ECONOMIC INSECURITY, MEMORY EFFECTS AND ALLOCATIONS CHOICES

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Abstract. This paper presents a macrodynamic model to examine economic insecurity, incorporating the effects of habit formation on consumption and investment decisions in the face of stochastic economic shocks. We extend the existing literature by formulating an intertemporal optimization framework in which agents face Poisson-type shocks and exhibit finite memory regarding their consumption habits. Our model quantifies economic insecurity through the optimal willingness to pay to reduce the intensity of these shocks, linking it, on the one hand, to macroeconomic parameters such as productivity and accumulated wealth, and on the other, to preferences and habit formation parameters.

Key findings reveal that consumption habits significantly affect economic insecurity, which, *ceteris paribus*, is higher in contexts where habits are stronger, more persistent, and where habit memory is longer. Insecurity is also more pronounced when the same level of wealth is associated with a higher level of habitual consumption. At the same time, economic insecurity increases with longer expected recovery times from shocks, underscoring the role of past financial behavior and wealth accumulation as buffers against economic uncertainty.

The model also enables an analysis of how shocks impact resource allocation, distinguishing agents' efforts to hedge against future shocks from their investments in productive activities. This distinction allows the model to reproduce more realistic investment dynamics compared to frameworks where precautionary saving primarily drives increases in aggregate investment.

From a technical perspective, to the best of our knowledge, we present the first fully solved optimal control problem derived from a dynamic economic model with forward-looking agents, finite memory (and thus a state equation with delay), and a Poisson process.

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JEL Classification: D81, E21, D14, C61.

1. Introduction

While the concept of economic insecurity is increasingly recognized in scholarly literature and reports by international organizations, its definition and the empirical methods to measure it vary significantly.

According to Osberg (1998) seminal paper, economic insecurity refers to *the anxiety produced by a lack of economic safety, i.e. by an inability to obtain protection against subjectively significant potential economic losses.* This characterization is quite consensual in the literature and can be found, for example, in Bossert and D'Ambrosio (2013) or in Stiglitz et al. (2009). This insecurity can stem from various sources, including job loss, health emergencies, inadequate social protections, or macroeconomic fluctuations.

Although the roots of economic insecurity lie in uncertainty about the economic future and potential economic shocks, the core notion rests on the sense of vulnerability (as Osberg says *anxiety*): It is a reality that is both objective (risk of major losses without an adequate safety net) and subjective (linked to individual preferences). Indeed, according to Hacker et al. (2014), economic insecurity, although based on objective data, implicitly incorporates the psychological dimension linked to the experience of economic losses, by focusing on the degree of protection of individuals against economic losses that generate difficulties. In the same line, Richiardi and He (2020) and Bossert and D'Ambrosio (2013)) reveal the multidimensional complexity of the concept, including aspects such as resource stability, anxiety, uninsured losses, and difficulty in mitigating the consequences of adverse events.

The concept of economic insecurity is now widely used as an analytical tool and for identifying development policies by major international institutions (see, for example, OECD, 2023; World Bank, 2023). However, its measurement remains quite heterogeneous. In empirical studies, direct measures of shocks that may underlie feelings of insecurity are often used; these include (see for instance Algan et al., 2017a; Foster and Frieden, 2017; Hijzen and Menyhert, 2016; Osberg and Sharpe, 2014) variations in unemployment, job stability, illness and health risks, or the risk of poverty for single parents. [1] Another approach, which directly examines the sensation of precarity, relies on self-reported economic insecurity from questionnaires, such as those used by Dominitz and Manski (1997), Mau et al. (2012), or

[1]The same proxies are used in the axiomatic theoretical approach developed by Bossert and D'Ambrosio (2013); Bossert et al. (2019).

Nau and Soener (2019). Each of these approaches has its advantages and disadvantages. Objective criteria (measures of shocks to macroeconomic variables) are easily measurable and well-defined but do not account for prior economic conditions, the influence of other macroeconomic parameters, or the role of individual preferences. Conversely, self-reported data provide limited insights into the causes and micro-economic dynamics underlying insecurity.

In this work, we present a novel perspective on the problem, examining it within a macrodynamic model featuring intertemporal optimization by agents. To our knowledge, this is the first study to adopt this perspective. In the model, agents face Poisson-type shocks, and their *anxiety* towards these shocks — their measure of fear regarding the potential effects — is quantified by their optimal (relative) willingness to pay to reduce the shocks' intensity and economic insecurity is understood as the chosen proactive measures individuals take to mitigate risks.

The explicit form of the insecurity measure in our model is given in Theorem 3.2: it is a complex but explicit function of all the parameters in the model: parameters related to the shock (intensity, frequency), macroeconomic parameters such as productivity and growth prospects, individual preferences (detailed below), and the level of previously accumulated wealth. Our definition of economic insecurity bridges the gap through the various concepts described above. Drawing from the principles of the Health Belief Model (HBM, Janz and Becker, 1984), this perspective incorporates both an economic reality—the measurable cost of preventive actions—and psychological factors, such as risk aversion and the need to preserve habitual consumption levels. Within this framework, individual perceptions play a central role: the decision to exert effort, whether in terms of expenses or other resources, is shaped by subjective risk evaluations. This conceptualization improves our understanding of economic insecurity by linking it with behavioral dynamics, where preventive decision-making is dependent not only on available resources but also on subjective beliefs shaping their use[2].

As highlighted by Bossert and D'Ambrosio (2013), economic insecurity involves the past, the past, and the future. Our structure includes the three

[2]In other terms, these insights underline that economic insecurity is not only a function of resource availability but also deeply influenced by subjective beliefs and psychological factors as claimed by Davinson and Sillence (2014).

- The past: a critical factor integrated into our model is the impact of consumption habits (so, of past consumption history) on economic insecurity. Individuals with different social levels may still experience insecurity because they are concerned about how shocks will affect their future consumption relative to their accustomed standard of living. As noted by Bossert and D'Ambrosio (2013) and Bossert et al. (2023), economic insecurity involves comparing current behavior with past experiences. Our model incorporates habit formation as described by Constantinides (1990), where saving behavior is influenced by past consumption habits. Consumers save to maintain their consumption habits over time, implying that, for a given level of wealth, the impact of shocks is more severe for those with higher habitual consumption. Consequently, *ceteris paribus*, individuals with a history of higher past consumption experience greater economic insecurity (see Proposition 4.9).
- The present: in our model, economic insecurity is influenced by actual wealth, which acts as a *buffer stock* that can be used to mitigate the negative effects of potential shocks (Bossert and D'Ambrosio, 2013), see Proposition 4.2. The importance of wealth in determining the state of insecurity has also been discussed by Hacker et al. (2014) and Osberg and Sharpe (2014).
- The future: in our model, economic insecurity depends on economic parameters that determine long-term expected growth, primarily factor productivity and future prospects. Economic insecurity increases when the expected recovery time from a potential shock is longer. This aligns with the idea described by United Nations (2008), who identify insecurity as the *inability to cope with and recover from the costly consequences of those events.* This concept will be further explored in Subsection 4.4.

From a technical standpoint, the model presents two main challenges: on the one hand, a stochastic dynamic dictated by Poisson shocks and, on the other, the presence of delayed equations as a consequence of habit formation, which necessitates comparing current consumption with past consumption. Therefore, the model constitutes an optimal control problem with finite delay (and thus infinite-dimensional) and Poisson jumps. To our knowledge, it is the first model of this kind in the economic literature.

We approach the problem (Appendix A.1) by reformulating it as a dynamic optimization problem in an infinite-dimensional Hilbert space setting. To this end, we rewrite the delayed equation as an evolution equation in an appropriate Hilbert space, a technique introduced by Vinter and Kwong (1981) and described in detail, for example, by Bensoussan et al. (2007). After reformulating the problem, we study the infinite-dimensional Hamilton-Jacobi-Bellman (HJB) equation, find an explicit solution (Proposition A.2), and we prove a verification theorem (Theorem A.5) to demonstrate that the feedback control suggested by the solution of the HJB equation is indeed optimal and that the solution of the HJB equation is also the value function of the problem.

The main results are presented in the main text in the formulation with delay (excluding the infinite-dimensional setting). Theorem 3.2 provides the explicit feedback solution of the control problem, characterizing the level of insecurity as well as the choices in terms of consumption and investment. A more technical result is found in Theorem 3.3, which describes the explicit evolution of a measure of habit-adjusted capital, $\Gamma(t)$.

The economic implications following from the main results are discussed in Section 4. We focus on four specific aspects:

(i) In Subsection 4.1, we examine the effects of the properties of the shocks by characterizing the implications of stronger or more frequent shocks on both insecurity and resource allocation. Consistent with empirical literature (see, for example, Hacker et al., 2014), even when all macroeconomic parameters remain constant, the sense of insecurity increases following a shock (Proposition 4.2). Owing to the model's ability to distinguish between the need to smooth consumption over time and the impact of shocks on investment incentives, in Proposition 4.3 we reproduce a documented fact in the literature (see e.g. Bachmann et al., 2013; Bloom, 2009): the reduction in investments driven by increased frequency or intensity of the shock.

(ii) In Subsection 4.2, we explore more directly the effects of habits on jump dynamics and welfare. To do so, we introduce a counterpart of the model without habits (Proposition 4.4) and compare the two in terms of the amplitude of shocks at equilibrium (Proposition 4.5) and the welfare levels (Proposition 4.6). We find that the presence of habits increases economic insecurity, while simultaneously lowering the maximum attainable welfare. This result is in line for instance with those of Carroll et al. (2000) and Dynan (2000) which show that the presence of habits in consumption makes individuals less adaptable to income shocks and amplifying the welfare losses due to deviations from habitual consumption levels.

(iii) Subsection 4.3 delves into the relationship between habits, memory, and insecurity, aiming to understand how preferences shape economic insecurity. Increasing agent risk aversion increases their sense of insecurity about future shocks (Proposition 4.7). Furthermore, parameters that strengthen the role of habits—such as their persistence, the length of memory, or their intensity—exacerbate insecurity and (Proposition 4.9) habits associated with higher consumption levels, given the same wealth, increase the level of insecurity. This fact can be explained by observing that, in habit-formation models à la Constantinides (1990) a minimum consumption level greater than zero that must be maintained.

(iv) In Subsection 4.4, we investigate the relationship between economic stagnation and insecurity. Specifically, we analyze how economic insecurity responds to changes in the productivity parameter α. As shown in Proposition 4.10, a lower α, which corresponds to lower total factor productivity (TFP) and, consequently, weaker economic growth, leads to higher levels of insecurity. This result is in line with what we previously underlined about the importance of the ability to recover in the determination of the insecurity condition.

Contribution to the literature. This article contributes to the literature in three ways

The first contribution directly concerns the literature on economic insecurity. Indeed, over the last thirty years, a vast literature has addressed economic insecurity and its effects on agents' behaviors, but to our knowledge, there is still no theory of economic insecurity in a macro-dynamic framework with intertemporal optimization by agents. In this article, we develop an intertemporal choice model in a context of economic insecurity that takes into account a key element in decisions in terms of insecurity perception (see also Bossert and D'Ambrosio, 2013): consumption habits. They play a role at different levels: reference levels of consumption considered acceptable, experience of previous shocks, and recovery prospects. We provide a dynamic model that enables to question the relationship between unpredictable economic losses (through Poisson processes), subjective perception of past behavior (through finite memory of consumption habit), and the ability to face these losses, through mitigation policies that may reduce the economic loss.

Second, we contribute to the habit formation literature. As we know, finite habit memory smooths consumption trajectories (Augeraud-Véron and Bambi, 2015), but in deterministic models, finite memory habit may lead to damping oscillations in the capital (Augeraud-Véron et al., 2017). Still, the interaction between habit formation and Poisson-driven stochastic dynamics remains largely unexplored. To the best of our knowledge, the only existing study in this direction is Bernasconi et al. (2020) (in a tax evasion model where Poisson shocks are due to random checks for fiscal evasion), but in this case, the authors only explore infinite memory habit processes.

The last strand of literature to which we contribute is that of dynamic economic models in infinite dimensions. In the case of deterministic delay equations, the techniques of rewriting and solving the problem as an optimal problem driven by an evolution equation in a Hilbert space have been successfully used, for example, by Fabbri and Gozzi (2008), Boucekkine et al. (2010) or Augeraud-Véron et al. (2017) and, in a stochastic framework, with Brownian motion, by Fabbri (2017) and Feo et al. (2024). This is the first macro-dynamic model in which a solution is found for a problem with finite delay and Poisson-type noise.

The structure of the article. The paper is organized as follows. Section 2 describes the model, including the accumulation law, habit formation, and impact of economic shocks through a Poisson process. In Section 3, we present the solution to the model, specifying the optimal conditions for consumption and the insecurity value. Section 4 discusses the results obtained, highlighting how the interaction between consumption habits and shock frequency influences the insecurity and stability of consumption. The conclusions are contained in Section 5, and the appendices mainly contain technical elements: Appendix A.1 describes how to rewrite the optimal control problem in a suitable infinite-dimensional setting, Appendix A.2 studies the infinite-dimensional HJB expression of the problem and provides a solution of the infinite-dimensional formulation of the problem while Appendix A.3 describes how to move from the infinite-dimensional setting to the original. Other proofs are contained in Appendix B.

