankings are attainable.

[3]In our Theorem 2 we characterize a subclass of these rules in our more general setting.

[4]On the other hand numerous literature is devoted to the problem of designing choice rules that would encourage agents to reveal their true preferences. The impossibility results about non-existence of strategy-proof and non-dictatorial rules of Gibbard (1973) and Satterthwaite (1975) were followed by a several possibilities using restricted domains of individual preferences in some frameworks (Barberà (2011).

preference never produces a social ordering that is between (in the sense of Grandmont's (1978) intermediate preferences) the original ordering and one's own preference.

In an earlier paper, Bossert and Storcken (1992) also formulate a definition applicable to aggregation rules, in the sense that a rule is strategy-proof if misrepresenting ones preference never results with a social ordering that is closer to ones own preference according to the Kemeny distance. Additional definitions of manipulability can be found in Sato (2013) who also provide several characterizations of aggregation rules based on these non-manipulability requirements.

The normative framework of certain peer ranking methods using analog properties to our anonymity, neutrality and monotonicity is studied in Ng and Sun (2003). They look on ranking methods based on agents points assigned by the rest of individuals and they show that no such method satisfies monotonicity, neutrality, and unanimity. Using similar settings as Ng and Sun (2003), Ohseto (2012) shows that when the candidates can select the evaluation points from a finite and large set of real numbers, there is no aggregation rule that is monotonic and unanimous.

Recent papers, like Amorós (2020) and Fujiwara-Greve et al. (2021), study the problem of choosing an ordered set of candidates by a subset of experts, and focusing on the problem when experts have interest in some candidates. Both papers consider an axiom also called Impartiality that although has the same spirit as ours, it is formally different. Also the theoretical contexts where those papers apply are different and the results obtained can not be directly compared.

Remainder

The structure of this paper goes as follows: In Section 2 we introduce the framework and the basic axiom we study: Impartiality. Section 3 contains our two impossibility results. A characterization result and a possibility result with some weakenings of Name Independence are presented in Section 4. Section 5 presents that Impartiality and a weak version of the anonymity property implies a constant social ranking function. Section 6 concludes with some remarks about our results. All the proofs are placed in the Appendix.

2 The Model

Let $N = \{1, \ldots, n\}$ be a finite set of agents. We denote by $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}_+$ the set of natural numbers, the set of real numbers and the set of non-negative real numbers. We define a ranking R as a complete, transitive and antisymmetric binary relation on N. We denote the set of possible rankings by $\mathcal{R}$. Given a ranking $R \in \mathcal{R}$ and a subset of agents $A \subset N$, we denote by $R_{|A}$ the ranking R restricted to the set of candidates A, while $\mathcal{R}_{|A}$ will denote all possible rankings of the agents of A. Given a ranking $R \in \mathcal{R}$ and an agent

$i \in N$, we denote by $U(R, i)$ the set of agents who are higher ranked than i by R, that is, $U(R, i) = \{j \in N \setminus \{i\} : j\, R\, i\}$. Let $\mathcal{M}_i$ be a set of possible messages that agent i can declare. Then, we define a profile of messages as a list of messages, one of each individual, $m = (m_1, \ldots, m_n) \in \times_{i \in N} \mathcal{M}_i$, being $\mathcal{M}_i$ any set without any structure imposed ex-ante. Some possibilities of different sets $\mathcal{M}_i$ are the following:

(*i*) the set of agents, $\mathcal{M}_i = N$, as in the case of nomination rules with self nomination (see Mckenzie, 2018).

(*ii*) the set of subsets of N, $\mathcal{M}_i = 2^N$, as in the approval rules (see Brams and Fishburn, 1978) in which each agent approves some candidates.

(*iii*) rankings, $\mathcal{M}_i = \mathcal{R}$ in which each agent ranks the possible candidates.

(*iv*) utility functions, $\mathcal{M}_i = \mathcal{U}$, where $\mathcal{U}$ is the set of functions $u : N \to \mathbb{R}_+$, as in the rating functions (see ???).

These examples are cases of common sets of possible messages; i.e., $\mathcal{M}_i = \mathcal{M}_j$ for all $i, j \in N$. It is also possible to define personalized sets of possible messages. Some interesting ones can be defined by eliminating from the previous examples of common sets the parts of the messages consisting on own evaluations. This would lead to the following personalized sets of possible messages:

(*i*) the set of agents different from oneself, $\mathcal{M}_i = N \setminus \{i\}$, as in the case of nomination rules without self nomination (see Holzman and Moulin, 2013).

(*ii*) the set of subsets of $N \setminus \{i\}$, $\mathcal{M}_i = 2^{N \setminus \{i\}}$, in which each agent approves some candidates different from herself.

(*iii*) rankings of the remaining agents, $\mathcal{M}_i = \mathcal{R}_{|N \setminus \{i\}}$.

(*iv*) utility functions, $\mathcal{M}_i = \mathcal{U}_{|N \setminus \{i\}}$, where $\mathcal{U}_{|N \setminus \{i\}}$ is the set of functions $u_i : N \setminus \{i\} \to \mathbb{R}_+$.

One of the innovations of our approach is that all of our results, except Theorem 4, work for any cartesian product of sets of possible messages.

We now introduce additional notation. Given a permutation σ of the set of agents N and a profile of messages m, we denote $m_\sigma = (m_{\sigma(1)}, \ldots, m_{\sigma(n)})$ as a profile of messages where agent i has the message agent $\sigma(i)$ had in m. That is, $\sigma(i)$ is the agent whose message in m is agent i's message in m_σ. Observe that m_σ does not necessarily belong to $\times_{i \in N} \mathcal{M}_i$ when the sets of messages are personalized. Given a permutation σ of the set of agents N and a ranking R, we define $\sigma(R)$ as a ranking such that $\sigma(i)\, \sigma(R)\, \sigma(j)$ whenever $i\, R\, j$ for all $i, j \in N$. Finally, given a set of agents $C \subset N$ and a profile of messages $m \in \times_{i \in N} \mathcal{M}_i$, we denote by m_C and m_{-C} the subprofiles of messages of the agents of C and $N \setminus C$, respectively. We will denote m_{-i} instead of $m_{-\{i\}}$.

The solution concept in this paper is a *social ranking function*, which is a mapping $f : \times_{i \in N} \mathcal{M}_i \to \mathcal{R}$. We say that f has *full range* if for any $R \in \mathcal{R}$, there exist $m \in \times_{i \in N} \mathcal{M}_i$ such that $f(m) = R$. Similarly, we say that f is *constant* if $f(m) = f(m')$ for all $m, m' \in \times_{i \in N} \mathcal{M}_i$. We are now ready to introduce the main axiom of this paper, Impartiality.

