Monopoly with Product Design*

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Abstract

We analyze a monopolist who designs and prices a product, with the design represented as a location on a Hotelling line. We characterize the sets of prices, consumer surpluses, and profits that can arise in the model across all distributions of consumer tastes, and compare them to those of the classical monopoly model without product design. Product design narrows the set of predictions: too low prices are never optimal, the seller's profit is bounded away from zero, and the maximal achievable consumer surplus is smaller than in the model without design. We also show that the consumer-optimal distributions, unlike in the classical monopoly problem without design, do not exhibit global unit elasticity, and the profit-minimizing distributions exhibit a type of uniformity property —they allocate equal probability mass to a finite number of partition cells of equal width.

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1 Introduction

Before entering the market, a product must be carefully designed and tailored to its target audience. For instance, after developing a promising new software, a company might need to decide: should it prioritize user-friendliness at the expense of speed, or opt for a faster but more complex interface? Similarly, a fashion brand might choose between a trendy or a more classic design for

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its latest item. An electronics company could design its device to primarily attract a younger audience or aim for broader appeal, risking that younger consumers might find it less attractive. The classical monopoly model sidesteps these product design considerations, focusing solely on the monopolist's pricing incentives.

This paper examines how product design interacts with pricing decisions in a monopoly setting and explores the implications for the allocation of surplus. Specifically, we investigate the predictions the model with product design can generate under different distributions of consumer tastes and demonstrate that some of them are markedly different from those of the classical model without product design.[1]

We consider a market populated by a continuum of consumers, each with unit demand, distributed on some interval. Each consumer's location corresponds to their preferred design. The further the product's design deviates from a consumer's ideal, the less they are willing to pay for it. Following the cannonical Hotelling model, we assume all consumers share the same maximum value for their ideal design and experience disutility based solely on the distance between the product's actual design and their preferred one. The monopolist decides on the design of the product (position on the line) and its price. The classical monopoly model emerges as a special case of this framework where the product's design is fixed exogenously: a consumer's willingness to pay decreases with their distance from the fixed design. The distribution of consumers' tastes, therefore, translates into a distribution of willingness to pay, shaping the resulting demand function.

We begin by identifying the set of prices that are optimal in the model with product design for some distribution of tastes.[2] We term such prices *rationalizable*. Crucially, we demonstrate that rationalizable prices have a strictly positive lower bound, meaning that prices below this threshold are not optimal for any distribution of tastes. Furthermore, we establish a strictly positive lower bound on rationalizable prices across all convex disutility functions. Prices below this bound are not optimal for any combination of convex disutility function and distribution of tastes. This finding contrasts with the predictions offered by the classical monopoly model. In the absence of product design, where consumer valuations are assumed to be distributed, for example, over the interval $[0, 1]$, any price within that range is rationalizable. By neglecting the role of product design, the classical monopoly model underestimates prices at the lower end.

Next, we study the effects of product design on consumer surplus and profit. The lowest con-

[1]The importance of product design has been recognized ever since the seminal work of Hotelling (1929). For further insights, see, among others, Lancaster (1966), Johnson and Myatt (2006), and Bar-Isaac et al. (2023). However, the research on product design has almost exclusively focused on oligopolistic markets; an exception is Bar-Isaac et al. (2023).

[2]We say that a price is optimal given the distribution of tastes if there exists a product design such that the price together with the product design maximizes the seller's profit across all price/design pairs.

sumer surplus and, simultaneously, the highest profit are attained when all consumers share the same preferred design. In this scenario, the seller adopts the preferred design and extracts the entire surplus. Of greater interest, however, is the highest attainable consumer surplus. High consumer surplus arises from a combination of high social welfare and low producer surplus. The former requires that consumers' preferred designs are close to the seller's design, thus minimizing the disutility from the mismatch. The latter requires that the seller charges a low price, which happens only when consumers' tastes are sufficiently dispersed. The consumer-optimal distribution strikes a balance between these opposing forces by ensuring that the seller is indifferent over a set of design-price combinations, with each price paired to a different design.

The characterization of the consumer-optimal distribution produces a couple of key insights. First, the maximal consumer surplus in the presence of product design is strictly smaller than the maximal consumer surplus in the model without design identified in Condorelli and Szentes (2020). The classical monopoly model, therefore, overestimates consumer surplus at the top. The ability to design the product benefits the seller but harms consumers. This result holds for any convex disutility function, with consumer surplus reaching its highest level when disutility is linear. Second, unlike in the monopoly model without product design, the seller is not indifferent between the equilibrium price and all prices above it. In other words, demand is not globally unit-elastic. Under the consumer-optimal distribution, the seller is indifferent over the equilibrium price and prices (coupled with particular designs) in some interval above the price, but prefers them to the prices at the very top.

We then turn our attention to the monopolist's profit. As mentioned earlier, the highest profit is achieved when all consumers share the same preferred design, allowing the monopolist to extract the entire surplus. More interesting is the question of what distribution of tastes minimizes the monopolist's profit. A lower bound on the profit is obtained by noting that the monopolist can always ensure a sale to at least mass $1/n$ of consumers by dividing the interval into n subintervals of equal width and selling on the subinterval with the highest concentration of customers. The seller's profit is, therefore, at least $1/n$ times the price required to cover the subinterval. By taking the supremum over n, one obtains a more precise lower bound. We demonstrate that this bound is tight—there exists a distribution of tastes under which the seller's maximum profit coincides with the bound.