2. The model

2.1. **Economic environment.** We consider a neoclassical macrodynamic model that features a representative agent endowed with wealth $k(t)$ (corresponding to the aggregate capital stock in the economy within a model featuring a linear production function of the type AK). Besides the usual intertemporal allocation problem of consumption and investments, the agent faces the problem of choosing the optimal effort to address a series of exogenous shocks. More precisely, we consider the dynamics of $k(t)$ governed by a one-dimensional stochastic differential equation with jumps of the following form:

$$\begin{cases} dk(t) = (\alpha k(t) - \chi e(t)\alpha k(t) - c(t))\, dt - (1 - \eta e(t))\,(1 - \xi)k(t)dP(t) \\ k(0) = k_0 \end{cases} \tag{1}$$

where P is a Poisson process (whose characteristics will be detailed shortly) used to model shocks. We consider two controls: $c(\cdot) > 0$ represents the consumption trajectory, while $e(\cdot) \in (0, 1/\eta)$ is our measure of economic insecurity, which is captured here through the agent's proactive measures (effort) to mitigate these risks. The parameters in the equation allow us to study situations with varying relative strengths of different processes: α is the interest rate (or the TFP - total factor productivity - in the aggregate version of the model), χ is the (unit) cost (in terms of yield or total production) of effort, η is the effectiveness of the effort, and ξ can be used to vary the intensity of the shocks.

The jumps modeled by the Poisson process P represent sudden, exogenous economic shocks that affect the agent's wealth $k(t)$. As discussed in the introduction, these shocks could capture, at the individual level, exogenous events such as variations in revenue due to unemployment, illness or the risk of poverty for single parents. At the aggregate level, they can be financial crises, abrupt changes in productivity, or significant losses in asset value. From a formal point of view $P\colon [0, +\infty) \times \Omega \to \mathbb{R}^+$ is a Poisson process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote by $\{\mathcal{F}_t\}_{t\geq 0}$ the filtration generated by P and we suppose that the conditional expectation and the conditional variance of the increment $dP(t)$ are given by

$$\mathbb{E}_t[dP(t)] = \lambda\, dt, \qquad \text{and} \qquad \mathbb{V}_t[dP(t)] = \lambda\, dt,$$

where λ is the rate parameter of the Poisson process, which represents the average number of events occurring per unit time.

If a jump occurs at time τ, we denote τ^- as the pre-jump time and τ^+ as the post-jump time. Then,

$$k\left(\tau^+\right) = \left(1 - \left(1 - \eta e\left(t\right)\right)\left(1 - \xi\right)\right) k\left(\tau^-\right).$$

2.2. **Preferences.** As mentioned in the introduction, an element characterizing the model is the presence of habits. We assume that the level of past consumption determines a certain reference level ("habit") for present consumption. More precisely, we assume (following Constantinides, 1990) that the habit level is given by

$$h(t) = z\int_{-T}^{0} e^{\beta s} c(t+s)\ ds = z\int_{-T+t}^{t} e^{\beta(s-t)} c(s)\ ds,$$

where, at any time t, $c(t+s)$, for $s \in [-T, 0)$, represents the consumption in the past interval $[t-T, t]$, T is the memory of the process, $\beta > 0$ is the persistence of the habit, and $z > 0$ is the intensity of the habit, which models the relative importance of past consumption relative to current consumption.

The concept of habits reflects the idea that an agent's current consumption preferences are influenced by their past consumption levels. This habit formation mechanism implies that individuals derive satisfaction not only from their absolute level of consumption but also from maintaining a standard relative to their historical consumption. Consequently, the model captures a type of "consumption inertia" where deviations from past consumption levels generate disutility. This feature encourages agents to smooth consumption over time, as abrupt changes can lower utility due to the gap between current and accustomed consumption levels.

Thus,

$$\frac{dh}{dt} = zc(t) - ze^{-\beta T}c(t-T) - \beta z\int_{-T+t}^{t} e^{\beta(s-t)}c(s)\,ds = -\beta h(t) + zc(t) - ze^{-\beta T}c(t-T).$$

The values of c in the interval $[-T, 0)$ are the initial data of the equation for the evolution of the habits, while $c(t)$, for $t \geq 0$ (together with $e(t)$, for $t \geq 0$), is a control variable. The previous integral delay equation can thus be rewritten as:

$$\begin{cases} \frac{dh}{dt} = -\beta h(t) + zc(t) - ze^{-\beta T} c(t-T), \\ c(s) = c_0(s), \text{ for } s \in [-T, 0) \text{ (given)}, \\ h(0) = z \int_{-T}^{0} e^{\beta s} c(s)\, ds. \end{cases} \tag{2}$$

We denote by $L^2_{loc,P}$ the following set of real-valued processes:

$$L^2_{loc,P} := \left\{ x\colon [0, +\infty) \times \Omega \to \mathbb{R} \ : \ \mathcal{F}_t\text{-adapted and } \mathbb{E} \int_0^T |x(t)|^2 \, dt < +\infty, \ \forall T > 0 \right\}. \tag{3}$$

Given two processes $c(\cdot), e(\cdot) \in L^2_{loc,P}$ with $e(\cdot)$ taking values in $[0, 1/\eta]$, the equation (1) has a unique solutions among $\mathcal{F}_t$-adapted and cádlág processes while equation (2) has unique solutions defined trajectory-by-trajectory (see respectively Theorem 6, page 249 in Protter, 2004 and Theorem 2.3, page 44 in Hale and Lunel, 2013). We will denote these solutions by $\left(k_{k_0,c(\cdot),e(\cdot)}(\cdot), h_{k_0,c(\cdot),e(\cdot)}(\cdot)\right)$, or simply by $k(\cdot), h(\cdot)$ when the dependencies are clear from the context.

The set of admissible controls, denoted as $\mathcal{C}\left(k_0, c_0\left(\cdot\right)\right)$, is defined as:

$$\mathcal{C}\left(k_0, c_0\left(\cdot\right)\right) = \left\{ \begin{array}{c} \left(c\left(\cdot\right), e\left(\cdot\right)\right) \in L^2_{loc,P} \times L^2_{loc,P} \ : \ c\left(\cdot\right) \text{ is positive, } e\left(\cdot\right) \text{ is } [0, 1/\eta]\text{-valued, and} \\ \text{s.t. } k_{k_0,c(\cdot),e(\cdot)}(t) \geq 0 \text{ and } c(t) \geq h_{k_0,c(\cdot),e(\cdot)}(t) \text{ a.s. for all } t \geq 0 \end{array} \right\}. \tag{4}$$

The target of the agents is to choose $(c(t), e(t)) \in \mathcal{C}\left(k_0, c_0\left(\cdot\right)\right)$ to maximize the following utility:

$$J(k_0, c_0\left(\cdot\right); c\left(\cdot\right), e\left(\cdot\right)) := \mathbb{E} \int_0^{+\infty} e^{-\rho t} \frac{\left(c(t) - h_{k_0,c(\cdot),e(\cdot)}(t)\right)^{1-\sigma}}{1-\sigma} \, dt.$$

In this utility function, the agent's satisfaction depends on the difference between their current consumption $c(t)$ and their habit level $h_{k_0,c(\cdot),e(\cdot)}(t)$. This formulation implies that the agent derives utility not solely from absolute consumption but rather from how current consumption compares to the habitual reference level. As a result, the agent is incentivized to maintain a consumption path that closely aligns with their established habits. In particular, this structure requires the constraint $c(t) > h(t)$.

We define the value function $V\left(k_0, c_0\left(\cdot\right)\right)$ as:

$$V\left(k_0, c_0\left(\cdot\right)\right) = \sup_{(c(\cdot),e(\cdot)) \in \mathcal{C}(k_0, c_0(\cdot))} J(k_0, c_0\left(\cdot\right); c\left(\cdot\right), e\left(\cdot\right)).$$

Note that the choice of e and the dynamics of k influence the functional via the positivity constraint on k and, consequently, the possible choices of $c(\cdot)$.

3. Solution of the model

In this section, we present the solution to the model. We begin by detailing the necessary conditions and assumptions that govern the model's dynamics, focusing on the constraints that ensure feasible and optimal behavior in response to economic shocks. We then derive the optimal control strategy for $c(\cdot)$ and $e(\cdot)$, examining its uniqueness and expressing it in feedback form. Additionally, we analyze the dynamics of a particular linear combination of the model's variables, which serves as a constant of motion along the optimal trajectories: this approach greatly simplifies the analysis by allowing us to bypass the complexities associated with delay terms. The full proofs of the results, including the infinite-dimensional framework required to solve the model, are provided in Appendix A.

We will work under the following hypotheses:

Hypothesis 3.1 *We suppose that*

- (h.1) $\alpha, \chi, \eta, \lambda, z, \beta, T > 0$, $\xi \in (0,1)$, $\sigma > 0$ *(and* $\sigma \neq 0$*)*
- (h.2) $c_0(s)$, *for* $s \in [-T, 0)$ *is non-negative for all* s *(other constraints on the controls and on the trajectories are given in the definition of the set of admissible controls)*
- (h.3) $\pi := \beta + \alpha \left(1 - \frac{\chi}{\eta}\right) > 0$ *(but* $\gamma := \alpha \left(1 - \frac{\chi}{\eta}\right)$ *can be negative)*
- (h.4) $\nu^{-1/\sigma} := \frac{(\rho+\lambda)}{\sigma} - \frac{(1-\sigma)\alpha}{\sigma}\left(1 + \frac{\chi\xi}{\eta(1-\xi)}\right) - \frac{\chi\alpha}{\eta(1-\xi)}\left(\frac{\chi\alpha}{\eta(1-\xi)\lambda}\right)^{-1/\sigma} > 0,$
- (h.5) $m := \left(1 - z\int_{-T}^{0} e^{\pi s} ds\right) > 0$
- (h.6) $n := \left(1 - \left(\frac{\lambda(1-\xi)\eta}{\chi\alpha}\right)^{1/\sigma}\right) > 0$

Hypotheses 3.1 (h.1)-(h.2) are natural modeling assumptions that follow from the meaning of the various parameters described in Section 2. Hypotheses 3.1-(h.3) ensure the feasibility of the control strategy within the given constraints. Hypotheses 3.1-(h.4) will be useful to prove that the value function of the problem remains bounded in the set of admissible controls. Assumption Hypotheses 3.1-(h.5) is needed to avoid a too strong impact of the habits, which risks giving a negative value for k with non-zero probability for all trajectories where consumption is at least h in each period. Finally, if condition Hypotheses 3.1-(h.6) is violated, one of the trade-offs of our construction is violated since it can be rewritten as $\lambda(1-\xi)\eta < \chi\alpha$, which is the condition that ensures increasing e reduces the growth rate of the main aggregate variable Γ used in the description of the dynamics.

The presence of the habit imposes, given the structure of the chosen utility, a minimal level of consumption. Indeed, the infimum of admissible strategies on consumption is given (in feedback form) by

$$c(s) = h(s) = z\int_{-T}^{0} e^{\beta s} c(t+s)\ ds.$$

For this reason, only initial conditions with sufficient initial capital have a non-void set of admissible controls. Given a certain initial history of consumption $c(s)$, $s\in[-T,0]$, for low levels of capital, the positivity constraint would be violated for all choices of the control. For this reason, we introduce a restriction on the initial data that we will suppose to belong to the following set[3]

$$\mathcal{I} := \left\{ (k, c(\cdot)) \in \mathbb{R}\times L^2(-T,0)\ :\ mk > \pi^{-1}\left(z\int_{-T}^{0} \left[e^{\beta s} - e^{-\beta T} e^{\gamma(-T-s)} \right] c(s) ds \right) \right\}. \quad (5)$$

Our argument will show that this condition is sufficient to ensure that the set of admissible controls is non-void.

The following theorem establishes the existence and uniqueness of the optimal control strategy for the model under the given assumptions. By leveraging the structure of the problem, we express the optimal consumption $c(t)$ and economic insecurity measure $e(t)$ in feedback form, allowing them to respond dynamically to the agent's current wealth level $k(t)$ and habit component $h(t)$. This feedback representation highlights the dependency of the optimal control on past consumption values over a finite horizon, thus incorporating memory effects into the optimal strategy. Additionally, we derive an explicit formula for the value function of the problem defined in (2.2).