Definition 1 *A social ranking function f is Impartial if for all $i \in N$, all $m \in \times_{i \in N} \mathcal{M}_i$ and all $m'_i \in \mathcal{M}_i$, $U(f(m), i) = U(f(m'_i, m_{-i}), i)$.*

Impartiality requires that if an agent changes her message, ceteris paribus, her outcome, understood as which agents are socially ranked higher than her and which don't, does not change. Then, under Impartiality, the message of an agent could change the ranking of the agents that are in her upper (respectively, lower) contour set between them, but cannot pass an agent from her upper contour set to the lower one, or vice versa. These upper and lower contour sets should be determined, therefore, by the messages of the other agents.

3 An impossibility result

Social choice theory normally defines two symmetry axioms, anonymity and neutrality, stating respectively that voters and candidates should be treated symmetrically in the aggregation procedure. Since here voters and candidates are the same, this idea of symmetry across agents can be explained by only one axiom called Name Independence (see Mackenzie, 2015, for a similar idea in social choice functions).[5] This axiom states that a social ranking function should not depend on the name of each agent. Since our framework allows any structure of the set of possible messages of each agent, our version of this property needs to be very general and applicable to all of them. We start introducing the formal definition of the property and, after that, we explain how it can be applied to different cartesian products of sets of possible messages.

Definition 2 *A social ranking function f is Name Independent if for any pair $\{i, j\} \subseteq N$ and any permutation $\sigma : N \to N$ such that $\sigma(k) = k$ for each $k \in N \setminus \{i, j\}$, there is a mapping $\gamma_\sigma : \bigcup_{k \in N} \mathcal{M}_k \to \bigcup_{k \in N} \mathcal{M}_k$ such that*

- *$f(m) = \sigma(f(\gamma_\sigma(m_{\sigma(1)}), \ldots, \gamma_\sigma(m_{\sigma(n)})))$ for each $m \in \times_{i \in N} \mathcal{M}_i$; and*
- *$\gamma_\sigma(m_{\sigma(k)}) \in \mathcal{M}_k$ for each $k \in N$ and each $m_{\sigma(k)} \in \mathcal{M}_{\sigma(k)}$.*

The axiom states that, for any pair of agents i and j, the outcome of i in a message profile m should be the same as the outcome of j if we jointly permute the identity of voters i and j with the permutation σ and the messages of all agents with the permutation γ_σ. Observe that, meanwhile σ permutes the roles of i and j as voters, the objective of γ_σ is to permute the roles of i and j as candidates in the messages of all agents. This implication on f is the first point of the definition. Since the sets of possible messages could be personalized in

[5] See also the use of this property in Mackenzie (2020) for an interesting analysis of the papal conclave.

some applications, we also need to impose that each one of the new messages obtained after the permutation belongs to the set of admissible messages of the individual that transmit it in the new message profile; i.e., $\gamma_\sigma(m_{\sigma(k)})$ should belong to $\mathcal{M}_k$. This is exactly the second point of the definition.

Since the formal definition of the property can be a bit complicated, let us now discuss how this definition can be interpreted for different sets of possible messages. It is important to mention first that, although the axiom only requires the existence of one mapping γ_σ satisfying the two parts of the definition of the property, each configuration of the sets of possible messages of each agent has an associated natural definition of γ_σ. To see it, consider the following examples, all of them constructed assuming that $N = \{1, 2, 3\}$, $i = 1$ and $j = 2$.

Example 1 Consider that, for each $k \in N$, $\mathcal{M}_k = N \setminus \{k\}$, as in the case of nomination rules without self nomination (see Holzman and Moulin, 2013). In that case, the natural definition of γ_σ is to make it equal to σ. Observe that this guarantees the second point of the definition; that is, $\gamma_\sigma(m_{\sigma(k)}) \in N \setminus k = \mathcal{M}_k$ for all $k \in N$ and all $m_{\sigma(k)} \in \mathcal{M}_{\sigma(k)}$. To analyze the first point of the axiom for this specification of γ_σ, consider the message profile $m = (3, 1, 1)$. Then, the message profile $(\gamma_\sigma(m_{\sigma(1)}), \gamma_\sigma(m_{\sigma(2)}), \gamma_\sigma(m_{\sigma(3)}))$ is such that agent 1 nominates $\gamma_\sigma(m_{\sigma(1)}) = \gamma_\sigma(m_2) = \gamma_\sigma(1) = \sigma(1) = 2$, agent 2 nominates $\gamma_\sigma(m_{\sigma(2)}) = \gamma_\sigma(m_1) = \gamma_\sigma(3) = \sigma(3) = 3$ and agent 3 nominates $\gamma_\sigma(m_{\sigma(3)}) = \gamma_\sigma(m_3) = \gamma_\sigma(1) = \sigma(1) = 2$. Then, the outcome of for example agent 1 in a name independent social ranking function with the message profile $m = (3, 1, 1)$ should be the same as the outcome of agent $\sigma(1) = 2$ with the message profile $(\gamma_\sigma(m_{\sigma(1)}), \gamma_\sigma(m_{\sigma(2)}), \gamma_\sigma(m_{\sigma(3)})) = (2, 3, 2)$. That is, if for instance agent 1 is ranked first in the social ranking in the message profile $(3, 1, 1)$, then agent 2 is also ranked first in the social ranking if the message profile is $(2, 3, 2)$.

Observe that this version of our axiom for social ranking functions is the adaptation of the one that Mackenzie (2015) states for social choice functions. There are only two minor differences. First, Mackenzie (2015) allows for any permutation σ of N, while we impose the condition only for those permutations that swap two agents. This difference is innocuous since requiring the condition only for swaps ensure the condition for any permutation. Second, meanwhile Mackenzie (2015) states the implication of the axiom for the specific case $\gamma_\sigma = \sigma^{-1}$, our version only requires that such implication occurs for some γ_σ.[6] Then, our axiom is, in logical terms, weaker than the direct adaptation of the one of Mackenzie (2015). We cannot be more specific in our definition since we pretend to introduce a valid definition for all possible sets of possible messages of each agent and not only for nominations. Given that, as it will be seen later on, our definition is going to generate impossibility results, the adoption of stronger versions of the property will not change them. □

Example 1 has been explained for a personalized set of messages in which each agent should nominate other agents different from herself. The analysis would be the same if

[6] Observe that, for the case of swaps, $\sigma^{-1} = \sigma$.

a common set of messages in which self-nominations are possible is assumed. We now introduce another example with a common set of possible messages.