The characterization of the minimal profit leads to a couple of important insights. First, the monopolist's ability to design the product guarantees her a profit above a strictly positive threshold, irrespective of the distribution of tastes. This contrasts with the predictions of the model without product design, where the seller's profit can be driven down to zero. In this regard, the basic monopoly model underestimates profits at the bottom. Second, the characterization sheds light on the types of distributions that minimize the monopolist's profit. A profit-minimizing distribution must limit the seller's ability to exploit both pricing and product design. The uniform distribution eliminates the benefits of product design, but may result in insufficiently low prices. The profit-minimizing distribution shares a characteristic similar to the uniform distribution: a finite number of equally wide partition cells each carry the same probability mass. However, it induces the seller to charge a lower price than the uniform distribution.

Related Literature. Product design has been studied extensively, typically utilizing the Hotelling framework. Indeed, Hotelling (1929) himself offered an interpretation of his model in terms of product design, using the sweetness of cider as a leading example. Most studies, however, focus on oligopoly settings and typically assume a uniform distribution of consumers' tastes. A notable exception is Anderson et al. (1997), who study the condition on the distribution of tastes under which a pure-strategy equilibrium exists in the duopoly setting.[3]

The study of monopoly within the Hotelling framework has gained attention in recent research. For example, Hidir and Vellodi (2021) examine consumer-optimal information revelation in a monopoly setting.[4] Bar-Isaac et al. (2023) analyze a product design scenario where consumers are distributed along a circle, and the monopolist selects a design in the same circle. Both of these studies assume a uniform distribution of tastes. Meanwhile, Kim and Kos (2023) investigate the monopolist's optimal design and pricing when the monopolist has no information about the distribution of tastes, addressing the robustness problem.[5]

Johnson and Myatt (2006) examine a monopoly model with product design, where the monopolist first selects a demand function and then sets the price, with the demand functions ordered according to a rotation order. In contrast, in our model, each design corresponds to a specific demand function but, because we do not impose restrictions on the distribution of consumer tastes, the resulting demand functions are not generally ordered by rotations.

The analysis of consumer surplus in monopoly markets is an increasingly active area of research. Bergemann et al. (2015) characterize the possible combinations of consumer and producer surplus when the seller can engage in third-degree price discrimination based on consumer infor-

[3] Rhodes and Zhou (2022) investigate the effects of personalized pricing on consumer surplus under the general distribution of tastes in the Hotelling oligopoly framework; but they do not allow for product design.

[4] Ali et al. (2023) consider a related information disclosure problem in a duopoly setting but do not allow for product design.

[5] The Hotelling model has also found compelling applications in the study of innovation; see Callander et al. (2021) and Callander et al. (2022).

mation.[6] Roesler and Szentes (2017) examine a monopoly setting where the buyer decides how much information to acquire about their value, and the seller sets a price after observing the buyer's information acquisition strategy. Condorelli and Szentes (2020), among other results, identify the distribution of valuations that maximizes consumer surplus in the classical monopoly framework. Unlike our paper, this body of work takes the product's design as exogenously given.

The remainder of this paper is organized as follows. Section 2 formally introduces our model. Section 3 characterizes the set of rationalizable prices. Section 4 studies the set of attainable consumer surplus, focusing on the maximal consumer surplus. Section 5 determines the set of profits that can arise in our model; in particular, it identifies the monopolist's lowest profit and a distribution that produces it. Section 6 obtains the Pareto payoff frontier, and Section 7 concludes.

2 The Model

A monopolist is facing a unit mass of consumers whose heterogeneous tastes are distributed over $[-1, 1]$ according to some distribution F. Given F, the seller chooses her product design (position) $\ell \in [-1, 1]$ and a price $p \geq 0$. Each consumer's willingness-to-pay for the seller's product depends on the distance between his and the seller's positions. Specifically, given ℓ, a consumer at $x \in [-1, 1]$ values the product at $1 - c(|x - \ell|)$, where $c : \mathcal{R}_+ \to \mathcal{R}_+$ is strictly increasing, convex, and differentiable, with $c(0) = 0$. A consumer's willingness-to-pay for the preferred design is normalized to 1, while $c(|x - \ell|)$ captures disutility from preference misalignment (or "transportation cost"). The seller's marginal cost of production is 0.

For each $p \leq 1$, let $\Delta_p := c^{-1}(1 - p)$ denote the maximal distance from the product's design at which a consumer is still willing to buy the product. Since Δ_p is strictly decreasing in p, the seller can be interpreted as choosing the *reach*, Δ, instead of the price, p. Whenever the alternative interpretation is more convenient, we denote the seller's strategy by (ℓ, Δ), instead of (ℓ, p), and use p_Δ to denote the corresponding price (i.e., $p_\Delta := 1 - c(\Delta)$). We also adopt the following notation: For any $\ell \leq \ell'$, $F([\ell, \ell']) := F(\ell') - F_-(\ell)$, where $F_-(x) := \lim_{x' \uparrow x} F(x)$, represents the measure of consumers that lie on $[\ell, \ell']$.