To distinguish the future values of the consumption (which are part of the control choices of the agent) and the past of c (which is part of the state) we introduce the notation c_t which will represent the "recent" past of the consumption when we are at time t (so c_t will

[3]Observe that $\left(h(t) - ze^{-\pi T}\int_{t-T}^{t} e^{\gamma(t-s)}c(s)ds\right) = \left(z\int_{-T}^{0}\left[e^{\beta s} - e^{-\beta T}e^{\gamma(-T-s)}\right]c(s)ds\right) = \left(z\int_{-T}^{0} e^{-\gamma s}\left[e^{(\beta+\gamma)s} - e^{-(\beta+\gamma)T}\right]c(s)ds\right)$, and under the Hypothesis 3.1-(h.3), $\pi > 0$ and then the integrand is always positive.

characterize, together with $k(t)$, the state of the system over time t):

$$\begin{cases} c_t \colon [-T, 0) \to \mathbb{R}^+ \\ c_t(s) := c(t+s). \end{cases}$$

THEOREM 3.2 *Suppose that Hypothesis 3.1 is verified and that the initial datum* $(k_0, c_0(\cdot))$ *belongs to* $\mathcal{I}$. *Then, at any time* $t \geq 0$, *the optimal control is unique and can be expressed in feedback form, in terms of the value of* $k(t)$ *and on the history of the consumption* $c(t+s)$, *for* $s \in [-T, 0)$ *as follows*

$$c(t) = \varphi(k(t), c_t) := h(t) + \nu^{-1/\sigma} m k(t) - \frac{\nu^{-1/\sigma}}{\pi} \left(z \int_{-T}^{0} \left[e^{\beta s} - e^{-\beta T} e^{\gamma(-T-s)} \right] c(s+t)\, ds \right) \tag{6}$$

and

$$\begin{aligned} e(t) = \phi(k(t), c_t) &:= \frac{1}{\eta} - n \frac{k(t) - \frac{1}{m\pi}\left(z \int_{-T}^{0} \left[e^{\beta s} - e^{-\beta T} e^{\gamma(-T-s)} \right] c(s+t) ds \right)}{\eta(1-\xi)k(t)} \\ &= \frac{1}{\eta}\left(1 - \frac{n}{1-\xi}\right) + \frac{nz}{\eta m \pi (1-\xi)} \frac{\int_{-T}^{0} \left[e^{\beta s} - e^{-\beta T} e^{\gamma(-T-s)} \right] c(s+t) ds}{k(t)}. \end{aligned} \tag{7}$$

Moreover, the maximal utility obtained using the optimal strategy is given by

$$V(k_0, c_0(\cdot)) = \frac{\left(1 - z \int_{-T}^{0} e^{\pi s}\, ds\right)^{1-\sigma} \nu}{1-\sigma} \Gamma(k_0, c_0(\cdot))^{1-\sigma}, \tag{8}$$

where:

$$\Gamma(k_0, c_0(\cdot)) := k_0 - \frac{1}{m\pi} \left(z \int_{-T}^{0} \left[e^{\beta s} - e^{-\beta T} e^{\gamma(-T-s)} \right] c_0(s)\, ds \right). \tag{9}$$

Proof. See Appendix A.3. □

The previous theorem represents the main result of this paper. In addition to providing the feedback form solution for consumption, it also determines the level of e that corresponds to the given situation, linking it on one hand to the various parameters of the model, and on the other to the past consumption history of the agents. As already mentioned in the introduction it is an (explicit) function of the model's parameters, including shock-related factors (intensity, frequency), macroeconomic variables such as productivity and growth prospects, individual preferences (detailed below), and the level of previously accumulated wealth. This explicit expression is used to analyze how this measure of insecurity responds to changes in the context (see Section 4, particularly Subsections 4.1, 4.3, 4.4).

The next theorem focuses on the dynamics of the variable $\Gamma(t)$. Γ is defined in (9) as a function of the state (in a sense "in a feedback way"), while in the following theorem we examine its dynamics along the optimal trajectory. Γ is a specific linear combination of k and the history of consumption. Its dynamics along the optimal trajectory have the property of being describable in a straightforward manner as the evolution of a non-delayed linear stochastic differential equation with jumps, where the coefficients are explicit expressions of the model parameters. In this way, the evolution of Γ and its expected value can both be written explicitly. In particular, the theorem provides insights into the long-term behavior of the optimal control strategy and the stability of the system in response to economic shocks.

THEOREM 3.3 *Suppose that Hypothesis 3.1 is verified and that the initial datum* $(k_0, c_0(\cdot))$ *belongs to* $\mathcal{I}$. *Define*[4]

$$\Gamma(t) := k(t) - \frac{1}{m\pi}\left(z\int_{-T+t}^{t}\left[e^{\beta s} - e^{-\beta T}e^{\gamma(-T-s)}\right]c(s+t)\,ds\right).$$

Then, along the optimal trajectory, we have

$$\begin{cases} d\Gamma(t) = \left(\gamma + n\dfrac{\chi\alpha}{\eta(1-\xi)} - \nu^{-1/\sigma}\right)\Gamma(t)\,dt - mn\Gamma(t)\,dP(t), \\ \Gamma(0) = k_0 - \dfrac{1}{m\pi}\left(z\displaystyle\int_{-T}^{0}\left[e^{\beta s} - e^{-\beta T}e^{\gamma(-T-s)}\right]c_0(s)\,ds\right). \end{cases}$$

In particular, all trajectories of $\Gamma(t)$ *will remain strictly positive (they start positive since the initial datum is in* $\mathcal{I}$*) and*[5]

$$\Gamma(t) = \Gamma(0)e^{\left(\gamma+n\frac{\chi\alpha}{\eta(1-\xi)}-\nu^{-1/\sigma}\right)t}(1-mn)^{P(t)},$$

$$\mathbb{E}[\Gamma(t)] = \Gamma(t) = \Gamma(0)e^{\left(\gamma+n\frac{\chi\alpha}{\eta(1-\xi)}-\nu^{-1/\sigma}\right)t}e^{-\lambda mnt}.$$

Proof. See Appendix A.3. □

[4]Using this definition of Γ, the condition contained in the definition (5) of $\mathcal{I}$ reads s $\Gamma(0) > 0$.

[5]It can be seen that

$$\gamma + n\frac{\chi\alpha}{\eta(1-\xi)} - \nu^{-1/\sigma} = \frac{1}{\sigma}\left(\left(1 + \frac{\chi\xi}{\eta(1-\xi)}\right) + \alpha - (\rho+\lambda)\right).$$

.

Remark that, although Γ is a (specific) linear combination of k and of the history of c, we can still express its dynamics as a (non-delayed) stochastic differential equation which can be expressed only in terms of Γ itself. This formulation provides a more manageable way to analyze the evolution of Γ over time and it is significant because it allows us to avoid the complexities associated with delay differential equations.

4. Discussion of the results

This section analyzes the economic implications of our model's results, focusing on the role of consumption habits and the perception of economic insecurity. We first examine how the characteristics of economic shocks influence consumption and investment decisions, next, we compare the impact of habit formation on equilibrium dynamics and welfare, we then explore the relationship between memory, individual preferences, and economic insecurity. Finally, we analyze the connection between economic stagnation and insecurity.

4.1. **The effects of shocks on insecurity and resource allocation.** In this subsection, we aim to understand how shocks influence, on the one hand, economic insecurity and, on the other, agents' choices regarding investment and consumption.

We first look at how changes in the parameters describing the characteristics of shocks affect the perception of insecurity, as measured by e:

Proposition 4.1 *Suppose that Hypothesis 3.1 is verified. Then*[6]*, in terms of the optimal feedback defined in (7), we have*

$$\frac{de}{d\lambda} > 0, \qquad \frac{de}{d\xi} < 0,$$

Proof. See Appendix B. □

The proposition indicates that insecurity increases with the frequency and intensity of shocks: the greater the likelihood that shocks may negatively impact future income flows or their intensity, the stronger the economic insecurity, understood as the effort agents are willing to make to try to reduce their impact. These facts are consistent with evidence from

[6]Note that in equation (1), the magnitude of the shock is proportional to $(1-\xi)$, so it is the reduction of ξ that increases the strength of the shocks. This explains why the derivative with respect to ξ has a negative sign.

the literature on social deprivation and economic insecurity. For instance, Bossert et al. (2007) highlight how increased exposure to shocks exacerbates feelings of deprivation and exclusion, while Smith and Thomas (2003) show, in the context of migration histories, that the memory of past shocks intensifies perceptions of future risk and economic insecurity.

This result is further confirmed when we examine the impact of a single shock on insecurity. Suppose that a shock occurs at time $\bar{t}$. Immediately after a shock, the state of the economy in terms of past consumption trajectories remains unchanged, while the level of k jumps from $k(\bar{t}^-)$ to $k(\bar{t}^+) = (1-\xi)k(\bar{t}^-)$. This change simultaneously generates a positive jump in the level of insecurity, as described by the following proposition.

PROPOSITION 4.2 *Suppose that Hypothesis 3.1 is verified and that the initial datum* $(k_0, c_0(\cdot))$ *belongs to* $\mathcal{I}$. *Suppose that a shock hits the system (evolving under the equilibrium characterized by the optimal choices described in Theorem 3.2) at a certain time* $\bar{t}$. *Then, the value of* $e(t)$ *jumps from* $e(\bar{t}^-)$ *to* $e(\bar{t}^+)$ *with*

$$e(t^+) > e(t^-).$$

Proof. See Appendix B. □

Next, we analyze how shocks affect the agents' resource allocation. The income of agents at a given period t (or the country's income if we interpret the model at an aggregate level) is equal to $\alpha k(t)$. As previously mentioned, agents decide each period how to allocate this income among three possible uses: consumption, saving (which, in the context of our closed economy, is equivalent to investment if we interpret the model as a description of the entire economy), and buffer against economic shocks: $\alpha k(t) = c(t) + s(t) + \chi e(t)\alpha k(t)$. We can rewrite this equation in terms of rates:

$$1 = \frac{c(t)}{\alpha k(t)} + \frac{s(t)}{\alpha k(t)} + \chi e(t) = \theta_c(t) + \theta_s(t) + \theta_m(t)$$

where we denoted $\theta_c(t) := \frac{c(t)}{\alpha k(t)}$ the consumption rate, $\theta_s(t) := \frac{s(t)}{\alpha k(t)}$ the saving/investment rate and $\theta_m(t) := \chi e(t)$ the mitigation effort rate, which is just proportional to insecurity. The following proposition, where, as in Proposition 4.1 we consider a certain time t with a certain history c_t and a certain level of capital $k(t)$ and the feedback expressions of e and

c provided in (7) and (6), describes how the three rates react to changes in the frequency and intensity of shocks.

PROPOSITION 4.3 *Suppose that Hypothesis 3.1 is verified. Then we have*

$$\begin{aligned} \frac{d\theta_m}{d\lambda} > 0, \qquad \frac{d\theta_m}{d\xi} < 0, \\ \frac{d\theta_s}{d\lambda} < 0, \qquad \frac{d\theta_s}{d\xi} > 0. \end{aligned} \tag{10}$$

and

$$\frac{d\theta_c}{d\lambda} < 0, \qquad \frac{d\theta_c}{d\xi} \textit{ has the sign of } (1-\sigma). \tag{11}$$

Proof. See Appendix B. □

The proposition shows that, unsurprisingly, the insecurity and then the mitigation effort θ_m rises with the frequency and severity of shocks: as the likelihood or intensity of shocks increases, so does the effort agents are willing to exert to mitigate their impact.

Regarding the impact on saving/investment rate, we observe that the model does not exhibit a precautionary saving effect. This is because the mitigation effort channel allows the agent to distinguish between two impulses: on the one hand, the need to smooth consumption over time, driven by the concavity of the instantaneous utility function; on the other, the influence of future returns on investment. The need to smooth consumption (which becomes stronger in the presence of more frequent or intense shocks) is addressed through the mitigation effort channel. Meanwhile, the effect of increased volatility and reduced expected returns on investment (as shocks are only negative in this model, their higher frequency or intensity reduces expected returns) is incorporated into the agent's behavior through a decrease in θ_s. This result is consistent with empirical observations (see, for instance, Bloom, 2009), where an increase in volatility is regularly associated with a decline in investment.

Finally, the ambiguous sign of the response of θ_c to an increase in ξ reflects this dual effect: as explained above, an increase in ξ decreases θ_m on one hand, while increasing θ_s on the other. Depending on which of these two effects dominates, we will observe one outcome or the other for $\theta_c = 1 - \theta_m - \theta_s$. In the case of changes in λ, one effect always dominates the other.

4.2. **Comparison with the no-habit situation: jumps and welfare.** We start by exploring how habits affect optimal control strategies, welfare, and the overall dynamics of the system by comparing the results obtained in the previous section with those obtained in the no-habit counterpart of the model. Indeed, to eliminate the effects of habits, it is sufficient to set $z = 0$ in the previously described model. In this case, as a corollary of Theorem 3.2, we obtain the following result.

PROPOSITION 4.4 *Suppose that Hypothesis 3.1 is verified with $z = 0$ (no-habit case) and that the initial datum $(k_0, c_0(\cdot))$ belongs to*[7] *$\mathcal{I}$. Then the optimal control, expressed in feedback form, expressed in the feedback form, is*

$$c(t) = \nu^{-1/\sigma} k(t)$$

$$e = \frac{1}{\eta}\left(1 - \frac{n}{1-\xi}\right).$$

The explicit form of the value functions is

$$V(k_0) = \frac{\nu}{1-\sigma} k_0^{1-\sigma}. \tag{12}$$

Once we have the expression of the optimal control in feedback form for both cases - with and without habits - we can compare their behavior. As a first fact, in the following proposition, we observe that, consistent with results obtained in stochastic models with Brownian motion (see, e.g., Detemple and Zapatero, 1991), the presence of habits tends to smooth consumption over time, reducing the sensitivity to shocks and dampening fluctuations. More precisely, while jumps $c(t^+) - c(t^-)$ occur at the moment of a shock in our jump-diffusion setting, we find that the presence of habits mitigates these abrupt changes, leading to a more stable consumption path.