Example 2 Consider that, for each $k \in N$, $\mathcal{M}_k = \mathcal{R}$. In that case, the natural definition of γ_σ would be the following: $\gamma_\sigma(R) = R'$ such that for all $k, l \in N$, $k\, R\, l \Leftrightarrow \sigma(k)\, R'\, \sigma(l)$. Since the set of possible messages is common, the second point of the definition of the axiom is satisfied; that is, $\gamma_\sigma(R_{\sigma(k)}) \in \mathcal{R} = \mathcal{M}_k$ for all $k \in N$ and all $m_{\sigma(k)} \in \mathcal{R} = \mathcal{M}_{\sigma(k)}$. To see the implication of the first point of the axiom for this specification of γ_σ, consider a message profile such that $1\, R_1\, 2\, R_1\, 3$, $1\, R_2\, 3\, R_2\, 2$, and $2\, R_3\, 3\, R_3\, 1$. Then, the message profile $(\gamma_\sigma(R_{\sigma(1)}), \gamma_\sigma(R_{\sigma(2)}), \gamma_\sigma(R_{\sigma(3)})) = (\gamma_\sigma(R_2), \gamma_\sigma(R_1), \gamma_\sigma(R_3)) = (R'_1, R'_2, R'_3)$ is such that $2\, R'_1\, 3\, R'_1\, 1$, $2\, R'_2\, 1\, R'_2\, 3$ and $1\, R'_3\, 3\, R'_3\, 2$. Then, the outcome of for example agent 1 in a name independent social ranking function with the message profile (R_1, R_2, R_3) should be the same as the outcome of agent $\sigma(1) = 2$ with the message profile (R'_1, R'_2, R'_3).

Again, our axiom does not specify that this is exactly the structure of the mapping γ_σ and only requires the existence of some γ_σ satisfying the conditions. The reason is, as we have commented in Example 1, that we pretend to have a definition that is valid not only for this particular set of possible messages of each agent but for any set of possible messages one can consider. However, the specification of any particular γ_σ would only strengthen the axiom, which does not change the impossibility results. $\square$

The last example shows the interpretation of this axiom for the case of rating functions.

Example 3 Consider that, for each $k \in N$, $\mathcal{M}_k = \mathcal{U}$, where $\mathcal{U}$ is the set of functions $u : N \to \mathbb{R}_+$. In that case, the natural definition of γ_σ would be the following: $\gamma_\sigma(u) = u'$ such that for all $k \in N$, $u'(k) = u(\sigma(k))$. Since the set of possible messages is common, the second point of the definition of the axiom is satisfied; that is, $\gamma_\sigma(m_{\sigma(k)}) \in \mathcal{U} = \mathcal{M}_k$ for all $k \in N$ and all $m_{\sigma(k)} \in \mathcal{U} = \mathcal{M}_{\sigma(k)}$. To see the implication of the first point of the axiom for this specification of γ_σ, consider any utility profile $(u_1, u_2, u_3) \in \mathcal{U}$. Then, the message profile $(\gamma_\sigma(u_{\sigma(1)}), \gamma_\sigma(u_{\sigma(2)}), \gamma_\sigma(u_{\sigma(3)})) = (\gamma_\sigma(u_2), \gamma_\sigma(u_1), \gamma_\sigma(u_3)) = (u'_1, u'_2, u'_3)$ is such that, for each $k, l \in N$, $u'_k(l) = u'_{\sigma(k)}(\sigma(l))$. Then, as in the previous examples, the outcome of for example agent 1 in a name independent social ranking function with the message profile (u_1, u_2, u_3) should be the same as the outcome of agent $\sigma(1) = 2$ with the message profile (u'_1, u'_2, u'_3).

As in the previous examples, our axiom does not specify that this is exactly the structure of the mapping γ_σ since the axiom must be weak enough to be applied to any sets of possible messages of each agent. $\square$

Although Name Independence is a quite natural axiom, our first result shows that it is incompatible with Impartiality.

Theorem 1 *There is no social ranking function satisfying Impartiality and Name Independence.*

Observe that the impossibility holds in a very general setting that allows for any definition of the set of possible messages of each agent and for a very weak version of Name Independence in which we have given full flexibility in the choice of γ_σ. The proof of this impossibility rests on the facts that, on the one hand, Name Independence implies that all rankings of $\mathcal{R}$ should be in the range of the social ranking function and, on the other hand, Impartiality implies that some rankings should not be feasible. In fact, Impartiality does not only imply that full range is not possible, but also (see Lemma 1 in the Appendix) that all agents that are situated in the first position in a ranking that appears in the range of the social ranking function never appears last in any other ranking of the range of this function.

4 Possibility results

We have seen that Impartiality is incompatible with Name Independence. However, there exist social ranking functions satisfying Impartiality, like the constant ones. To present other non-constant social ranking function satisfying Impartiality, suppose that $N = \{1, 2, 3\}$ and let f be such that 1 and 2 are always above 3 (that is, 1 and 2 occupy always the first two positions and 3 the last one). To choose which agent occupies the first position in each profile of messages, f applies a mapping $f^{\{3\}}_{\{1,2\}}$ by which only the message of agent 3 is considered to rank agents 1 and 2. Observe that f is impartial, since the messages of agents 1 and 2 do not affect the social ranking and the message of agent 3 only determines the social ranking of the other pair of agents that are both always placed above 3. We now define a family of social ranking functions inspired in this example and that satisfy Impartiality. To do that, we introduce some notation and definitions.

We define an *ordered partition of a set of agents* $A \subseteq N$, with $|A| > 1$, as an ordered set of groups of agents $(A_1, \ldots, A_t)$ such that $t > 1$, $A_i \neq \emptyset$ for all $i \in \{1, \ldots, t\}$, $\bigcup_{i=1}^t A_i = A$ and $A_i \cap A_j = \emptyset$ for all $i, j \in \{1, \ldots, t\}$. Observe that an ordered partition of a set of agents can be interpreted as a partial order of the candidates of A: the agents of A_1 are ranked higher from those of A_2, the agents of A_2 are ranked higher from those of A_3, and so on, but there is no comparison established from agents of the same set of the ordered partition. We denote, for any $A, B \subseteq N$, by $f^B_A : \times_{i\in B}\mathcal{M}_i \to \mathcal{R}_{|A}$ a function that aggregates the messages of the agents of B and determines a ranking of the agents of A.