Given (ℓ, p), the seller's demand and profit are, respectively, given by

$$D(\ell, p; F) := F\left([\ell - \Delta_p, \ell + \Delta_p]\right) \text{ and } \pi(\ell, p; F) := pD(\ell, p; F).$$

[6] Terstiege and Vigier (2023) uncover a connection between Condorelli and Szentes (2020) and Bergemann et al. (2015), showing that a multi-product version of the latter's framework can be approximated by the former's as the number of products increases.

The seller chooses (ℓ, p) to maximize the profit:

$$\pi(F) := \max_{(\ell,p)} \pi(\ell, p; F).$$

We use $B(F)$ to denote the set of the seller's optimal strategies given F, that is, $B(F) := \{(\ell, p) : \pi(\ell, p; F) = \pi(F)\}$. Finally, we write $CS(F)$ for the maximal consumer surplus under the distribution F subject to the seller's profit maximization, that is,

$$CS(F) := \max_{(\ell,p)\in B(F)} \int \max\{1 - c(|x - \ell|) - p, 0\} dF(x).$$

Relation to the classical monopoly model. Our model differs from the classical monopoly model, in that consumers have horizontally differentiated *tastes*, rather than vertically different values, and the seller designs the product. The two models, however, are much closer than these differences seemingly suggest. In fact, the basic model can be interpreted as a special case of our model in which the product's design is exogenously given.

To see this formally, fix the product's design at $\ell \in [-1, 1]$. Since a consumer's value for the product is $1 - c(|x - \ell|)$, the distribution of consumers' willingness-to-pay, denoted G, is given by

$$G(p) := 1 - Pr\{1 - c(|x - \ell|) > p\} = 1 - \int_{(\ell-\Delta_p, \ell+\Delta_p)} dF(x).$$

The resulting demand function is

$$D(p) := 1 - G_{-}(p) = F\left([\ell - \Delta_p, \ell + \Delta_p]\right).$$

The seller chooses p to maximize $pD(p)$, just as in the classical monopoly problem.

Conversely, consider the monopoly model (without design) with a downward-sloping demand function $D(p)$. Fix the product's design to $\ell = 0$, and consider some disutility function c such that $c(1) \geq 1$.[7] The given demand function can be generated by a symmetric distribution of tastes F such that

$$F(y) = \frac{D(\Delta^{-1}(y)) + 1}{2}, \text{ for all } y \in [0, 1]$$

and $F(-y) = 1 - F_{-}(y)$ for all $y \in [0, 1]$. It should be noted that any distribution of tastes F

[7] Given $\ell = 0$, the consumers that are farthest away from the seller (those at -1 or 1) are willing to pay $1 - c(1)$. Therefore, the assumption $c(1) \geq 1$ ensures that consumers can have arbitrarily small willingness-to-pay for the seller's product.

(given a fixed design) generates a single demand function D. However, any demand D might be generated by several distributions of tastes. The following result summarizes the argument.

Proposition 1 *Fix a disutility function $c(\cdot)$ with $c(1) \geq 1$.*[8] *For any non-increasing demand function $D : [0,1] \to [0,1]$, there exists a distribution of tastes F that generates the demand function given the design $\ell = 0$. Conversely, given the fixed design, any distribution of tastes F results in some non-increasing demand function $D : [0,1] \to [0,1]$.*

3 Pricing

In the monopoly model without product design, any non-negative price is optimal for some distribution of consumers' values; it suffices to consider degenerate distributions at each value. In this section, we demonstrate that product design fundamentaly changes this basic prediction. For any price $p < 1$, the seller can design a product that appeals to a strictly positive mass of consumers. Therefore, too low a price (close to 0) is never optimal. Building on this reasoning, we demonstrate that there exists a strictly positive price such that no price below the threshold is optimal for *any* distribution of tastes. The following terminology will be of use.

Definition 1 *A price p is* rationalizable *if there exists a distribution of tastes, F, and a design, ℓ, such that (ℓ, p) is optimal for the seller given the distribution F.*[9]

Suppose that given a distribution F, it is optimal for the seller to choose (ℓ, p), or equivalently (ℓ, Δ_p). A *necessary* condition for Δ_p to be optimal is that the deviation to serve either $[\ell - \Delta_p, \ell]$ or $[\ell, \ell + \Delta_p]$ —a strategy that entails a higher price while allowing the seller to capture at least a half of consumers from $[\ell - \Delta_p, \ell + \Delta_p]$—is not profitable, that is,

$$(1 - c(\Delta_p))\, F\left([\ell - \Delta_p, \ell + \Delta_p]\right) \geq \left(1 - c\left(\frac{\Delta_p}{2}\right)\right) \frac{F\left([\ell - \Delta_p, \ell + \Delta_p]\right)}{2},$$

which simplifies to $2(1 - c(\Delta_p)) \geq 1 - c(\Delta_p/2)$. In simple terms, the price required to cover half of the interval cannot exceed twice the original price. The following result ensues.