PROPOSITION 4.5 *Suppose that Hypothesis 3.1 holds both for $z = 0$ and for some $z = \bar{z} > 0$, keeping all other parameters fixed. Additionally, assume that the initial datum $(k_0, c_0(\cdot))$*

[7]Indeed, when $z = 0$, the value of c_0 does not enter into the solution. However, in the following, we will compare the results for the model with habits and without habits, so we keep a setting where both are well defined.

belongs to $\mathcal{I}$ for both $z = 0$ and $z = \bar{z} > 0$ (note that the definition of $\mathcal{I}$ depends on the value of z). Then, the consumption path is smoother in the case $z = \bar{z} > 0$ (the habit case) compared to the case $z = 0$ (no-habit case): if we consider the jump $c(t^+) - c(t^-)$ at the moment of a shock, this quantity is lower when $z = \bar{z} > 0$.

Proof. See Appendix B. □

The rationale behind these differing behaviors is intuitive: in the habit model, the utility function at each point is effectively a shifted version of the utility function in the no-habit model. Consequently, any decrease in consumption following a shock results in a larger utility loss, as it corresponds to a portion of the instantaneous utility function with a steeper derivative. For this reason, in the habit model, the agent has a stronger incentive to smooth consumption over time.

In Figure 1, we illustrate sample trajectories of consumption dynamics for the parameter set $(\alpha, \beta, \chi, \eta, T, \lambda) = (0.3, 2, 0.99, 1, 0.3, 5, 0.5)$ and initial conditions $(k_0, c_0(s)) = (15, 2)$, where $c_0(s) = 2$ for all past values of s. The comparison is made between two cases: the model without habits ($z = 0$) and the model with habits ($z = 1.5$). In the plot, red lines represent the no-habit case and blue lines represent the habit model; within each color, solid lines show consumption dynamics, while dashed lines show the trajectory of $e(t)$.

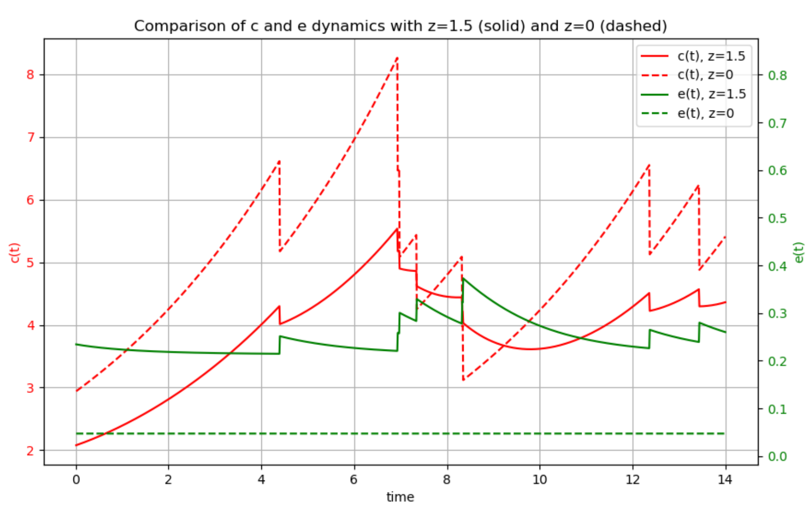

FIGURE 1. Control variables with and without habit for $(\alpha, \beta, \chi, \eta, T, z, \lambda) = (0.3, 2, 0.99, 1, 0.3, 5, 1.5, 0.5)$ and $(k_0, c_0(s)) = (15, 2)$.

We now compare welfare in the two cases. The following proposition shows that the welfare of the representative agent in the model with habits is always lower than in the model without habits (with $z = 0$). This result is not surprising, as, for fixed h and c the instantaneous utility function decreases with increases in the habit parameter z.

PROPOSITION 4.6 *Suppose that Hypothesis 3.1 holds both for* $z = 0$ *and for some* $z = \bar{z} > 0$*, with all other parameters kept fixed. Assume also that the initial datum* $(k_0, c_0(\cdot))$ *belongs to* $\mathcal{I}$ *for both* $z = 0$ *and* $z = \bar{z} > 0$*. Then, the welfare when* $z = 0$ *(no habit) is always greater than the welfare when* $z > 0$ *(with habit).*

Proof. See Appendix B. □

Our findings on the effects of habit formation on consumption dynamics and welfare confirm, in the context of the insecurity, the conclusions of Constantinides (1990), Abel (1990) and Detemple and Zapatero (1991). As in Constantinides (1990) habit formation can increase agents' sensitivity to fluctuations, as deviations from habitual consumption levels lead to larger utility losses. We show that a similar behavior holds even in a setting with jump-diffusion shocks. Abel (1990)'s work highlights the stabilizing effect of habit formation, suggesting that agents may prefer smoother consumption paths to avoid the disutility associated with falling behind past consumption levels or the "Catching Up with the Joneses" effect. This finding aligns with our observation that, under habit formation, agents reduce the amplitude of consumption jumps following shocks. Similarly, Detemple and Zapatero (1991) observe that habit persistence tends to stabilize consumption and influence investment choices by lowering the agent's tolerance to consumption volatility. Our model captures this stabilizing effect by reducing abrupt consumption changes in response to shocks.

4.3. **Memory, habits, and insecurity.** In the previous section, we aimed to understand how the presence of memory and habits influences certain key characteristics of the model, such as the magnitude of jumps and welfare.

In this section (and the following ones), we focus on our primary variable of interest, insecurity e, and how it is affected by the structure of memory and habits.

As discussed in the introduction, our characterization of insecurity combines two types of determinants: (i) some have an "objective" nature (e.g., probabilities of job loss, health emergencies) and are captured in the model through parameters describing the frequency, intensity, and effects of shocks. These determinants were analyzed in Section 4.1; (ii) others have a subjective nature, as they are related to the perception of insecurity and are therefore influenced by the structure of preferences in the model. These subjective determinants can further be divided into two categories: those depending on the preference parameters and those depending on the state/history of the system. In this section, we investigate the impact of both on our measure of insecurity. We begin by examining the effect of σ.

PROPOSITION 4.7 *Suppose that Hypothesis 3.1 is verified. Then, in terms of the optimal feedback defined in (7), we have*

$$\frac{de}{d\sigma} > 0.$$

Proof. See Appendix B. □

An increase in σ in the model can be interpreted as an increase in (relative) risk aversion. In this sense, the interpretation of the result is quite straightforward: as risk aversion increases, we observe a heightened sense of insecurity, reflecting greater anxiety about the uncertainty generated by potential future shocks.

When we turn to the parameters related to how agents account for past behavior (namely β, z, and T), the results are more consistent:

PROPOSITION 4.8 *Suppose that Hypothesis 3.1 is verified. Then, in terms of the optimal feedback defined in (7), we have*

$$\frac{de}{d\beta} < 0, \qquad \frac{de}{dz} > 0, \qquad \frac{de}{dT} > 0.$$

Proof. See Appendix B. □

In the model, β represents the rate at which old habits lose importance (the persistence of the habit): the higher β, the faster this loss. The parameter z represents the intensity of the habit, i.e., the relative importance of past consumption, while T represents the memory of the process. The comparative statics with respect to these three parameters convey the same message: as the importance of habits increases, the sense of insecurity grows.

To understand this result, it suffices to realize that in a habit-formation model like that of Constantinides (1990), habits impose a minimum level of consumption. The constraints imposed by the social context and/or the inertia of expenses required by a given lifestyle prevent consumption from falling below a certain minimum. The stronger the habits, the stricter this constraint becomes (a similar mechanism is described, in a different model of habit formation by Campbell and Cochrane, 1999, see in particular the first paragraph of page 221). Therefore, under the same budgetary conditions, it is evident that an individual more influenced by habits is more likely to approach this limit and thus feels greater insecurity in the face of potential abrupt future changes.

These insights are further supported by the final result of this subsection, described in the following proposition: the higher the level of consumption to which the agent is accustomed, the higher, *ceteris paribus*, the sense of insecurity regarding potential future shocks.

PROPOSITION 4.9 *Suppose that Hypothesis 3.1 is verified. Suppose to have, for a fixed time $\bar{t}$, two different consumption histories, c^1 and c^2 with $c_1 < c_2$ and the same initial capital. Than, if we consider the levels of insecurity e found in the optimal feedback defined in (7), we obtain that:*

$$e(k, c^1) < e(k, c^2)$$

Proof. See Appendix B. □

4.4. **Insecurity, growth and stagnation.** We now examine how economic insecurity depends on the parameter α. Our main result is the following:

PROPOSITION 4.10 *Suppose that Hypothesis 3.1 is verified. Then, in terms of the optimal feedback defined in (7), we have*

$$\frac{de}{d\alpha} < 0.$$

Proof. See Appendix B. □

This result can be intuitively understood by recalling that the parameter α represents total factor productivity (TFP) in the system, which directly influences the economy's growth rate. Lower values of α correspond to lower productivity and, consequently, weaker economic growth. In such a scenario, the economy experiences slower recovery from shocks, leading to prolonged periods of heightened economic insecurity. This mechanism explains the result stated in Proposition 4.10.

The link between economic stagnation and rising economic insecurity has already been emphasized by Hacker et al. (2013) and, as noted in the introduction, aligns with the perspective of United Nations (2008) which defines insecurity in terms of resilience in recovering from shocks.

As a point of reflection for future research, we observe that this relationship (as discussed by Inglehart and Norris, 2019) could play an important role in explaining the connection between economic stagnation and the rise of populism (Funke et al., 2016), as empirical evidence suggests that increasing economic insecurity fuels support for populist movements (Algan et al., 2017b).

5. Conclusions

This paper introduces a novel macrodynamic framework to analyze economic insecurity through intertemporal optimization in the presence of habits, finite memory, and Poisson shocks. By explicitly deriving a feedback solution for the optimal control problem, we establish a direct connection between insecurity—measured as agents' willingness to mitigate shocks—and a broad range of economic and behavioral determinants.

Our findings show that economic insecurity increases with the frequency and intensity of shocks, aligning with empirical evidence on the psychological and economic impacts of heightened uncertainty. Furthermore, the model demonstrates how individual preferences and consumption habits can amplify feelings of insecurity: agents with higher risk aversion, stronger habit persistence, higher habit intensity, or longer memory horizons are more anxious about potential future shocks. These results underscore the dual role of habits in promoting consumption stability while simultaneously deepening perceived vulnerability to abrupt changes.

In terms of resource allocation, the mitigation effort rate rises with more frequent or severe shocks, while the saving/investment rate decreases. This dual response reflects the tension between addressing immediate risks and the diminishing attractiveness of future returns. These findings align with empirical evidence on the negative effects of volatility on investment behavior and highlight the intricate dynamics between insecurity, resource allocation, and economic shocks.

From a technical perspective, this article presents the first dynamic model to combine forward-looking agents, finite memory (and thus a state equation with delay), and a Poisson process, with a fully solved optimal solution expressed in feedback form.

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Appendix A. Solution of the optimal control problem

A.1. **The Hilbert space setting.** As a first step we show how we can rewrite the optimal control problem in a suitable Hilbert space formulation.

We introduce first the Hilbert space M^2 defined as

$$M^2 := \mathbb{R} \times L^2([-T,0);\mathbb{R}),$$

with scalar product

$$\left\langle \left(\psi^0, \psi^1(\cdot)\right), \left(\varphi^0, \varphi^1(\cdot)\right) \right\rangle = \psi^0\varphi^0 + \int_{-T}^{0} \psi^1(s)\, \varphi^1(s)\, ds,$$

and the unbounded operator

$$\begin{cases} D(A) := \{(\psi^0, \psi^1) \in M^2 \ : \ \psi^1 \in W^{1,2}(-T,0;\mathbb{R}),\ \psi^0 = \psi^1(0)\} \\ A\colon D(A) \to M^2 \\ A(\psi^0,\psi^1) := (-\beta\psi^0, \frac{\mathrm{d}}{\mathrm{d}s}\psi^1). \end{cases}$$

We also introduce the operator B given by

$$B\colon D(A) \to \mathbb{R}, \qquad B(\psi(0),\psi) = B\psi = z\psi(0) - ze^{-\beta T}\psi(-T).$$

From classical results on existence and uniqueness respectively of stochastic differential equation with Levy processes and delay differential equations (see Theorem 6, page 249 in Protter, 2004 and Theorem 2.3, page 44 in Hale and Lunel, 2013), one can see that, given a couple of initial data $c \in L^2([-T,0);\mathbb{R})$ and $k(0) \in \mathbb{R}$, given two processes $c(\cdot), e(\cdot) \in L^2_{loc,P}$ (defined in (3)), with $e(\cdot)$ $[0,1/\eta]$-valued, the state equations (1) and (2) have a unique $\mathcal{F}_t -$ adapted and cádlág solution (where the solution of (2) is defined trajectory-by-trajectory). We will denote them by $\left(k_{k_0,c(\cdot),e(\cdot)}(\cdot), h_{k_0,c(\cdot),ec(\cdot)}(\cdot)\right)$ or simply by $k(\cdot), h(\cdot)$.