Definition 3 *A social ranking function f is a* partition ordering *if there is an ordered partition of N, $(N_1, \ldots, N_t)$, and a set of functions $f^{N\setminus N_1}_{N_1}, f^{N\setminus N_2}_{N_2}, \ldots, f^{N\setminus N_t}_{N_t}$ such that for any $i, j \in N$, with $i \in N_p$ and $j \in N_q$, and for any $m \in \times_{k\in N}\mathcal{M}_k$:*

$$i \in U(f(m), j) \text{ if } p < q \text{ or } [p = q \text{ and } i \in U(f^{N\setminus N_p}_{N_p}(m_{-N_p}), j].$$

Observe that the previous example is included in the family of partition orderings: it corresponds with $N_1 = \{1, 2\}$ and $N_2 = \{3\}$. Similarly, any constant social ranking function is also included in this family by fixing $t = n$. All partition orderings are impartial. More interestingly, the following theorem shows that the partition orderings are the unique social ranking functions satisfying Impartiality if the number of agents is lower than 4.

Proposition 1 *Let $n < 4$. Then, a social ranking function f satisfies Impartiality if and only if f is a partition ordering.*

If the number of agents is higher than 3, there are other impartial social ranking functions different from the partition orderings. For example, suppose that $N = \{1, 2, 3, 4\}$ and consider a social ranking function f that works as follows. First, agent 4 is always ranked last and her message determines who is ranked third. Second, the messages of the agents that have been determined to occupy the last two positions (that is, 4 and the one 4 has determined to be third) are aggregated to obtain who is in first position and who is in second position. Observe that f is impartial since the message of each agent is only considered to determine the ranking of the agents that are going to be situated higher than her. However, it is not a partition ordering because, although all agents of $N_1 = \{1, 2, 3\}$ are always ranked higher than $N_2 = \{4\}$, the ranking of the agents of N_1 does not depend exclusively on the message of the agent outside N_1.

This example is going to inspire us to define a bigger set of impartial social ranking functions when there are more than 3 agents. The social ranking functions of this family first partition the set of agents into some subsets, $N_1, \ldots, N_t$, such that independently of the messages, the agents of any N_p are always ranked higher than the agents of any N_q with $p < q$. The difference with respect to the partition orderings is the determination of the ranking of the agents inside each N_p. Observe that if we fix the messages of the agents outside N_p, we have a subproblem in which the messages of the agents of N_p should decide a ranking of them. Then, we could apply any impartial subrule and this is what this family of social ranking functions do. Exactly, each social ranking function f in this subclass defines for each message subprofile of the agents outside N_p, m_{-N_p}, an impartial function, $f_{m_{-N_p}}$, that determines the final ranking of the agents of N_p for each message subprofile of the agents inside N_p, m_{N_p}.

Definition 4 *A social ranking function f is an* impartial partition ordering *if there is an ordered partition of N, $(N_1, \ldots, N_t)$, and a set of impartial functions $f_{m_{-N_p}} : \times_{i \in N_p} \mathcal{M}_i \to \mathcal{R}_{|N_p}$, one for each set N_p of the ordered partition and each $m_{-N_p} \in \times_{i \in N \setminus N_p} \mathcal{M}_i$, such that for any $i, j \in N$, with $i \in N_p$ and $j \in N_q$, and any $m \in \times_{k \in N} \mathcal{M}_k$:*

$$i \in U(f(m), j) \text{ if } p < q \text{ or } [p = q \text{ and } i \in U(f_{m_{-N_p}}(m_{N_p}), j)].$$

All partition orderings are impartial partition orderings. In particular, they correspond with the cases in which all the functions $f_{m_{-N_p}}$ are constant. However, the selection of other impartial non-constant function $f_{m_{-N_p}}$ allows to define impartial partition orderings that are not partition orderings.

It can be seen that all social ranking functions of the family of impartial partition orderings satisfy Impartiality. We now introduce another property satisfied by this family. In Section

3, we have shown that Name Independence implies full range of the social ranking function and this directly creates an impossibility with Impartiality. Then, to obtain some possibility results, it is necessary to relax Name Independence. We opt to propose a property that, in such a way, maintains the spirit of Name Independence in the maximum possible extent and, at the same time, does not necessarily imply full range of the social ranking function: the property requires that the social ranking function is symmetric with respect to any pair of agents that the function ranks in different ways in rankings of its range. The idea is that, since the impossibility comes because some rankings are not possible by Impartiality, we have to apply the full force of Name Independence only to those pairs of agents whose comparison is not fixed by the force of the Impartiality axiom.

To define the weaker version of Name Independence, we need the following concept. A pair of agents $\{i, j\} \subseteq N$ is a *free pair at* f if there exist $m, m' \in \times_{k\in N}\mathcal{M}_k$ such that $i \in U(f(m), j)$ and $j \in U(f(m'), i)$. That is, two agents constitute a free pair at a social ranking function if it is possible that the function ranks one above the other and the other way around. As examples, all pairs of agents are free at any name independent social ranking function, while there are no free pairs at constant social ranking functions. Using this concept, we can define the axiom of Weak Name Independence.

Definition 5 *A social ranking function f is Weakly Name Independent if for any of its free pairs $\{i, j\} \subseteq N$ and any permutation $\sigma : N \to N$ such that $\sigma(k) = k$ for each $k \in N \setminus \{i, j\}$, there is a mapping $\gamma_\sigma : \bigcup_{k\in N} \mathcal{M}_k \to \bigcup_{k\in N} \mathcal{M}_k$ such that*

- *$f(m) = \sigma(f(\gamma_\sigma(m_{\sigma(1)}), \ldots, \gamma_\sigma(m_{\sigma(n)})))$ for each $m \in \times_{k\in N}\mathcal{M}_k$; and*
- *$\gamma_\sigma(m_{\sigma(k)}) \in \mathcal{M}_k$ for each $k \in N$ and each $m_{\sigma(k)} \in \mathcal{M}_{\sigma(k)}$.*

The property of Weak Name Independence applies the symmetry idea of Name Independence only to those pairs of agents that are free in the social ranking function. Observe that, although the axiom is weaker than Name Independence, it does not preclude that some pairs are not free. It allows to have any set of free pairs, but at the same time does not force to have all pairs as free. It is a weakening of the original axiom thought to maintain its spirit at the maximum possible extent combined with other axioms, like Impartiality, that restrict the range.

The following theorem shows that all social ranking functions satisfying Impartiality and Weak Name Independence are impartial partition orderings.