Proposition 2 *Let $\overline{\Delta}$ be the maximal value of $\Delta \in [0,1]$ such that $1 - c(\Delta) \geq (1 - c(\Delta/2))/2$. No price $p < \underline{p} := 1 - c(\overline{\Delta})$ is rationalizable.*

[8]The condition can be relaxed to $c(2) \geq 1$ by assuming that the design is fixed at -1 or 1. We fix ℓ to 0 so as to make this result directly comparable to the subsequent analysis.

[9]Armstrong and Zhou (2022) and Lang and Wasser (2024) use similar concepts. They refer to such prices as *implementable*.

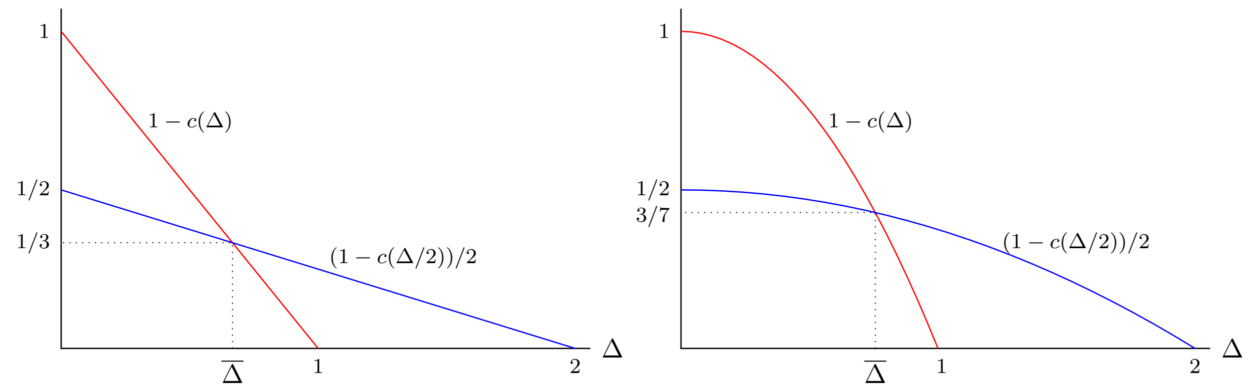

Figure 1: This figure illustrates Proposition 2. The left panel depicts the case where $c(y) = y$, while the right panel is for the case where $c(y) = y^2$.

Proof. In the appendix, we show that $\overline{\Delta}$ is well defined, and $1 - c(\Delta) \geq (1 - c(\Delta/2))/2$ holds if and only if $\Delta \leq \overline{\Delta}$. ■

The result holds because $1 - c(\Delta)$ decreases faster than $(1 - c(\Delta/2))/2$ at any Δ, implying that they can intersect at most once; see Figure 1. If the intersection occurs at a $\Delta < 1$, then $\overline{\Delta}$ corresponds to the crossing point. Otherwise (i.e., $1 - c(\Delta)$ is uniformly above $(1 - c(\Delta)/2)/2$ over $[0, 1]$), $\overline{\Delta} = 1$.

In Section 4, we establish the converse to Proposition 2: for any $p \in [\underline{p}, 1]$, there are a distribution F and a design ℓ such that (ℓ, p) is the seller's optimal strategy given F. In consequence, price p is rationalizable if and only if $p \in [\underline{p}, 1]$. In what follows, we refer to $\underline{p}$ as the *minimal rationalizable price*. The following result establishes bounds on $\underline{p}$ that do not depend on the details of disutility $c(\cdot)$ (beyond its value at 1).

Corollary 1 *The following holds:*

$$1/3 \leq \underline{p} \leq \max\{1/2, 1 - c(1)\}.$$

Proof. By its definition, $\overline{\Delta}$ satisfies

$$1 - c(\overline{\Delta}) \geq \frac{1}{2}\left(1 - c\left(\frac{\overline{\Delta}}{2}\right)\right).$$

Since c is convex and $c(0) = 0$, $c\left(\overline{\Delta}/2\right) \leq c\left(\overline{\Delta}\right)/2$. Therefore, the above inequality implies that

$$1 - c(\overline{\Delta}) \geq \frac{1}{2}\left(1 - \frac{c(\overline{\Delta})}{2}\right),$$

which can be rewritten as

$$1 - c(\overline{\Delta}) \geq \frac{1}{3}.$$

The second result holds because either $\overline{\Delta} = 1$ or

$$1 - c(\overline{\Delta}) = \frac{1}{2}(1 - c(\overline{\Delta}/2)) \leq \frac{1}{2}.$$

■

The lower bound in the above result is striking, as it means that a price below $1/3$ can *never* be optimal for the seller, regardless of the distribution F and the disutility function c. The result lends itself to a geometric interpretation. Suppose c is linear and consider $p < 1/3$ combined with some design ℓ. The seller serves consumers on $[\ell - \Delta_p, \ell + \Delta_p]$. Instead, the seller could choose the more populous side between $[\ell - \Delta_p, \ell]$ and $[\ell, \ell + \Delta_p]$, thereby reaching at least half of the original demand with a higher price $p_{\Delta_p/2} = (1 + p)/2$.[10] If $p < 1/3$ then the price more than doubles, while the demand at most halves. Examining power disutility functions further elucidates the result.