Using these solutions (in particular the solution of (2)) we define, for each time $t \geq 0$, an element of M^2, that we will call, following Vinter and Kwong (1981), *structural state*, defined as

$$X(t) = \left(x^0(t), x^1(t)\right) = \left(h(t), s \mapsto -ze^{-\beta T}c(t-T-s)\right) \in \mathbb{R} \times L^2([-T,0);\mathbb{R}) =: M^2. \tag{13}$$

It will allow to describe the state of our system, for any time $t \geq 0$, as a couple $(k,x) \in \mathbb{R} \times M^2$. We have the following result which connect, via the structural state, the solution of the equations (1) and (2) (in particular the solution of 2) with the solution of an evolution equation in the Hilbert space $\mathbb{R} \times M^2$.

THEOREM A.1 *We consider a couple of initial data $c \in L^2([-T,0);\mathbb{R})$ and $k(0) \in \mathbb{R}$ and two controls $c(\cdot), e(\cdot) \in L^2_{loc,P}$ with $e(\cdot)$ $[0,1/\eta]$-valued. We denote by $k(\cdot)$ and $h(\cdot)$ the unique solution of (1) and (2), we introduce the structural state X as in (13). Then the couple $(k(\cdot), X(\cdot))$ is the unique solution in*

$$\Pi \overset{def}{=} \left\{ f \in C(0,T;M^2) \ : \ \frac{d}{dt} j^* f \in L^2(0,T;D(A)') \right\} \tag{14}$$

to the following evolution equation in $\mathbb{R} \times M^2$:

$$\begin{cases} dk(t) = (\alpha k(t) - \chi\alpha e(t)k(t) - c(t))\, dt - (1-\eta e(t))\,(1-\xi)\, k(t) dP(t) \\ \frac{d}{dt} x(t) = A^* x(t) + B^* c(t) \\ (k(0), x(0)) = \left(k(0),\ z\int_{-T}^0 e^{\beta(s-t)} c(s)\ ds,\ s \mapsto -z e^{-\beta T} c(-T-s) \right). \end{cases} \tag{15}$$

Proof. The first equation of (15) (and the initial datum of k) is just equation (1) rewritten. The second equation follows by the general theory to transform linear controlled DDE into an evolution equation in Hilbert spaces introduced by Vinter and Kwong (1981) and described by Bensoussan et al. (2007) Theorem 5.1 page 258. □

The previous result allows as to rewrite the state equation(s) of the original optimization problem driven by a system containing a delay differential equation into a new system driven by an evolution equation in an Hilbert space. To completely rewrite in terms of the state variables (k,x) and the controls (c,e), we also need to write the target functional, the set of admissible controls, the value function and the set of admissible initial data using the new variables. This can easily done, the functional can be expressed as[8]

$$J((k,x),(c(\cdot),e(\cdot))) = \mathbb{E}\int_0^{+\infty} e^{-\rho t} \frac{(c(t)-x^0(t))^{1-\sigma}}{1-\sigma} dt, \tag{16}$$

and the set of the admissible controls by

$$\mathcal{C}\left(k(0),(x^0(0),x^1(0)(\cdot))\right) = \left\{ \begin{array}{c} c(\cdot), e(\cdot) \in L^2_{loc,P} \ : \ c(\cdot) \text{ is positive, } e(\cdot) \text{ is } [0,1/\eta]\text{-valued and} \\ k(t) \geq 0 \text{ and } c(t) \geq x^0(t)\ a.s. \text{ for all } t \geq 0 \end{array} \right\}. \tag{17}$$

The value function, now defined on $\mathbb{R} \times M^2$ is then

$$V(k,x) = \sup_{(c(\cdot),e(\cdot)) \in \mathcal{C}(k,x)} \mathbb{E}\int_0^{+\infty} e^{-\rho t} \frac{(c(t)-x^0(t))^{1-\sigma}}{1-\sigma} dt \tag{18}$$

while the set $\mathcal{I}$ of admissible initial data can be rewritten as

$$\mathcal{I} := \left\{ (k,(x_0,x_1(\cdot))) \in \mathbb{R} \times M^2 \ : \ \left(1 - z\int_{-T}^0 e^{\pi s} ds\right) k(t) > \pi^{-1} \left(x^0 - z\int_{-T}^0 e^{-\beta T} e^{\gamma s} x^1(s) ds \right) \right\} \tag{19}$$

Finally, we get the HJB of the problem which is given by

[8]To avoid introduce too many notations, and since the two problems are equivalent, we will keep the same notations that we have in the main text (J, V, $\mathcal{C}$ and $\mathcal{I}$). This fact does not create confusion because: in the main text we always refer to the problem in its initial form while in the appendix we always refer to the Hilbert space formulation.

$$\rho V(k,x) = \sup_{c\geq 0, e\in[0,1/\eta]} \Bigg(\frac{1}{1-\sigma}(c-x^0)^{1-\sigma} + \frac{\partial V(k,x)}{\partial k}(\alpha k - e\chi\alpha k - c) +$$

$$\langle x, AD_x V(k,x)\rangle_{M^2} + cBD_x V(k,x) + \Big(V\big((1-(1-\eta e(t))(1-\xi))k, x\big) - V(k,x)\Big)\lambda \Bigg). \quad (20)$$

Denoting by $H_{CV}(k,x,e,c)$ the argument of the maximization on the right-hand side of the equation we can rewrite the previous equation as

$$\rho V(k,x) = H(k,x) := \sup_{c\geq 0, e\in[0,1/\eta]} H_{CV}(k,x,e,c). \quad (21)$$

We will call *current value Hamiltonian* the function $H_{CV}(k,x,e,c)$ and *maximum value Hamiltonian* the function $H(k,x)$.

A.2. **The solution of the optimal control problem in infinite dimension.** We start by looking for an (explicit) solution of the HJB equation (20) that we will characterize in the following as the value function of the problem:

PROPOSITION A.2 *Suppose that Hypothesis 3.1 is verified. Then the function* $v(k,x) = \nu\frac{G(k,h)^{1-\sigma}}{1-\sigma}$ *is solution of the HJB equation (20) on the set* $\mathcal{I}$*, with*

$$\nu = \left(\frac{(\rho+\lambda)}{\sigma} - \frac{(1-\sigma)\alpha}{\sigma}\left(1 + \frac{\chi\xi}{\eta(1-\xi)}\right) - \frac{\chi\alpha}{\eta(1-\xi)}\left(\frac{\chi\alpha}{\eta(1-\xi)\lambda}\right)^{-1/\sigma}\right)^{-\sigma}, \quad (22)$$

$$G(k,x) = \left(1 - z\int_{-T}^0 e^{\pi s}ds\right)k - \pi^{-1}x_0 - \pi^{-1}\int_{-T}^0 e^{\gamma s}x_1(s)ds, \quad (23)$$

where, using the same notation of the main text, we fixed $\pi = \beta + \alpha\left(1 - \frac{\chi}{\eta}\right)$.

Proof. We try to find a solution in the following form:

$$v(k,x) = a\frac{\left(k + bx_0 + b\int_{-T}^0 e^{\gamma s}x_1(s)ds\right)^{1-\sigma}}{1-\sigma}$$

for some (positive) a and γ and some (negative) b.

To simplify the computations we will use the notation

$$\Gamma(k,x) := \left(k + bx_0 + b\int_{-T}^0 e^{\gamma s}x_1(s)ds\right).$$

With this specification we have

$$\frac{\partial v}{\partial k}(k,x) = a\Gamma^{-\sigma},$$

$$D_x v(k,x) = ab\left(1, s\mapsto e^{\gamma s}\right)\Gamma^{-\sigma},$$

We can observe that $D_x v(k,x)$ belongs to the domain of the operator A, defined in (A.1), so we can apply on it the operators A and B defined above. We obtain

$$AD_x v(k,x) = ab\,(-\beta, s \mapsto \gamma e^{\gamma s})\,\Gamma^{-\sigma},$$
$$\langle x, AD_x v(k,x)\rangle_{M^2} = ab\left(-\beta x_0 + \gamma \int_{-T}^{0} e^{\gamma s} x_1(s) ds\right)\Gamma^{-\sigma}$$
$$BD_x v(k,x) = zab\left(1 - e^{-(\beta+\gamma)T}\right)\Gamma^{-\sigma}.$$

The first order condition on c gives

$$0 = (c - x_0)^{-\sigma} - \frac{\partial v}{\partial k}(k,x) + BD_x v(k,x) = (c - x^0)^{-\sigma} - a\Gamma^{-\sigma} + abz\left(1 - e^{-(\beta+\gamma)T}\right)\Gamma^{-\sigma}$$

i.e.

$$(c - x_0) = a^{-1/\sigma}\left[1 - bz\left(1 - e^{-(\beta+\gamma)T}\right)\right]^{-1/\sigma}\Gamma \tag{24}$$

so that

$$cBD_x v(k,x) = x_0 abz\left(1 - e^{-(\beta+\gamma)T}\right)\Gamma^{-\sigma} + a^{1-1/\sigma} zb\left(1 - e^{-(\beta+\gamma)T}\right)\left[1 - bz\left(1 - e^{-(\beta+\gamma)T}\right)\right]^{-1/\sigma}\Gamma^{1-\sigma}.$$

From the first order condition on e we get

$$0 = -\chi\alpha k \frac{\partial v}{\partial k}(k,x) + \lambda\,(1-\xi)\,\eta k \frac{\partial v}{\partial k}((1 - (1 - \eta e)\,(1 - \xi))k, x)$$

which gives

$$a\left(((1 - (1 - \eta e)\,(1 - \xi))k) + bx_0 + b\int_{-T}^{0} e^{\gamma s} x_1(s) ds\right)^{-\sigma} = \frac{a\chi\alpha}{\lambda\,(1-\xi)\,\eta}\left(k + bx_0 + b\int_{-T}^{0} e^{\gamma s} x_1(s) ds\right)^{-\sigma} \tag{25}$$

and then

$$v\big(((1 - (1 - \eta e)\,(1 - \xi))k, x\big) = \frac{a}{1-\sigma}\left(((1 - (1 - \eta e)\,(1 - \xi))k) + bx_0 + b\int_{-T}^{0} e^{\gamma s} x_1(s) ds\right)^{1-\sigma}$$
$$= \frac{a}{1-\sigma}\left(\frac{\chi\alpha}{\lambda\,(1-\xi)\,\eta}\right)^{-(1-\sigma)/\sigma}\left(k + bx_0 + b\int_{-T}^{0} e^{\gamma s} x_1(s) ds\right)^{1-\sigma} = \left(\frac{\chi\alpha}{\lambda\,(1-\xi)\,\eta}\right)^{-(1-\sigma)/\sigma} v\big(k,x\big). \tag{26}$$

From (25) we also get

$$\left((1 - (1 - \eta e)\,(1 - \xi))k + bx_0 + b\int_{-T}^{0} e^{\gamma s} x_1(s) ds\right) = \left(\frac{\chi\alpha}{\lambda\,(1-\xi)\,\eta}\right)^{-1/\sigma}\left(k + bx_0 + b\int_{-T}^{0} e^{\gamma s} x_1(s) ds\right)$$

i.e.

$$-(1 - \eta e)\,(1 - \xi)\,k + \Gamma = \left(\frac{\chi\alpha}{\lambda\,(1-\xi)\,\eta}\right)^{-1/\sigma}\Gamma$$

i.e.

$$e = \frac{1}{\eta} - \frac{\left(1 - \left(\frac{\chi\alpha}{\lambda(1-\xi)\eta}\right)^{-1/\sigma}\right)\Gamma}{\eta\left(1-\xi\right)k}$$

So our HJB equation becomes

$$\begin{aligned}
(\rho+\lambda)\frac{a}{1-\sigma}\Gamma^{1-\sigma} &= \frac{a^{1-1/\sigma}}{1-\sigma}\left[1 - bz\left(1 - e^{-(\beta+\gamma)T}\right)\right]^{-(1-\sigma)/\sigma}\Gamma^{1-\sigma} + a\alpha k\Gamma^{-\sigma} \\
&- \frac{\chi\alpha}{\eta}ka\Gamma^{-\sigma} + \chi\alpha a\frac{\left(1 - \left(\frac{\chi\alpha}{\lambda(1-\xi)\eta}\right)^{-1/\sigma}\right)}{\eta\left(1-\xi\right)}\Gamma^{1-\sigma} - a^{1-1/\sigma}\Gamma^{1-\sigma}\left[1 - bz\left(1 - e^{-(\beta+\gamma)T}\right)\right]^{-1/\sigma} - a\Gamma^{-\sigma}x_0 \\
&+ a\left(-\beta bx_0 + \gamma b\int_{-T}^{0} e^{\gamma s}x_1(s)ds\right)\Gamma^{-\sigma} + x_0 abz\left(1 - e^{-(\beta+\gamma)T}\right)\Gamma^{-\sigma} \\
&+ a^{1-1/\sigma}zb\left(1 - e^{-(\beta+\gamma)T}\right)\left[1 - bz\left(1 - e^{-(\beta+\gamma)T}\right)\right]^{-1/\sigma}\Gamma^{1-\sigma} \\
&+ \left[\left(\frac{\chi\alpha}{\lambda\left(1-\xi\right)\eta}\right)^{-(1-\sigma)/\sigma}\frac{a}{1-\sigma}\Gamma^{1-\sigma}\right]\lambda. \qquad (27)
\end{aligned}$$