Theorem 2 *A social ranking function f satisfies Impartiality and Weak Name Independence if and only if f is an impartial partition ordering.*

Observe that the family of impartial partition orderings incorporate Impartiality as a property in its own definition. It says that, fixing the messages of the agents outside a set of the ordered partition, the ranking of the agents inside the set should be done aggregating the messages of the agents inside the set with a subrule that is impartial. Thus, the class of impartial partition ordering is not closely defined and requires a recursive argument. We proceed now to introduce some concepts and notation that we need to define a subfamily with a definition in a closed form.

First, we introduce the concept of a *social ranking partition with set of candidates* $A \subseteq N$ as a mapping $g_A : \times_{i \in N \setminus A} \mathcal{M}_i \to \mathcal{R}^*_A$ being $\mathcal{R}^*_A$ the set of all possible ordered partitions of a set of agents A. That is, any g_A establishes an ordered partition of the set of candidates A using the messages of the agents outside the set A.

Second, we introduce some additional notation. Suppose that we have an initial ordered partition of N, say $(N_1, \ldots, N_t)$, and for each $A \subseteq N_p$, with $p \in \{1, \ldots, t\}$, we have a social ranking partition g_A. Then, for any message profile $m \in \times_{i \in N} \mathcal{M}_i$ and any N_p with $|N_p| > 1$, we use $g_{N_p}(m_{-N_p})$ to partition N_p in smaller subsets ordered as $(N_{(p,1)}, \ldots, N_{(p,v)})$. We denote by $N_{(p,s)}(m)$ the s-th subset of agents of the partition of N_p implemented by $g_{N_p}(m_{-N_p})$. If any of these subsets $N_{(p,s)}(m)$ is not a singleton yet, then we use $g_{N_{(p,s)}}(m_{-N_{(p,s)}})$ to partition it into small subsets and ordered as $(N_{(p,s,1)}, \ldots, N_{(p,s,z)})$. In that case, we denote by $N_{(p,s,u)}(m)$ the u-th subset of agents of the partition of $N_{(p,s)}$ implemented by $g_{N_{(p,s)}}(m_{-N_{(p,s)}})$. Proceeding similarly and in a recursive way, given an initial ordered partition $(N_1, \ldots, N_s)$ and a social ranking partition g_A for each $A \subseteq N_p$, with $p \in \{1, \ldots, t\}$, we assign for each message profile $m \in \times_{i \in N} \mathcal{M}_i$ an identification of each agent $k \in N$ as a vector of numbers $\vec{n}_k(m)$ such that $N_{\vec{n}_k(m)} = \{k\}$.

Finally, we denote by $<^L$ the lexicographic order of vectors of natural numbers such that $\vec{n} <^L \vec{n'}$ if there is $l \in \mathbb{N}$ such that $n_i = n'_i$ for all $i < l$ and $n_l < n'_l$.

We are now ready to introduce the following family of social ranking functions:

Definition 6 *A social ranking function f is a* sequential impartial partition ordering *if there is an ordered partition of N, $(N_1, \ldots, N_t)$, and a social ranking partition g_A for each $A \subseteq N_p$ and for any $p \in \{1, \ldots, t\}$, such that for any $i, j \in N$ and any $m \in \times_{k \in N} \mathcal{M}_k$:*

$$i \in U(f(m), j) \Leftrightarrow \vec{n}_i(m) <^L \vec{n}_j(m).$$

Note that, like impartial partition orderings, sequential impartial partition orderings coincide with partition orderings in the fact that all of them initially partition the set of agents in such a way that all agents belonging to N_p are always ranked above any agent of N_q for all $q > p$. However, both classes differ from partition orderings in that the ranking of the agents inside N_p may not only depend on the messages of the agents outside that set but also on those of the agents inside N_p. In particular, for the sequential impartial partition orderings, the messages of the agents outside N_p only partition this set into ordered subsets such that the agents of $N_{(p,i)}$ are going to be ranked above the agents of $N_{(p,j)}$ with $i < j$, but the order of the agents of the same subset $N_{(p,i)}$ are not decided by only the messages of the agents outside N_p, but also by the messages of the agents of other subsets of N_p different from $N_{(p,i)}$. Then, this process is replicated iteratively until all subsets are singletons and a complete ranking has been reached.

Observe also that the sequential impartial partition orderings are a subfamily of the impartial partition orderings. Then, they satisfy Impartiality and Weak Name Independence. In fact, they also satisfy a property stronger than Weak Name Independence. This property is going to require that when we fix the messages of some agents and we have a subproblem in which the messages of the remaining agents should decide a ranking of them, the subrule used satisfies Weak Name Independence. To define the property in formal terms, we need additional notation. We say that a pair of agents $\{i, j\} \subseteq N$ is a *free pair at f for the message subprofile $m_A \in \times_{k\in A}\mathcal{M}_k$*, with $i, j \notin A$, if there exist $m'_{-A}, m''_{-A} \in \times_{k\in N\setminus A}\mathcal{M}_k$ such that $j \in U(f(m_A, m'_{-A}), i)$ and $i \in U(f(m_A, m''_{-A}), j)$. Observe that, when $A = \emptyset$, the definition of being a free pair for a message subprofile coincides with our previous definition of being a free pair. We are now ready to introduce the new property.

Definition 7 *A social ranking function f is Consistently Weak Name Independent if for any $A \subseteq N$, any $m_A \in \times_{k\in A}\mathcal{M}_k$, any free pair $\{i, j\} \subseteq N$ at f for message subprofile m_A and any permutation $\sigma : (N\setminus A) \to (N\setminus A)$ such that $\sigma(k) = k$ for each $k \in N\setminus(A\cup\{i, j\})$, there is a mapping $\gamma_\sigma : \bigcup_{k\in N\setminus A}\mathcal{M}_k \to \bigcup_{k\in N\setminus A}\mathcal{M}_k$ such that*

- $f(m)_{|(N\setminus A)} = \sigma(f(m_A, \gamma_\sigma(m_{\sigma(N\setminus A)})))_{|(N\setminus A)}$ *for each $m \in \times_{k\in N}\mathcal{M}_k$; and*[7]
- $\gamma_\sigma(m_{\sigma(k)}) \in \mathcal{M}_k$ *for each $k \in N \setminus A$ and each $m_{\sigma(k)} \in \mathcal{M}_{\sigma(k)}$.*

The following theorem shows that any social ranking function satisfying Impartiality and Consistent Weak Name Independence is a sequential impartial partition ordering.