Power Disutility. Suppose $c(y) = ty^\alpha$ for some $t > 0$ and $\alpha \geq 1$. Then

$$\overline{\Delta} = \min\left\{\left(\frac{1}{t\left(2 - 1/2^\alpha\right)}\right)^{1/\alpha}, 1\right\}.$$

The resulting minimal rationalizable price is

$$\underline{p} = \max\left\{\frac{2^\alpha - 1}{2^{\alpha+1} - 1}, 1 - t\right\}.$$

[10]Due to linear disutility, $p = 1 - t\Delta_p$, that is, $\Delta_p = (1 - p)/t$. Then,

$$p_{\Delta_p/2} = 1 - t\frac{\Delta_p}{2} = 1 - \frac{1 - p}{2} = \frac{1 + p}{2}.$$

Observe that this minimal rationalizable price is equal to $1/3$ when $\alpha = 1$, which implies that the lower bound of $\underline{p}$ in Corollary 1 is achievable when c is linear. It increases and converges to $\max\{1/2, 1-t\}$ as α tends to ∞, thus establishing that the upper bound in Corollary 1 is also binding.

Comparison to the classical monopoly model. In the monopoly model without product design, the monopolist distorts the outcome by setting a price above marginal cost. Depending on the demand and cost structure, this price can be arbitrarily low or even zero. Corollary 1 highlights a significant departure in the model with product design. Here, the seller can always strategically designs the product so that the resulting demand makes it optimal to charge a price of at least $1/3$. This interaction between pricing and design ensures that pricing distortions are substantial, especially since the marginal cost of production in this model is zero.[11]

4 Consumer Surplus

Consumer surplus plays a pivotal role in the theory of monopoly by shedding light on the economic implications of market power and pricing strategies employed by monopolistic firms. In this section, we characterize the range of consumer surplus that the model with product design can generate. The minimal consumer surplus, zero, is attained when the distribution of tastes is a Dirac distribution. The seller designs the product all the consumers prefer and extracts all the surplus by charging price 1. Most of the subsequent analysis revolves around characterizing the maximal attainable consumer surplus. Any level between the minimal and the maximal can also be realized.

4.1 Maximal Consumer Surplus with a Fixed Design

We start by examining the case where the design of the product is exogenously fixed at $\ell = 0$. This case, as illustrated in Section 2, is analogous to the problem without design.[12] In what follows, we develop a solution method to the problem with fixed design which is also applicable to the problem with design. In what follows, we develop a solution method for the problem with fixed design which is then extended to the problem with design.

[11] The analysis of rationalizable prices readily extends to the setting where the seller incurs a constant marginal cost of production $m \in [0, 1]$. In this case, the minimal rationalizable price across all disutility functions becomes $1/3 + (2/3)m$. This result implies that when the marginal cost m is small, the Lerner index ($(p - m)/p$) in the model with design is close to 1, irrespective of the distribution of consumer tastes.

[12] There are two minor differences. First, given a fixed design, multiple distributions of tastes can lead to the same distribution over valuations. Second, depending on the disutility function c, the monopolist might wish to cover the entire space, in which case the lower bound on the price at $1 - c(1)$ is binding.

To begin with, notice that consumer surplus under the distribution F and the seller's strategy (ℓ, Δ) can be written as follows:

$$\begin{aligned} CS(\ell, \Delta; F) &= \int_{[\ell-\Delta,\ell+\Delta]} (1 - c(|x-\ell|) - p_\Delta)\, dF(x) \\ &= \int_{[-\Delta,\Delta]} (1 - c(|z|) - (1 - c(\Delta)))\, dF(\ell + z) \\ &= \int_{(0,\Delta]} F\left([\ell - y, \ell + y]\right) dc(y), \end{aligned} \tag{1}$$

where the second equality is obtained by changing the variable with $z = x - \ell$ and the third by integrating by parts and changing the variable with $y = -z$.

A key intermediate step in our solution method is to establish an upper bound on the integrand $F\left([\ell - y, \ell + y]\right)$ in the above representation, using the optimality of Δ given the design. For ease of notation, fix $\ell = 0$. Then, the fact that the seller should prefer serving $[-\Delta, \Delta]$ to $[-x, x]$ for any x leads to

$$F\left([-x, x]\right) \leq \frac{1 - c(\Delta)}{1 - c(x)} F\left([-\Delta, \Delta]\right) \leq \frac{1 - c(\Delta)}{1 - c(x)}, \tag{2}$$

where the last inequality holds because $F\left([-\Delta, \Delta]\right) \leq 1$.