We simplify $a\Gamma^{-\sigma}$ and we rewrite the equation as follows:

$$\begin{aligned}
&\begin{pmatrix}
(\rho+\lambda)\frac{1}{1-\sigma} - \frac{a^{-1/\sigma}}{1-\sigma}\left[1 - bz\left(1 - e^{-(\beta+\gamma)T}\right)\right]^{-(1-\sigma)/\sigma} \\
-\chi\alpha\frac{\left(1 - \left(\frac{\chi\alpha}{\lambda(1-\xi)\eta}\right)^{-1/\sigma}\right)}{\eta(1-\xi)} + a^{-1/\sigma}\left[1 - bz\left(1 - e^{-(\beta+\gamma)T}\right)\right]^{-1/\sigma} \\
-a^{-1/\sigma}zb\left(1 - e^{-(\beta+\gamma)T}\right)\left[1 - bz\left(1 - e^{-(\beta+\gamma)T}\right)\right]^{-1/\sigma} \\
-\left(\frac{\chi\alpha}{\lambda(1-\xi)\eta}\right)^{-(1-\sigma)/\sigma}\frac{\lambda}{1-\sigma}
\end{pmatrix}\Gamma = \alpha k\left(1 - \frac{\chi}{\eta}\right) \\
&\qquad + bx_0\left(\frac{-1}{b} - \beta + z\left(1 - e^{-(\beta+\gamma)T}\right)\right) + \gamma b\int_{-T}^{0} e^{\gamma s}x_1(s)ds. \qquad (28)
\end{aligned}$$

If

$$\alpha\left(1 - \frac{\chi}{\eta}\right) = \frac{-1}{b} - \beta + z\left(1 - e^{-(\beta+\gamma)T}\right) = \gamma$$

the right hand side of (28) can be expressed as $\alpha\left(1 - \frac{\chi}{\eta}\right)\Gamma$ so we look for a solution where this relation is verified and we impose

$$\gamma = \alpha\left(1 - \frac{\chi}{\eta}\right)$$

and

$$\alpha\left(1 - \frac{\chi}{\eta}\right) = \frac{-1}{b} - \beta + z\left(1 - e^{-(\beta+\gamma)T}\right) \qquad (29)$$

i.e.

$$b = -\left(\beta + \alpha\left(1 - \frac{\chi}{\eta}\right) - z\left(1 - e^{-(\beta+\gamma)T}\right)\right)^{-1}.$$

So we fixed now (depending on the parameters of the system) b and γ. We only need to characterize a. If we use the previous expression into (28) we get

$$\left(\begin{array}{c} (\rho+\lambda)\frac{1}{1-\sigma}-\frac{a^{-1/\sigma}}{1-\sigma}\left[1-bz\left(1-e^{-(\beta+\gamma)T}\right)\right]^{-(1-\sigma)/\sigma} \\ -\chi\alpha\frac{\left(1-\left(\frac{\chi\alpha}{\lambda(1-\xi)\eta}\right)^{-1/\sigma}\right)}{\eta(1-\xi)}+a^{-1/\sigma}\left[1-bz\left(1-e^{-(\beta+\gamma)T}\right)\right]^{-1/\sigma} \\ -a^{-1/\sigma}zb\left(1-e^{-(\beta+\gamma)T}\right)\left[1-bz\left(1-e^{-(\beta+\gamma)T}\right)\right]^{-1/\sigma} \\ -\left(\frac{\chi\alpha}{\lambda(1-\xi)\eta}\right)^{-(1-\sigma)/\sigma}\frac{\lambda}{1-\sigma} \end{array}\right)=\alpha\left(1-\frac{\chi}{\eta}\right)$$

that we can rewrite as

$$-\frac{\sigma}{1-\sigma}a^{-1/\sigma}\left[1-bz\left(1-e^{-(\beta+\gamma)T}\right)\right]^{1-1/\sigma} = \alpha\left(1-\frac{\chi}{\eta}\right)-\frac{\rho+\lambda}{1-\sigma}+\frac{\chi\alpha}{\eta(1-\xi)}\left(1+\frac{\sigma}{1-\sigma}\left(\frac{\chi\alpha}{\eta(1-\xi)\lambda}\right)^{-1/\sigma}\right) \tag{30}$$

From (29) we have

$$1-bz\left(1-e^{-(\beta+\gamma)T}\right)=-\beta b-\alpha b+\alpha\frac{\chi}{\eta}b$$

so (30) becomes

$$-\frac{\sigma}{1-\sigma}a^{-1/\sigma}\left[-\beta b-\alpha b+\alpha\frac{\chi}{\eta}b\right]^{1-1/\sigma} = \alpha\left(1-\frac{\chi}{\eta}\right)-\frac{\rho+\lambda}{1-\sigma}+\frac{\chi\alpha}{\eta(1-\xi)}\left(1+\frac{\sigma}{1-\sigma}\left(\frac{\chi\alpha}{\eta(1-\xi)\lambda}\right)^{-1/\sigma}\right) \tag{31}$$

so that

$$a=\left(\frac{-\sigma\left(-\beta b-\alpha b+\alpha\frac{\chi}{\eta}b\right)^{1-1/\sigma}}{(1-\sigma)\alpha\left(1-\frac{\chi}{\eta}\right)-(\rho+\lambda)+\frac{\chi\alpha}{\eta(1-\xi)}\left(1-\sigma+\sigma\left(\frac{\chi\alpha}{\eta(1-\xi)\lambda}\right)^{-1/\sigma}\right)}\right)^{\sigma}.$$

Letting $\pi=\beta+\alpha\left(1-\frac{\chi}{\eta}\right)$, then

$$\begin{aligned} b &= -\pi^{-1}\left(1-z\int_{-T}^{0}e^{\pi s}ds\right)^{-1}, \\ a &= (-\pi b)^{(\sigma-1)}\left(-\frac{(1-\sigma)}{\sigma}\alpha\left(1-\frac{\chi}{\eta}\right)+\frac{(\rho+\lambda)}{\sigma}-\frac{\chi\alpha}{\eta(1-\xi)}\left(\frac{1-\sigma}{\sigma}+\left(\frac{\chi\alpha}{\eta(1-\xi)\lambda}\right)^{-1/\sigma}\right)\right)^{-\sigma}. \end{aligned}$$

The latter equation can be simplified as

$$a=\left(1-z\int_{-T}^{0}e^{\pi s}ds\right)^{(1-\sigma)}\left(\frac{(\rho+\lambda)}{\sigma}-\frac{(1-\sigma)}{\sigma}\alpha\left(1+\frac{\xi\chi}{\eta}\right)-\frac{\chi\alpha}{\eta(1-\xi)}\left(\frac{\chi\alpha}{\eta(1-\xi)\lambda}\right)^{-1/\sigma}\right)^{-\sigma}.$$

If we denote $\nu=a\left(1-z\int_{-T}^{0}e^{\pi s}ds\right)^{-(1-\sigma)}$ and $G(k,x)=\left(1-z\int_{-T}^{0}e^{\pi s}ds\right)\Gamma(k,x)$ we get the claim. □

Now we want to prove that the feedback that we obtain from the solution of the Hamilton-Jacobi-Bellman HJB equation that we found in Proposition A.2 is indeed optimal and, at the same time, that the solution of the HJB equation that we found is indeed the value function of the problem.

To prove that we need some preliminary work. I particular we will need two result: the first is a results which explicitly show how the quantity $\Gamma(t)$ evolves along the candidate-optimal trajectory and that is essential to show that the candidate-optimal trajectories are indeed admissible (Lemma A.3), the second (Lemma A.4) is an estimate about the maximal growth of the quantity $\mathbb{E}[k^{1-\sigma}(t)]$ (varying the control amount the set of admissible control). This estimate will the useful to estimate the evolution of the values of $\mathbb{E}[v(k(t), X(t))]$ along the trajectories and finally to prove the verification result in Theorem A.5.

First of all we introduce some notation: the closed-loop policy associated with the HJB equation (20) will be denoted $\varphi(k, x)$ for the one defining consumption and $\phi(k, x)$ for the one defining mitigation effort e:

$$\varphi(k,x) = x_0 + \nu^{-1/\sigma} G(k,x), \tag{32}$$

$$\phi(k,x) = \frac{1}{\eta} - \frac{\Gamma(k,x)}{\eta(1-\xi)k}\left(1 - \left(\frac{\lambda(1-\xi)\eta}{\chi\alpha}\right)^{1/\sigma}\right). \tag{33}$$

Lemma A.3 *Suppose that Hypothesis 3.1 is verified and that (k, x) is un initial datum in $\mathcal{I}$. Consider a solution $(k_{\varphi,\phi}(\cdot), X_{\varphi,\phi}(\cdot))$ of*

$$\begin{cases} dk(t) = (\alpha k(t) - \chi\alpha\phi(k(t), x(t))\, k(t) - \varphi(k(t), x(t)))\, dt - (1 - \eta\phi(k(t), x(t)))(1-\xi)\, k(t) dP(t) \\ \frac{d}{dt} x(t) = A^* x(t) + B^*(\varphi(k(t), x(t))) \\ (k(0), x(0)) = (k, x) \end{cases} \tag{34}$$

Then, defining $G(k, x)$ as in (23), along the trajectory of (34) we have

$$\begin{aligned} dG(k_{\varphi,\phi}, X_{\varphi,\phi}) = {} & \left(\gamma + \left(\frac{\chi\alpha}{\eta(1-\xi)}\left(1 - \left(\frac{\lambda(1-\xi)\eta}{\chi\alpha}\right)^{1/\sigma}\right) - \nu^{-1/\sigma}\right)\right) G(k_{\varphi,\phi}, X_{\varphi,\phi})\, dt \\ & - \left(1 - z\int_{-T}^{0} e^{\pi s} ds\right)^{-1}\left(1 - \left(\frac{\lambda(1-\xi)\eta}{\chi\alpha}\right)^{1/\sigma}\right) G(k_{\varphi,\phi}, X_{\varphi,\phi})\, dP(t) \end{aligned}$$

In particular, along the candidate optimal trajectories we have

$$G(k_{\varphi,\phi}(t), X_{\varphi,\phi}(t)) = G(k_{\varphi,\phi}(0), X_{\varphi,\phi}(0))\, e^{\left(\gamma + n\frac{\chi\alpha}{\eta(1-\xi)} - \nu^{-1/\sigma}\right)t}(1-mn)^{P(t)} \tag{35}$$

and

$$\mathbb{E}[G(k_{\varphi,\phi}(t), X_{\varphi,\phi}(t))] = G(k_{\varphi,\phi}(0), X_{\varphi,\phi}(0))\, e^{\left(\gamma + n\frac{\chi\alpha}{\eta(1-\xi)} - \nu^{-1/\sigma}\right)t} e^{-\lambda mnt}.$$

Since Γ is G times a (positive) multiplicative constant, the same equations hold for $\Gamma(k_{\varphi,\phi}(t), X_{\varphi,\phi}(t))$. Observe that, thanks to (35), $G(k_{\varphi,\phi}(t), X_{\varphi,\phi}(t)) > 0$ for all $t \geq 0$.