Theorem 3 *If a social ranking function f satisfies Impartiality and Consistent Weak Name Independence, then f is a sequential impartial partition ordering.*

Then, to obtain the characterization of all social ranking functions satisfying Impartiality and Consistent Weak Name Independence, we need to impose additional conditions on the structure of the set of social ranking partitions. For the sake of simplicity, we consider the case of a common set of messages, $\mathcal{M}_i = \mathcal{M}_j = \mathcal{M}$ for any $i, j \in N$.

Definition 8 *A social ranking partition g_A, with $A \subseteq N$, is* neutral *if for any permutation $\sigma : A \to A$, there is a permutation $\gamma_\sigma : \mathcal{M} \to \mathcal{M}$ such that*

$$g_A(m_{-A}) = \sigma(g_A(\gamma_\sigma(m_{-A}))) \text{ for each } m_{-A} \in \times_{i\in N\setminus A}\mathcal{M}.$$

We now define $\mathcal{R}^{**}_A$ the set of all possible ordered partitions of a set of agents A formed by two subsets, at least one being a singleton. Formally, $\mathcal{R}^{**}_A = \{(A_1, A_2) \in \mathcal{R}^*_A \,|\, |A_1| = 1 \text{ or } |A_2| = 1\}$. We require that, for each $A \subseteq N_p$, the range of g_A should be a (not necessarily proper) subset of $\mathcal{R}^{**}_A$.

[7] $(m_A, \gamma_\sigma(m_{\sigma(N\setminus A)}))$ denotes the message profile in which all agents of A maintains the same message as in m and the messages of the remaining agents have been permuted in such a way that the message of an agent $k \in N \setminus A$ in this profile is $\gamma_\sigma(m_{\sigma(k)})$.

Definition 9 *Let $A \subseteq N$ and $i, j \in N \setminus A$. The social ranking partitions $g_{A\cup\{i\}}$ and $g_{A\cup\{j\}}$ are* coherent *if for any bijective mapping $\sigma : A \cup \{i\} \to A \cup \{j\}$ such that $\sigma(k) = k$ for each $k \in A$, we have that for any message subprofile $m_{-(A\cup\{j\})} \in \times_{i\in N\setminus(A\cup\{j\})}\mathcal{M}$,*

$$g_{A\cup\{j\}}(m_{-(A\cup\{j\})}) = \sigma(g_{A\cup\{i\}}(m_{-(A\cup\{i,j\})}, m_{\sigma(i)})).^{8}$$

A natural question that arises from Theorems 2 and 3 is if there are other impartial social ranking functions apart from those that satisfy Weak Name Independence when the number of agents is higher than 3. It can be shown that, even in the case with four alternatives and a binary common message space for all agents, there are other impartial social ranking functions that do not belong to the family of impartial partition orderings.

Example 4 Suppose that $N = \{1, 2, 3, 4\}$ and consider that $\mathcal{M}_i = \{Y, N\}$ for all $i \in N$. The range of the social ranking function that we are going to construct has 6 strict rankings: $1\,R_1\,4\,R_1\,2\,R_1\,3$, $2\,R_2\,1\,R_2\,4\,R_2\,3$, $1\,R_3\,2\,R_3\,4\,R_3\,3$, $2\,R_4\,1\,R_4\,3\,R_4\,4$, $1\,R_5\,3\,R_5\,2\,R_5\,4$, and $1\,R_6\,2\,R_6\,3\,R_6\,4$. The social ranking function is such that:

- If $m_1 = m_3 = N$, then $f(m) = R_1$.
- If $m_1 = N$ and $m_3 = m_4 = Y$, then $f(m) = R_2$.
- If $m_1 = m_4 = N$ and $m_3 = Y$, then $f(m) = R_3$.
- If $m_1 = m_3 = m_4 = Y$, then $f(m) = R_4$.
- If $m_1 = Y$ and $m_4 = N$, then $f(m) = R_5$.
- Otherwise, $f(m) = R_6$.

It can be shown that f satisfies Impartiality, but not Weak Name Independence. □

5 Anonymity

Theorem 1 and, in particular, Lemma 1 implies that under Impartiality the range of the social ranking function is very limited. In particular, agents are partitioned into at least two groups: those that can obtain the first social position, and those that can be last ranked. As a consequence, we have shown that this property is incompatible with Name Independence or Candidate Neutrality.

Other important implication of Impartiality is, as Holzman and Moulin (2013) shows for social choice functions in which the set of possibles messages of each agent i is $N \setminus \{i\}$, its incompatibility with the classical property of Anonymity except in the constant rules. The

[8] $(m_{-(A\cup\{i,j\})}, m_{\sigma(i)})$ denotes the message subprofile of agents $N \setminus (A \cup \{i\})$ in which all agents of $N \setminus (A \cup \{i, j\})$ maintains the same message as in $m_{-(A\cup\{j\})}$ and the message of agent j is the message of agent i in the subprofile $m_{-(A\cup\{j\})}$.

reasoning behind this result is the following. Impartiality implies that the message of each agent cannot affect her own outcome and Anonymity implies that all messages are equally relevant for the outcome of each agent. Then, it can be deduced that no message can affect the outcome of any agent and the range of any impartial and anonymous function is a singleton. This result can be generalized to social ranking functions. In fact, it is possible to state an stronger and less obvious result. If Impartiality is imposed, instead of requiring that all messages are equally relevant for the outcome of each agent i, as Anonymity does, it has much more sense to require that this is the case for all messages except that of i (since Impartiality has imposed that this message is irrelevant for i's outcome). We call this axiom Weak Anonymity. To define it formally, we need to apply a permutation σ to the message profile m that maintains invariant the message of i and to require that the outcome of i with the social ranking function does not change. However, remember that, although $m \in \times_{i\in N}\mathcal{M}_i$, it is possible that $m_\sigma \notin \times_{i\in N}\mathcal{M}_i$ if the sets of messages are personalized. In fact, it could be the case that $\mathcal{M}_j \cap \mathcal{M}_k = \emptyset$ for all $j, k \in N \setminus \{i\}$ and, in that case, the property could not be defined. Therefore, it is necessary to define the property only for some cartesian products of sets of possible messages in which the message profiles can be permuted. In our result, we consider the case of common sets of possible messages.

Definition 10 *Suppose that $\mathcal{M}_i = \mathcal{M}_j$ for all $i, j \in N$. Then, we say that a social ranking function f is Weak Anonymous if for all $i \in N$, all $m \in \times_{i\in N}\mathcal{M}_i$, and all permutations σ of N with $\sigma(i) = i$, $U(f(m), i) = U(f(m_\sigma), i)$.*

The combination of this axiom with Impartiality also leads to the constant social ranking functions.