For each $\Delta \in (0, 1]$, we let $\overline{\mathcal{F}}^0_\Delta$ denote the class of distributions such that if $F \in \overline{\mathcal{F}}^0_\Delta$ then F satisfies

$$F\left([-x, x]\right) = \frac{1 - c(\Delta)}{1 - c(x)},$$

for all $x \in (0, \Delta]$. We use F^0_Δ to denote a generic distribution in $\overline{\mathcal{F}}^0_\Delta$ and $\overline{F}^0_\Delta$ to denote the distribution that is symmetric around 0 (i.e., $\overline{F}^0_\Delta(x) = 1 - \overline{F}^0_\Delta(-x)$ for $x \in (0, 1]$). Any F^0_Δ assigns probability mass $1 - c(\Delta)$ to $x = 0$, and its support is contained in $[-\Delta, \Delta]$. In addition, given any F^0_Δ, Δ is an optimal reach for the seller.[13]

The results so far imply that for any distribution F under which the seller's optimal reach is Δ

[13] If the seller chooses reach $\Delta' > \Delta$, then she serves all consumers on $[-\Delta, \Delta]$, so her profit is $1 - c(\Delta')(< 1 - c(\Delta))$. If she chooses $\Delta' \in [0, \Delta]$ then her profit is $(1 - c(\Delta'))F^0_\Delta\left([-\Delta', \Delta']\right) = 1 - c(\Delta)$, regardless of the value of Δ'. Therefore, Δ' is optimal if and only if $\Delta' \in [0, \Delta]$.

given $\ell = 0$, we have

$$\begin{aligned} CS(F) &= \int_{(0,\Delta]} F([-x,x])\, dc(x) \\ &\leq \int_{(0,\Delta]} \frac{1-c(\Delta)}{1-c(x)} dc(x) \\ &= -(1-c(\Delta))\log(1-c(\Delta)) \\ &= CS(F^0_\Delta). \end{aligned}$$

In other words, the distributions in $\overline{\mathcal{F}}^0_\Delta$ maximize consumer surplus among those distributions that induce the seller to choose Δ (equivalently, price p_Δ).

After having identified optimal distributions for each Δ, it remains to optimize over Δ.[14] Maximizing $CS(F^0_\Delta) = -(1-c(\Delta))\log(1-c(\Delta))$ leads to the following result, which corresponds to Theorem 1 of Condorelli and Szentes (2020).

Theorem 1 *If the product's design is fixed at $\ell = 0$, the maximally attainable consumer surplus is equal to $1/e$ when $1 - c(1) \leq 1/e$ and $-(1-c(1))\ln(1-c(1))$ otherwise.*[15] *It is attained by any distribution of tastes in $\overline{\mathcal{F}}^0_{\Delta_0}$ where $\Delta_0 = c^{-1}(1-1/e)$ if $1 - c(1) \leq 1/e$, and $\Delta_0 = 1$ otherwise.*

The distribution of willingness to pay that maximizes consumer surplus, provided that $c(1)$ is large enough, ensures that the seller is indifferent between reach Δ_0 and any smaller non-negative reach; equivalently, the seller is indifferent between price $p_{\Delta_0} = 1 - c(\Delta_0)$ and any price up to 1. Figure 2 depicts the symmetric distribution of tastes $\overline{F}^0_{\Delta_0}$ and its density conditional on $x \neq 0$ (right); probability mass at $\ell = 0$ is represented by the solid red pillar.

We examine more closely the symmetric distribution $\overline{F}^0_{\Delta_0}$, which allows for a simple graphical argument. The key characteristic of the distribution $\overline{F}^0_{\Delta_0}$ is that its density increases fast as x moves away from $\ell = 0$ (i.e., as $|x|$ increases). The property is necessary to maintain the seller's indifference across a range of reaches conditional on $\ell = 0$. However, it renders the distribution vulnerable to deviations in product design. In particular, despite the symmetric nature of $\overline{F}^0_{\Delta_0}$, $\ell = 0$ is *not* the seller's optimal product design. Therefore, if the seller can choose the design, consumer surplus will be smaller than in Theorem 1.

To understand why $\ell = 0$ is not the seller's optimal design under $\overline{F}^0_{\Delta_0}$, notice that the seller is indifferent between serving $[-\Delta_0, \Delta_0]$ and $\left[-\frac{\Delta_0}{2}, \frac{\Delta_0}{2}\right]$. Now, consider the seller's deviation from

[14] The method of characterizing a certain property for each price and subsequently maximizing over all prices was employed in Kos and Messner (2015), Armstrong and Zhou (2022), and Lang and Wasser (2024).

[15] The second case arises because $1 - c(1)$ is the lowest willingness to pay by consumers. This limits the extent to which the firm can be incentivized to lower its price, rendering the maximal consumer surplus $1/e$ out of reach.