Proof. Indeed, as $G(k,x) = \left(\left(1 - z\int_{-T}^{0} e^{\pi s}ds\right)k - \pi^{-1}x_0 - \pi^{-1}\int_{-T}^{0} e^{\gamma s}x_1(s)ds\right)$,

$$G(k,x) = \left(1 - z\int_{-T}^{0} e^{\pi s}ds\right)k + \langle x, \kappa\rangle,$$

with $\kappa = -\pi^{-1}(1, s \mapsto e^{\gamma s})$. Thus

$$\begin{aligned}
dG(X_{\varphi,\phi}) &= \left(1 - z\int_{-T}^{0} e^{\pi s}ds\right)dk + \langle dx(t), \kappa\rangle \\
&= \left(1 - z\int_{-T}^{0} e^{\pi s}ds\right)dk + \langle A^* x(t) + B^*(\varphi(k(t), x(t))), \kappa\rangle dt \\
&= \left(1 - z\int_{-T}^{0} e^{\pi s}ds\right)dk + \langle x(t), A\kappa\rangle dt + \left\langle \left(x_0 + \nu^{-1/\sigma}G(k,x)\right), B\kappa\right\rangle dt \\
&= \left(1 - z\int_{-T}^{0} e^{\pi s}ds\right)dk - \pi^{-1}\langle x(t), (-\beta, s\mapsto \gamma e^{\gamma s})\rangle dt - z\pi^{-1}\left(1 - e^{-\pi T}\right)\left(x_0 + \nu^{-1/\sigma}G(k,x)\right)dt \\
&= \left(1 - z\int_{-T}^{0} e^{\pi s}ds\right)\left(\alpha\left(1 - \frac{\chi}{\eta}\right)k + \frac{\chi\alpha}{\eta}\frac{\Gamma(k,x)}{(1-\xi)}\left(1 - \left(\frac{\lambda(1-\xi)\eta}{\chi\alpha}\right)^{1/\sigma}\right) - x_0 - \nu^{-1/\sigma}G(k,x)\right)dt \\
&\quad -\pi^{-1}\left(\left(-\beta + z\left(1 - e^{-\pi T}\right)\right)x_0 + \int_{-T}^{0}\gamma e^{\gamma s}x_1(s)\,ds\right)dt - z\nu^{-1/\sigma}G(k,x)\left(\int_{-T}^{0} e^{\pi s}ds\right)dt \\
&\quad -(1 - \eta\phi(k(t), x(t)))(1-\xi)k(t)dP(t).
\end{aligned}$$

We have that

$$\begin{aligned}
&\left(1 - z\int_{-T}^{0} e^{\pi s}ds\right)\left(\alpha\left(1 - \frac{\chi}{\eta}\right)k + \frac{\chi\alpha}{\eta}\frac{\Gamma(k,x)}{(1-\xi)}\left(1 - \left(\frac{\lambda(1-\xi)\eta}{\chi\alpha}\right)^{1/\sigma}\right) - x_0 - \nu^{-1/\sigma}G(k,x)\right) \\
&-\pi^{-1}\left(\left(-\beta + z\left(1 - e^{-\pi T}\right)\right)x_0 + \int_{-T}^{0}\gamma e^{\gamma s}x_1(s)\,ds\right) - z\nu^{-1/\sigma}G(k,x)\left(\int_{-T}^{0} e^{\pi s}ds\right), \\
= &\left(1 - z\int_{-T}^{0} e^{\pi s}ds\right)\gamma k - \pi^{-1}\gamma x_0 - \pi^{-1}\gamma\int_{-T}^{0} e^{\gamma s}x_1(s)\,ds \\
&+\left(\frac{\chi\alpha}{\eta(1-\xi)}\left(1 - \left(\frac{\lambda(1-\xi)\eta}{\chi\alpha}\right)^{1/\sigma}\right) - \nu^{-1/\sigma}\right)G(k,x).
\end{aligned}$$

Thus, as $G(k,x) = \left(\left(1 - z\int_{-T}^{0} e^{\pi s}ds\right)k - \pi^{-1}x_0 - \pi^{-1}\int_{-T}^{0} e^{\gamma s}x_1(s)ds\right)$,

$$\begin{aligned}
dG(X_{\varphi,\phi}) &= \left(\gamma + \left(\frac{\chi\alpha}{\eta(1-\xi)}\left(1 - \left(\frac{\lambda(1-\xi)\eta}{\chi\alpha}\right)^{1/\sigma}\right) - \nu^{-1/\sigma}\right)\right)G(k,x)\,dt \\
&\quad -\Gamma(k,x)\left(1 - \left(\frac{\lambda(1-\xi)\eta}{\chi\alpha}\right)^{1/\sigma}\right)dP(t).
\end{aligned}$$

Expressing $\Gamma(k,x)$ and a function of $G(k,x)$ yields to:

$$
\begin{aligned}
dG\left(X_{\varphi,\phi}\right) \quad &= \quad \left(\gamma + \left(\frac{\chi\alpha}{\eta(1-\xi)}\left(1-\left(\frac{\lambda\left(1-\xi\right)\eta}{\chi\alpha}\right)^{1/\sigma}\right) - \nu^{-1/\sigma}\right)\right) G\left(k,x\right)dt \\
&\quad - \left(1 - z\int_{-T}^{0} e^{\pi s}ds\right)^{-1}\left(1-\left(\frac{\lambda\left(1-\xi\right)\eta}{\chi\alpha}\right)^{1/\sigma}\right) G\left(k,x\right)dP(t)
\end{aligned}
$$

and then we have the claim. □

LEMMA A.4 *Suppose that Hypothesis 3.1 is verified and that (k, x) is an initial datum in $\mathcal{I}$. Then, for any $(c(\cdot), e(\cdot)) \in \mathcal{C}\left(k(0), (x^0(0), x^1(0)(\cdot))\right)$, denoted by $(k(\cdot), X(\cdot))$ the related trajectory, we have*

$$\lim_{t\to+\infty} e^{-\rho t}\mathbb{E}\left[\left(G\left(k(t), X(t)\right)\right)^{1-\sigma}\right] = 0.$$

Proof. Since $(c(\cdot), e(\cdot))$ is an admissible control then c remains positive and then x_1, which satisfies (13), remains negative. So, since we know, thanks to Theorem A.1, that

$$x_0 = -e^{\beta T}\int_{-T}^{0} e^{\beta(-T-s)}x_1(s)ds,$$

we have

$$\left(x_0 + \int_{-T}^{0} e^{\gamma s}x_1(s)ds\right) = \int_{-T}^{0}\left(e^{\gamma s} - e^{-\beta s}\right)x_1(s)ds$$

and then, since by Hypothesis 3.1(h.3) $\gamma > -\beta$, $\left(x_0 + \int_{-T}^{0} e^{\gamma s}x_1(s)ds\right)$ remain positive. The expression of $G(k,x)$ is

$$G\left(k,x\right) = \left(1 - z\int_{-T}^{0} e^{\pi s}ds\right)k - \pi^{-1}\left(x_0 + \int_{-T}^{0} e^{\gamma s}x_1(s)ds\right),$$

and then, along the admissible trajectories,

$$0 < G\left(k,x\right) \le \left(1 - z\int_{-T}^{0} e^{\pi s}ds\right)k,$$

so it we knew that

$$\lim_{t\to+\infty} e^{-\rho t}\mathbb{E}\left[k(t)^{1-\sigma}\right] = 0. \tag{36}$$

we would be sure that

$$\lim_{t\to+\infty} e^{-\rho t}\mathbb{E}\left[\left(G\left(k(t), X(t)\right)\right)^{1-\sigma}\right] = 0.$$

We concentrate our attention now proving (36). Given the structure of the stochastic equation for k it always remains positive. To estimate the higher possible growth of k we can consider the situation where $c \equiv 0$ since increasing c always reduces the future level of k (of course taking $c \equiv 0$ could not be an admissible choice but the growth of any admissible choice will be smaller than that). Imagine to reach a certain time t a level $k(t)$ and to take a control e which is constant on $(t, t+\varepsilon)$ for some small $\varepsilon > 0$[9]. At $t+\varepsilon$ the value of k will be

$$k(t+\varepsilon) = k(t)e^{(1-\chi e)\varepsilon}\left[1-(1-\eta e)(1-\xi)\right]^{P(\varepsilon)}$$

[9]The argument could be presented in a little more formal fashion but, since the final choice of e will not depend on ε, it will be clear that everything works.

So

$$k(t+\varepsilon)^{1-\sigma} = k(t)e^{(1-\chi e)\alpha(1-\sigma)\varepsilon}\left[1-(1-\eta e)(1-\xi)\right]^{(1-\sigma)P(\varepsilon)}$$

i.e. $y(s) = k(s)^{1-\sigma}$ is, for $s \in (t, t+\varepsilon)$ a solution of the following stochastic equation

$$dy(s) = (1-\chi e)\alpha(1-\sigma)y(s)ds - \tilde{\beta}dP(s)$$

where

$$\tilde{\beta} = 1 - \left[1-(1-\eta e)(1-\xi)\right]^{(1-\sigma)}.$$

In particular, for $s \in (t, t+\varepsilon)$,

$$\mathbb{E}[k(t+\varepsilon)^{1-\sigma}] = \mathbb{E}[y(t+\varepsilon)] = y(t)e^{(1-\chi e)\alpha(1-\sigma)\varepsilon} - \lambda\left(1-\left[1-(1-\eta e)(1-\xi)\right]^{(1-\sigma)}\right).$$

What is the choice of e which maximizes this quantity and then the growth of k? The necessary and sufficient first order condition gives

$$\left[1-(1-\eta e)(1-\xi)\right]^{(1-\sigma)} = \frac{\chi\alpha}{\eta(1-\xi)\lambda}$$

i.e.

$$e = \frac{1}{\eta(1-\eta)}\left[-\xi + \left(\frac{\chi\alpha}{\eta(1-\xi)\lambda}\right)^{-1/\sigma}\right].$$

The corresponding growth rate of $\mathbb{E}[k(t+\varepsilon)^{1-\sigma}]$, i.e. its highest possible growth rate will be

$$\alpha(1-\sigma) - \lambda - \chi e\alpha(1-\sigma) + \lambda\left(\xi + \eta(1-\xi)\right)^{1-\sigma}$$
$$= \alpha(1-\sigma) + \lambda + \sigma\left(\frac{\chi\alpha}{\eta(1-\xi)}\right)\left(\frac{\chi\alpha}{\eta(1-\xi)\lambda}\right)^{-1/\sigma} + \frac{\xi\chi\alpha(1-\sigma)}{\eta(1-\xi)}. \quad (37)$$

So we have the limit (36) if and only if ρ is higher than the highest possible growth rate of $\mathbb{E}\left[k(t)^{1-\sigma}\right]$ i.e.

$$\rho > \alpha(1-\sigma) + \lambda + \sigma\left(\frac{\chi\alpha}{\eta(1-\xi)}\right)\left(\frac{\chi\alpha}{\eta(1-\xi)\lambda}\right)^{-1/\sigma} + \frac{\xi\chi\alpha(1-\sigma)}{\eta(1-\xi)}$$

but this is exactly point (h.4) of Hypothesis 3.1 so it is verified and we get the claim. □

THEOREM A.5 *Suppose that Hypothesis 3.1 is verified and that (k,x) is an initial datum in $\mathcal{I}$. The optimal control for the problem (15)-(16)-(17) is characterized (as a closed loop control) as*

$$c = \varphi(k,x)$$

$$e = \phi(k,x)$$

where φ and ϕ are defined in (32) and (33). Moreover, for all (k,x) in $\mathcal{I}$, the solution of the HJB equation (20) characterized in Proposition A.2, is the value function (18) of the problem.

Proof. φ and ϕ are the closed-loop policy associated with the solution of the HJB equation (20) characterized in Proposition A.2 in the sense that they are the argmax of the current Hamiltonian.

Consider now a generic control $(c(\cdot), e(\cdot)) \in \mathcal{C}\left(k(0), (x^0(0), x^1(0)(\cdot))\right)$ and denote by $(k(\cdot), X(\cdot))$ the related trajectory.

Taking into account the function v defined in Proposition A.2 we introduce the function:

$$v_0(t,k,x)\colon \mathbb{R}\times\mathbb{R}\times M^2 \to \mathbb{R}, \qquad v_0(t,k,x) := e^{-\rho t} v(k,x).$$

Using that $(Dv(k(t), X(t))) \in D(A)$ and that the application $x \mapsto Dv(k,x)$ is continuous with respect to the norm of $\mathbb{R}\times D(A)$, we find:

$$\begin{aligned} dv_0(t,k(t),X(t)) &= -\rho v_0(t,k(t),X(t))dt \\ &\quad + D_k v_0(t,k(t),X(t))\left(\alpha k(t) - \chi\alpha e(t) - c(t)\right)dt \\ &\quad + \left[v_0(t-,k(t-),X(t-)) - v_0(t-,(1-(1-\eta e(t-)\left(k(t-),x(t-)\right))(1-\xi))k(t-),X(t-))\right]dP(t) \\ &\quad + \langle D_x v_0(t,k(t),X(t)) | A^* X(t) + B^* c(t)\rangle_{D(A)\times D(A)'} dt. \end{aligned} \tag{38}$$

If we use that

$$\begin{aligned} &\langle D_x v_0(t,k(t),X(t)) | A^* X(t) + B^* c(t)\rangle_{D(A)\times D(A)'} \\ &\qquad = \langle A D_x v_0(t,k(t),X(t)), X(t)\rangle_{M^2} + c(t) B D_x v_0(t,k(t),X(t)), \end{aligned} \tag{39}$$

integrating on $[0,\tau]$ and taking the expectation $\mathbb{E}$ we get, from (38),

$$\begin{aligned} &\mathbb{E} v_0(\tau,k(\tau),X(\tau)) - v_0(0,k,x) = \\ &= \mathbb{E}\int_0^\tau e^{-\rho t}\Big(-\rho v(k(t),X(t)) + \bigg(\frac{\partial V(k(t),x(t))}{\partial k}(\alpha k(t) - e(t)\chi\alpha k(t) - c(t)) + \langle x(t), A D_x V(k(t),x(t))\rangle_{M^2} \\ &\qquad + c(t) B D_x V(k(t),x(t)) + \Big(V\big((1-(1-\eta e(t))(1-\xi))k(t), x(t)\big) - V(k(t),x(t))\Big)\lambda\bigg)\,\mathrm{d}t \end{aligned} \tag{40}$$

If we add in each side the quantity

$$\mathbb{E}\int_0^\tau e^{-\rho t}\frac{(c(t) - x^0(t))^{1-\sigma}}{1-\sigma}dt$$

we obtain

$$\begin{aligned} &\mathbb{E} v_0(\tau,k(\tau),X(\tau)) + \mathbb{E}\int_0^\tau e^{-\rho t}\frac{(c(t)-x^0(t))^{1-\sigma}}{1-\sigma}dt - v_0(0,k,x) = \\ &\qquad = \mathbb{E}\int_0^\tau e^{-\rho t}\Big(-\rho v(k(t),X(t)) + H_{CV}(k(t),X(t),e(t),c(t))\,\mathrm{d}t. \end{aligned} \tag{41}$$