Theorem 4 *Suppose that $\mathcal{M}_i = \mathcal{M}_j$ for all $i, j \in N$ and let f be an impartial and weakly anonymous social ranking function. Then, f is constant.*

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Appendix

Proof of Proposition 1

We define $U(f) = \{i \in N : \exists m \in \times_{i \in N} \mathcal{M}_i$ such that $U(f(m), i) = \emptyset\}$ and $L(f) = \{i \in N : \exists m \in \times_{i \in N} \mathcal{M}_i$ such that $U(f(m), i) = N \setminus \{i\}\}$. That is, $U(f)$ (respectively, $L(f)$) is the set of agents that are in the first (respectively, last) position in the social ranking for at least one profile of messages. The proof is structured in two lemmas. The first lemma shows that Impartiality implies that there is no agent that belongs to both $U(f)$ and $L(f)$.

Lemma 1 *Let f be an impartial social ranking function. Then, $U(f) \cap L(f) = \emptyset$.*

Proof: Suppose by contradiction that there is an agent $i \in U(f) \cap L(f)$. Let $m = (m_1, \ldots, m_n)$ and $m' = (m'_1, \ldots, m'_n)$ be such that $U(f(m), i) = \emptyset$ and $U(f(m'), i) = N \setminus \{i\}$. Starting at m, we construct a sequence of message profiles such that each message profile is obtained changing from the previous message profile one agent's message from the one in m to the one in m' in the following order: we select in each step the agent who is ranked highest in the outcome by f in the last profile (among those agents whose message has not still changed). Observe that this sequence ends at message profile m'.

Since $U(f(m), i) = \emptyset$, the first two profiles of the sequence are m and (m'_i, m_{-i}). Since $U(f(m), i) = \emptyset$, we obtain by Impartiality that $U(f(m'_i, m_{-i}), i) = \emptyset$. Then, we now change the message of agent $j \in N$ such that $U(f(m'_i, m_{-i}), j) = \{i\}$. That is, the third profile of the sequence is $(m'_{\{i,j\}}, m_{-\{i,j\}})$. Since $U(f(m'_i, m_{-i}), j) = \{i\}$, we obtain by Impartiality that $U(f(m'_{\{i,j\}}, m_{-\{i,j\}}), j) = \{i\}$. We now change the message of agent k such that $U(f(m'_{\{i,j\}}, m_{-\{i,j\}}), k) = \{i, j\}$ and, then, the next profile of the sequence is $(m'_{\{i,j,k\}}, m_{-\{i,j,k\}})$. Since $U(f(m'_{\{i,j\}}, m_{-\{i,j\}}), k) = \{i, j\}$, applying Impartiality again we have that $U(f(m'_{\{i,j,k\}}, m_{-\{i,j,k\}}), k) = \{i, j\}$. Continuing this procedure we will obtain that $i \in U(f(m'), l)$ for some $l \in N$, which contradicts that $U(f(m'), i) = N \setminus \{i\}$. $\square$

The second lemma shows that Name Independence implies full range.

Lemma 2 *Let f be a social ranking function that satisfies Name Independence (or Candidate Neutrality). Then, f has full range.*

Proof: We do the proof only for Name Independence (the proof with Candidate Neutrality is similar). Consider any ranking $R \in \mathcal{R}$ and we have to show that R belongs to the range of f. Consider any $m \in \times_{i \in N} M_i$ and denote $f(m)$ by R'. Construct a permutation σ of N such that $|\{k \in N \mid k\, R\, i\}| = |\{k \in N \mid k\, R'\, \sigma(i)\}|$ for all $i \in N$. Observe that, by construction, $R' = \sigma(R)$. By Name Independence, there exists a mapping γ_σ such that $\gamma_\sigma(m_i) \in \mathcal{M}_{\sigma(i)}$ for all $i \in N$, and $f(m) = \sigma(f(\gamma_\sigma(m_\sigma)))$. Then, since $f(m) = R'$, we should have that $R = f(\gamma_\sigma(m_\sigma))$ and, thus, R belongs to the range of f. $\square$

It is obvious that the union of Lemmas 1 and 2 implies the impossibility of combining Impartiality with Name Independence (or Candidate Neutrality).

Proof of Theorem 1

We know by Lemma 1 that Impartiality implies that $U(f) \cap L(f) = \emptyset$. Suppose first that $n = 2$. Given that $U(f) \neq \emptyset \neq L(f)$, we have that $|U(f)| = |L(f)| = 1$. Thus, the unique impartial social ranking functions are the constant ones, which are partition orderings: they correspond with $s = 2$, $N_1 = U(f)$ and $N_2 = L(f)$.

Suppose now that $n = 3$. Then, either $|L(f)| = 1$ or $|U(f)| = 1$. Consider first the cases in which $|L(f)| = |U(f)| = 1$. Then, the range of f is a unique ranking and the social ranking function f is constant. Thus, f is a partition ordering: it corresponds with $s = 3$, $N_1 = U(f)$, $N_2 = N \setminus (U(f) \cup L(f))$ and $N_3 = L(f)$. Consider finally the cases in which

$|L(f)| = 1$ and $|U(f)| = 2$ (the proof of the case when $|L(f)| = 2$ and $|U(f)| = 1$ is similar and thus omitted). Let $L(f) = \{i\}$ and $U(f) = \{j, k\}$. Then, the range of f includes two strict rankings: $j\, R^1\, k\, R^1\, i$ and $k\, R^2\, j\, R^2\, i$. By Impartiality, the choice between P^1 and P^2 should only depend on the message of agent i. Then, f is partition ordering that corresponds with $s = 2$, $N_1 = U(f)$, $N_2 = L(f)$ and $f_{N_1}^{N_2}$ be such that for each $m_i \in \mathcal{M}_i$, $f_{N_1}^{N_2}(m_i) = f(m_i, m_j, m_k)$ for any $m_j \in \mathcal{M}_j$ and any $m_k \in \mathcal{M}_k$.

Proof of Theorem 2

Each impartial partition ordering satisfies Impartiality and Weak Name Independence. Then, we concentrate on the sufficiency part.