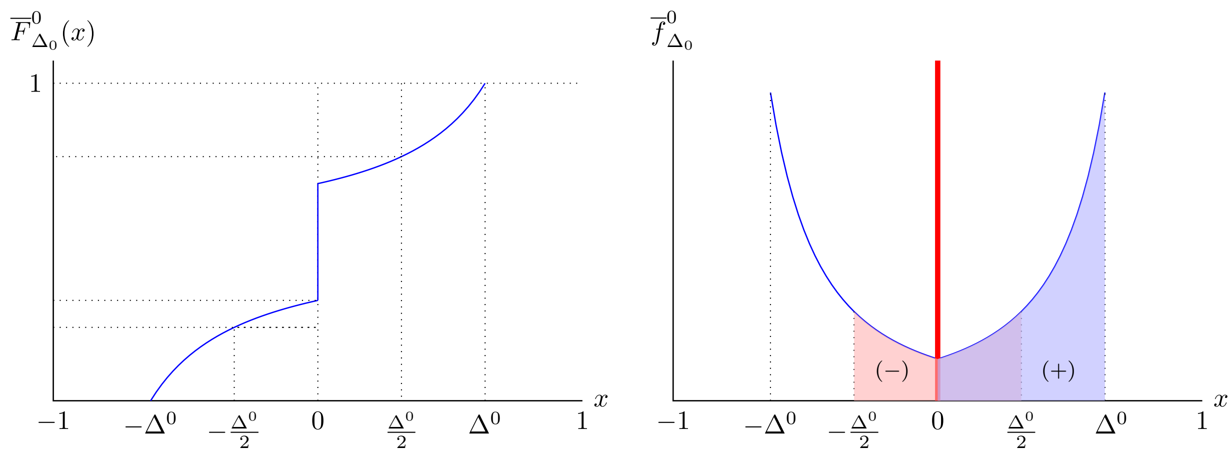

Figure 2: The symmetric distribution $\overline{F}^0_{\Delta_0}$ in $\overline{\mathcal{F}}^0_{\Delta_0}$. The left panel depicts the cumulative distribution function, while the right panel draws the probability density function, excluding the mass point at 0 (represented as the red line on 0). In this figure, $c(|x-\ell|) = |x-\ell|$.

the latter to $(\ell, \Delta) = \left(\frac{\Delta_0}{2}, \frac{\Delta_0}{2}\right)$; that is, suppose the seller opts to serve consumers on the interval $[0, \Delta^0]$, instead of $\left[-\frac{\Delta_0}{2}, \frac{\Delta_0}{2}\right]$. As visualized in the right panel of Figure 2, the deviation makes the seller lose consumers in the interval $\left[-\frac{\Delta_0}{2}, 0\right)$ but gain those in $\left(\frac{\Delta_0}{2}, \Delta_0\right]$. Due to the shape of the distribution, the gain in the latter interval outweighs the loss in the former, leading to an overall increase in the quantity sold. Since the reach remains unchanged, the deviation strictly benefits the seller.

While the above argument shows only that $l = 0$ is not optimal under the symmetric distribution of tastes $\overline{F}^0_{\Delta_0} \in \mathcal{F}^0_{\Delta_0}$, the conclusion applies more broadly.

Proposition 3 *The product design $\ell = 0$ is not optimal for any distribution of tastes $F^0_{\Delta_0} \in \mathcal{F}^0_{\Delta_0}$.*

Proof. See the appendix. ■

4.2 Attainable Consumer Surplus with Product Design

In this subsection, we consider the monopoly model with product design and characterize the maximal attainable consumer surplus and a distribution that achieves it. Our analysis proceeds much like in the previous subsection.

Fix a $\Delta \in [0, \overline{\Delta}]$. The corresponding price $p_\Delta \in [\underline{p}, 1]$ is rationalizable. Suppose a distribution F rationalizes p_Δ. For ease of notation, we assume that the optimal design is 0; as clarified later, this incurs no loss of generality for the purpose of characterizing the maximal consumer surplus.

As in Section 4.1, no deviation of the form $(\ell, \Delta) = (0, x)$ should be profitable, leading to the bounds in (2). However, the subsequent discussion in Section 4.1 indicates that these deviations alone do not impose sufficient restrictions on the distribution F, as deviating to an alternative design $\ell \neq 0$ can be strictly profitable. We establish more stringent bounds on the distribution.

Drawing insights from the relevant deviations in the problem without design—where the seller's indifference over prices (only) resulted in the density increasing as one moves away from the design—we start by examining the following restricted classes of deviations:

$$X_+ = \{(\ell, \Delta') : \ell = \Delta - \Delta', \Delta' \in [\Delta/2, \Delta]\},$$
$$X_- = \{(\ell, \Delta') : \ell = -\Delta + \Delta', \Delta' \in [\Delta/2, \Delta]\}.$$

The set X_+ represents the seller's deviations to serve consumers in intervals of the form $[-x, \Delta]$ for some $x \in [0, \Delta]$, while X_- represents symmetric deviations of covering intervals of the form $[-\Delta, x]$. The requirement that the seller should weakly prefer $(0, \Delta)$ to all deviations in $X_+ \cup X_-$ provides the following bound for $F([-x, x])$, similar to (2) in the model without design.