Thank to Lemma A.4 know that

$$\lim_{\tau\to+\infty} \mathbb{E} v_0(\tau,k(\tau),X(\tau)) = 0$$

and we can pass to the limit in (41) getting

$$\mathbb{E}\int_0^{+\infty} e^{-\rho t}\frac{(c(t)-x^0(t))^{1-\sigma}}{1-\sigma}dt - v_0(0,k,x) =$$
$$= \mathbb{E}\int_0^{+\infty} e^{-\rho t}\Big(-\rho v(k(t),X(t)) + H_{CV}(k(t),X(t),e(t),c(t))\,\mathrm{d}t. \quad (42)$$

If we use in previous expression the fact that v is a solution of (21) we get

$$\mathbb{E}\int_0^{+\infty} e^{-\rho t}\frac{(c(t)-x^0(t))^{1-\sigma}}{1-\sigma}dt - v_0(0,k,x) =$$
$$= \mathbb{E}\int_0^{+\infty} e^{-\rho t}\Big(H_{CV}(k(t),X(t),e(t),c(t)) - H_{CV}(k(t),X(t))\,\mathrm{d}t \quad (43)$$

now observe that $H_{CV}(k,X,e,c) \le H(k,X)$, which, according to equation (43), implies that for every admissible control

$$\mathbb{E}\int_0^{+\infty} e^{-\rho t}\frac{(c(t)-x^0(t))^{1-\sigma}}{1-\sigma}dt \le v_0(0,k,x).$$

On the other hand the feedback strategy $\varphi(k,x)$, $\phi(k,x)$ satisfies the same expression with the equality so its utility is higher than the utility of any other control so it is optimal. Since $v_0(0,k,x) = v(k,x)$ is the value of the target function at the optimal control it is the value function. □

A.3. **From the Hilbert space setting back to the original problem: the proofs of Theorem 3.2 and 3.3.**

Proof of Theorem 3.2. The claim follows by Theorem A.5 rewriting the result with the delay differential equation notation that we use in Section 3. □

Proof of Theorem 3.3. The claim follows by Lemma A.3 rewriting the result with the delay differential equation notation that we use in Section 3. □

Appendix B. Other proofs

Proof of Proposition 4.1. Observe that, from (7), we have:

$$e = \frac{1}{\eta}\left(1 - \frac{n}{1-\xi}\right) + \frac{nz}{m\pi(1-\xi)}\frac{\int_{-T}^0\left[e^{\beta s} - e^{-\beta T}e^{\gamma(-T-s)}\right]c(s+t)ds}{k(t)}.$$

We then compute the derivative of e.

$$\frac{de(t)}{d\lambda} = \frac{1}{\sigma}\left(\frac{\lambda(1-\xi)\eta}{\chi\alpha}\right)^{\frac{1}{\sigma}-1}\cdot\frac{(1-\xi)\eta}{\chi\alpha}\cdot\frac{k - \frac{1}{m\pi}\left(z\int_{-T}^0\left[e^{\beta s} - e^{-\beta T}e^{\gamma(-T-s)}\right]c(s+t)\,ds\right)}{\eta(1-\xi)k} > 0.$$

Similarly, if we now consider the impact of a variation the shock magnitude we have

$$\frac{de(t)}{d\xi} = -\frac{k - \frac{1}{m\pi}\left(z\int_{-T}^0\left[e^{\beta s} - e^{-\beta T}e^{\gamma(-T-s)}\right]c(s+t)ds\right)}{\eta\,(1-\xi)^2}\left(\left(\frac{\lambda(1-\xi)\eta}{\chi\alpha}\right)^{\frac{1}{\sigma}}\left(\frac{1}{\sigma}-1\right)+1\right).$$

It can be easily seen, according to the properties of n that whatever σ, the latter bracket is positive. Thus, as expected, the greater the magnitude of the shock (the less ξ), the greater the e. □

Proof of Proposition 4.2. Observe that, from (7), we have:

$$e = \frac{1}{\eta}\left(1 - \frac{n}{1-\xi}\right) + \frac{nz}{m\pi(1-\xi)} \frac{\int_{-T}^{0}\left[e^{\beta s} - e^{-\beta T} e^{\gamma(-T-s)}\right] c(s+t)ds}{k(t)}.$$

In this expression, everything remains constant before and after the shock except for the value of k, which decreases. Consequently, the value of e increases. □

Proof of Proposition 4.3. We start looking at the effects on θ_m. Its derivatives, a part for a positive constant, is the same as the derivatives of e.

Concerning consumption,

$$c(t) = h(t) + \nu^{-1/\sigma} mk(t) - \frac{\nu^{-1/\sigma}}{\pi}\left(z \int_{-T}^{0}\left[e^{\beta s} - e^{-\beta T} e^{\gamma(-T-s)}\right] c(s+t)ds\right)$$

$$\frac{d\theta_c\left(t\right)}{d\lambda} = \frac{1}{\alpha k(t)}\frac{dc\left(t\right)}{d\lambda} = \frac{-\left(\frac{\alpha\chi}{\eta\lambda(1-\xi)}\right)^{-\frac{1}{\sigma}+1}\left(1-\left(\frac{\alpha\chi}{\eta\lambda(1-\xi)}\right)^{\frac{1}{\sigma}-1}\right)}{\sigma\alpha k(t)} \times \left(mk(t) - \frac{1}{\pi}\left(z\int_{-T}^{0}\left[e^{\beta s} - e^{-\beta T} e^{\gamma(-T-s)}\right] c(s+t)ds\right)\right) < 0 \quad (44)$$

and

$$\frac{d\theta_c\left(t\right)}{d\xi} = \frac{1}{\alpha k(t)}\frac{dc\left(t\right)}{d\xi} = \frac{\left(1-\sigma\right)\alpha\chi\left(\frac{\alpha\chi}{\eta\lambda(1-\xi)}\right)^{-\frac{1}{\sigma}}\left(1-\left(\frac{\alpha\chi}{\eta\lambda(1-\xi)}\right)^{\frac{1}{\sigma}}\right)}{\eta\sigma\left(1-\xi\right)^2\alpha k(t)} \times \left(mk(t) - \frac{1}{\pi}\left(z\int_{-T}^{0}\left[e^{\beta s} - e^{-\beta T} e^{\gamma(-T-s)}\right] c(s+t)ds\right)\right). \quad (45)$$

So, while θ_c always decrease with respect to shock frequency, it decreases with shock magnitude if and only if $\sigma < 1$.

Similarly, saving s is

$$\begin{aligned} s &= \alpha k - \chi e \alpha k - c \\ &= \alpha k - \frac{\alpha\chi}{\eta}\left(1 - n\frac{\Pi}{(1-\xi)}\right) - h(t) - \nu^{-1/\sigma} m\Pi \end{aligned}$$

where $\Pi = k - \frac{1}{m\pi}\left(z\int_{-T}^{0}\left[e^{\beta s} - e^{-\beta T} e^{\gamma(-T-s)}\right] c(s+t)ds\right)$. Then

$$\begin{aligned}\frac{d\theta_s}{d\lambda} &= -\frac{1}{\alpha k(t)}\frac{\Pi}{\sigma} < 0,\\ \frac{d\theta_s}{d\xi} &= \frac{1}{\alpha k(t)}\frac{\alpha\xi\Pi}{\eta\sigma\left(1-\xi\right)^2} > 0\end{aligned}$$

and so an increase in the shock frequency or magnitude implies an increase in precautionary saving. This concludes the proof. □

Proof of Proposition 4.5. Assume a shock happens at time t and that consumption and capital before the shock are respectively $c\left(t\right)$ and $k\left(t\right).$

In the model without habit $\left(z=0\right)$, then

$$\begin{aligned}c\left(t^+\right)-c\left(t^-\right) &= \nu^{-1/\sigma}k(t^+)-\nu^{-1/\sigma}k(t^-),\\ &= -\nu^{-1/\sigma}nk\left(t^-\right),\\ &= -nc\left(t^-\right).\end{aligned}$$

In the model with habit, then

$$c\left(t^+\right)=h(t^+)+\nu^{-1/\sigma}m\Gamma(t^+).$$

However, h is a deterministic variable, thus $h(t^+)=h\left(t\right).$ Moreover, $\Gamma(t^+)=\left(1-mn\right)\Gamma(t^-)$, thus the following equation holds.

$$\begin{aligned}c\left(t^+\right)-c\left(t^-\right) &= h(t)+\nu^{-1/\sigma}m\left(1-mn\right)\Gamma(t^-)-\left(h(t)+\nu^{-1/\sigma}m\Gamma(t^-)\right),\\ &= -m^2nc\left(t^-\right)+\nu^{-1/\sigma}m^2nh\left(t\right).\end{aligned}$$

As $0<m^2n<n$, the result holds. □

Proof of Proposition 4.6. The proof is based on the computation developed by Lucas. We indeed consider μ, such that

$$\begin{aligned}V\left(\left(1-\mu\right)k_0,c_0(\cdot),0\right) &= V\left(k_0,c_0(\cdot),z\right),\\ \mu &= 1-m\left(z\right)\left(1-\frac{1}{m\left(z\right)\pi}\left(z\int_{-T}^{0}\left[e^{\beta s}-e^{-\beta T}e^{\gamma(-T-s)}\right]\frac{c_0(s)}{k_0}ds\right)\right)>0.\end{aligned}$$

This expression, in addition to showing (due to its positivity) that welfare in the case of the model without habit is always superior to that of the model with habit, also measures the loss of utility in "real" terms. □

Proof of Proposition 4.7. In the expression (7) the unique quantity which depends on σ is n. It is easy to see that when σ increases n decreases and then e increases. □

Proof of Proposition 4.8. The value of β influences the expression (7) in three ways, all of which point in the same direction: (i) increasing β changes the integrand, where β appears explicitly. Referring to Footnote 3, we can rewrite the integral as

$$\left(z\int_{-T}^{0}\left[e^{\beta s}-e^{-\beta T}e^{\gamma(-T-s)}\right]c(s)ds\right)=\left(z\int_{-T}^{0}e^{-\gamma s}\left[e^{(\beta+\gamma)s}-e^{-(\beta+\gamma)T}\right]c(s)ds\right).$$

It is straightforward to verify that the term $\left[e^{(\beta+\gamma)s}-e^{-(\beta+\gamma)T}\right]$ decreases with β. (ii) Increasing β raises π (see the definition in (h.3) of Hypothesis 3.1). (iii) Increasing β raises m (see the definition in (h.5) of Hypothesis 3.1, noting that s in the integrand is always negative). All three channels lead to a reduction in the expression for e in (7) (see particularly the final expression).

Increasing z has both a direct effect in (7) (which increases e) and an indirect effect via m (see the definition in (h.5) of Hypothesis 3.1), which decreases and thereby also increases e. Thus, we find $\frac{de}{dz}>0$.

A similar argument applies to T. This concludes the proof. □

Proof of Proposition 4.9. It follows immediately by (7) since the consumption only enters in the integral expression. □

Proof of Proposition 4.10. We have that

$$e=\frac{1}{\eta}-n\frac{k-\frac{z}{\pi m}\int_{-T}^{0}\left(e^{\beta s}-e^{-\beta T}e^{-\gamma(-T-s)}\right)c\left(s+t\right)ds}{\eta\left(1-\xi\right)k\left(t\right)}$$

where

$$\begin{aligned}n&=1-(C\alpha)^{-1/\sigma}\\ \frac{\partial n}{\partial\alpha}&=\frac{1}{\alpha\sigma}(C\alpha)^{-1/\sigma}\end{aligned}$$

and

$$\pi m=\pi\left(1-z\int_{-T}^{0}e^{-\pi s}ds\right)$$

$$\frac{\partial\left(\pi m\right)}{\partial\alpha}=\frac{\partial\pi}{\partial\alpha}\left(m+\pi z\int_{-T}^{0}se^{-\pi s}ds\right).$$

Thus

$$\begin{aligned}\frac{\partial e}{\partial\alpha}&=-\frac{\partial n}{\partial\alpha}\frac{k-\frac{z}{\pi m}\int_{-T}^{0}\left(e^{\beta s}-e^{-\beta T}e^{-\gamma(-T-s)}\right)c\left(s+t\right)ds}{\eta\left(1-\xi\right)k\left(t\right)}\\ &-\frac{n}{\eta\left(1-\xi\right)k\left(t\right)}\frac{\partial\pi}{\partial\alpha}\left(\begin{array}{c}\frac{z\left(m+\pi z\int_{-T}^{0}se^{-\pi s}ds\right)}{(\pi m)^{2}}\int_{-T}^{0}\left(e^{\beta s}-e^{-\beta T}e^{-\gamma(-T-s)}\right)c\left(s+t\right)ds\\ +\frac{z}{\pi m}\int_{-T}^{0}\left(T+s\right)\left(e^{-\beta T}e^{\gamma(T+s)}\right)c\left(s+t\right)ds\end{array}\right)<0.\end{aligned}$$

So we have the claim. □

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