By Lemma 1, $U(f) \neq N$. Denote $U(f)$ by N_1. We are going to show first that for any $i \in N_1$ and any $j \in N \setminus N_1$, we have that $i \in U(f(m), j)$ for all $m \in \times_{i \in N}\mathcal{M}_i$. Suppose otherwise that there exists $m \in \times_{i \in N}\mathcal{M}_i$ such that $j \in U(f(m), i)$. Since $i \in N_1$, there is a message profile $m' \in \times_{i \in N}\mathcal{M}_i$ such that $U(f(m'), i) = \emptyset$ and, then, we can deduce that $\{i, j\}$ is a free pair at f. Consider a permutation σ of N such that $\sigma(i) = j$, $\sigma(j) = i$ and $\sigma(k) = k$ for all $k \in N \setminus \{i, j\}$. Then, applying Weak Name Independence, there is a permutation σ_γ of $\bigcup_{i \in N} \mathcal{M}_i$ such that $\gamma_\sigma(m_i') \in \mathcal{M}_{\sigma(i)}$ for all $i \in N$, and $f(m') = \sigma(f(\gamma_\sigma(m_\gamma')))$. Then, since $U(f(m'), i) = \emptyset$ and $\sigma(j) = i$, we have that $U(f(\gamma_\sigma(m_\gamma')), j) = \emptyset$. This implies that $j \in N_1$, which is a contradiction.

We can now define N_2 as $N_2 = \{i \in N \mid \exists\, m \in \times_{i \in N}\mathcal{M}_i$ such that $U(f(m), i) = N_1\}$. In a similar way as in the previous paragraph, it is easy to prove that for any $i \in N_2$ and any $j \in N \setminus (N_1 \cup N_2)$, we have that $i \in U(f(m), j)$ for all $m \in \times_{i \in N}\mathcal{M}_i$. Proceeding in this way until finishing with all agents, we have constructed a partition of N, $\{N_1, \ldots, N_s\}$ such that for all $i \in N_p$ and $j \in N_q$, with $p < q$, $i \in U(f(m), j)$ for all $m \in \times_{i \in N}\mathcal{M}_i$. Observe also that if i and j belong to the same set of the partition, they are a free pair.

If $|N_1| = \ldots = |N_s| = 1$, then the proof is finished. Otherwise, consider a set N_p of the partition such that $|N_p| > 1$. Consider any set of messages of the agents outside this set, $m_{-N_p} \in \times_{i \in N \setminus N_p}\mathcal{M}_i$, and construct a mapping $g_{N_p}^{m_{-N_p}} : \times_{i \in N_p}\mathcal{M}_i \to \mathcal{R}_{|N_p}$ such that $g_{N_p}^{m_{-N_p}}(m_{N_p}) = f(m_{N_p}, m_{-N_p})_{|N_p}$. Observe that Impartiality of f implies that $g_{N_p}^{m_{-N_p}}$ should be impartial as well. Then, the proof is finished.

Proof of Theorem 3

Given that Consistent Weak Name Independence implies Weak Name Independence, we know by Theorem 2 that f is an impartial partition ordering. Then, we only need to determine the exact structure of each $g_{N_p}^{m_{-N_p}}$. Observe that Impartiality of f implies that each $g_{N_p}^{m_{-N_p}}$ should be impartial as well. Similarly, Consistent Weak Name Independence of f implies that each $g_{N_p}^{m_{-N_p}}$ should be weak name independent. Thus, we can apply the arguments in the proof of Theorem 2 to show that $g_{N_p}^{m_{-N_p}}$ has the structure of an impartial partition ordering. Then, we obtain an ordered partition of each N_p, $(N_{(p,1)}, \ldots, N_{(p,t)})$, and we can apply the same reasoning to each $N_{(p,k)}$ such that $|N_{(p,k)}| > 1$. Repeating iteratively this process we will arrive at a complete ranking of the set of agents such that for each $i,j \in N$, there exist $\vec{n}_i, \vec{n}_j$ with $N_{\vec{n}_i} = \{i\}$ and $N_{\vec{n}_j} = \{j\}$ and such that $i \in U(f(m),j) \Leftrightarrow \vec{n}_i <^L \vec{n}_j$.

Proof of Theorem 4

Observe first that all constant social ranking functions satisfy Impartiality and Weak Anonymity. Suppose now by contradiction that there is a social ranking function f satisfying these axioms that is not constant. Then, there exist $m, m' \in \times_{i\in N}\mathcal{M}_i$ such that $f(m) \neq f(m')$. Starting at m, construct a sequence of message profiles such that in each step of the sequence the message of an agent $i \in N$ is changed from m_i to m'_i such that the sequence ends at m'. Consider the last message profile of the sequence in which f selects $f(m)$, denoted by (m'_A, m_{-A}), and the next message profile of the sequence, $(m'_{A\cup\{j\}}, m_{-(A\cup\{j\})})$. Then, $f(m'_A, m_{-A}) \neq f(m'_{A\cup\{j\}}, m_{-(A\cup\{j\})})$. Therefore, there are two agents $k, k' \in N$ such that $k \in U(f(m'_A, m_{-A}), k')$ and $k \notin U(f(m'_{A\cup\{j\}}, m_{-(A\cup\{j\})}), k')$. Consider the permutation σ of N such that $\sigma(j) = k$, $\sigma(k) = j$ and $\sigma(i) = i$ for all $i \in N \setminus \{j,k\}$. Then, construct the message profiles $(m'_A, m_{-A})_\sigma$ and $(m'_{A\cup\{j\}}, m_{-(A\cup\{j\})})_\sigma$. By Weak Anonymity we deduce that $U(f((m'_A, m_{-A})_\sigma), k') = U(f(m'_A, m_{-A}), k')$ and that $U(f((m'_{A\cup\{j\}}, m_{-(A\cup\{j\})})_\sigma), k') = U(f(m'_{A\cup\{j\}}, m_{-(A\cup\{j\})}), k')$. Then, $k \in U(f((m'_A, m_{-A})_\sigma), k')$ and $k \notin U(f((m'_{A\cup\{j\}}, m_{-(A\cup\{j\})})_\sigma), k')$. Or, equivalently, $k' \notin U(f((m'_A, m_{-A})_\sigma), k)$ and $k' \in U(f((m'_{A\cup\{j\}}, m_{-(A\cup\{j\})})_\sigma), k)$. Observe that the message profiles $(m'_A, m_{-A})_\sigma$ and $(m'_{A\cup\{j\}}, m_{-(A\cup\{j\})})_\sigma$ only differ in the message of agent k and, then, Impartiality implies that $U(f((m'_A, m_{-A})_\sigma), k) = U(f((m'_{A\cup\{j\}}, m_{-(A\cup\{j\})})_\sigma), k$ We have arrived at a contradiction and, thus, the theorem is proved.