Lemma 1 *Suppose* $(0, \Delta)$ *is the seller's optimal strategy given the distribution* F*, where* $\Delta \in [0, \overline{\Delta}]$*. Then, for any* $x \in (0, 1]$*,*

$$F([-x, x]) \leq 2\frac{1 - c(\Delta)}{1 - c(\frac{x+\Delta}{2})} - 1.$$

Proof. See the appendix. ∎

Compared to the corresponding condition with a fixed design in (2), the inequality in Lemma 1 is more stringent. Specifically,

$$2\frac{1 - c(\Delta)}{1 - c(\frac{x+\Delta}{2})} - 1 \leq \frac{1 - c(\Delta)}{1 - c(x)},$$

for every $x \in [0, \Delta]$.[16] Intuitively, the condition in (2) is based only on the seller's deviation to serve $[-x, x]$ (i.e., excluding $[-\Delta, -x]$ and $[x, \Delta]$ simultaneously), while the condition in Lemma 1

[16] To see this, observe that this inequality can be rewritten as

$$\frac{1}{1 - c\left(\frac{x+\Delta}{2}\right)} \leq \frac{1}{2}\left(\frac{1}{1 - c(\Delta)} + \frac{1}{1 - c(x)}\right).$$

This always holds because convexity of $c(\cdot)$ implies the same property for $1/(1 - c(\cdot))$.

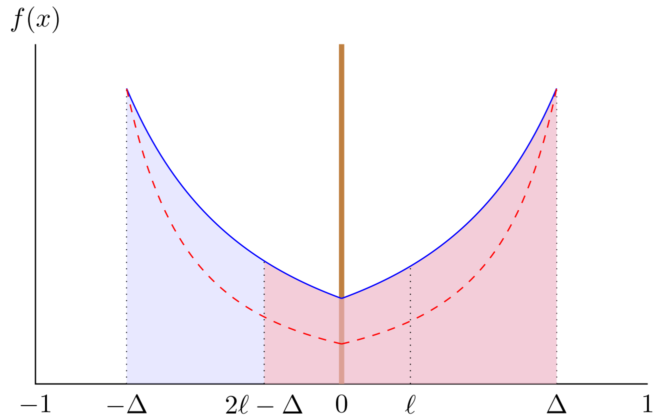

Figure 3: This figure depicts the density function of F_Δ (blue solid) and that of $\overline{F}^0_\Delta$ in Section 4.1 (red dashed), each conditional on $x \neq 0$. The brown solid pillar represents probability mass at 0. In this figure, $c(|x-\ell|) = |x-\ell|$ and $\Delta = \frac{e-1}{e} \approx 0.6321$.

reflects two potential deviations to serve $[-x, \Delta]$ or $[-\Delta, x]$ (i.e., excluding $[-\Delta, -x]$ and $[x, \Delta]$ separately), thus the latter yields a stronger restriction on $F([-x, x])$.

Let F_Δ denote the distribution such that all deviations in $X_+ \cup X_-$ are binding and $supp(F_\Delta) = [-\Delta, \Delta]$. In the following result, we show that the constraints uniquely pin down a distribution, that the distribution is symmetric, and that it rationalizes p_Δ. By construction, it also follows that the condition in Lemma 1 is binding:

$$F_\Delta([-x, x]) = 2\frac{1-c(\Delta)}{1-c(\frac{x+\Delta}{2})} - 1,$$

for every $x \in [0, \Delta]$. As depicted in Figure 3, due to the difference between (2) and Lemma 1, F_Δ assigns more probability to all intervals of the form $[-x, x]$ than $\overline{F}^0_\Delta$.

Lemma 2 *For each $\Delta \in [0, \overline{\Delta}]$, F_Δ is well defined and symmetric around 0. Moreover, given F_Δ, strategy $(0, \Delta)$ is optimal for the seller.*

Proof. See the appendix. ■

Note that $(0, \Delta)$ should be more profitable than any arbitrary strategy (ℓ', Δ'). A key step in establishing Lemma 2 is to show that conditional on reach $\Delta' \in [\Delta/2, \Delta]$, $\ell = \Delta - \Delta'$ is the seller's optimal design, that is, it is optimal for the seller to serve the right-most (or left-most) subinterval of $[-\Delta, \Delta]$, represented by the red region in Figure 3. This is because, as for $\overline{F}^0_\Delta$, the density function conditional on $x \neq 0$ increases as $|x|$ rises, so the seller's demand is maximized when she serves the right-most (or left-most) interval. Meanwhile, by the construction of F_Δ, the seller is indifferent between serving $[-\Delta, \Delta]$ and $[-x, \Delta]$ for any $x \geq 0$. This implies that $\pi(0, \Delta) = \pi(\Delta - \Delta', \Delta') \geq \pi(\ell', \Delta')$ for any ℓ' and $\Delta' \in [\Delta/2, \Delta]$.[17]

Lemmas 1 and 2 together imply that for any distribution F that rationalizes p_Δ,[18]

$$\begin{aligned} CS(F) &= \int_{(0,\Delta]} F\left([-x, x]\right) dc(x) \\ &\leq \int_{(0,\Delta]} \left(2\frac{1 - c(\Delta)}{1 - c(\frac{x+\Delta}{2})} - 1 \right) dc(x) \\ &= CS(F_\Delta). \end{aligned}$$

In other words, F_Δ maximizes consumer surplus among those distributions under which Δ is an optimal reach for the seller. Optimizing over Δ yields the following result.

Theorem 2 *Let Δ^* be the value of Δ that solves*

$$\max_{\Delta \in [0, \overline{\Delta}]} CS(F_\Delta) = -c(\Delta) + 2\int_0^\Delta \frac{1 - c(\Delta)}{1 - c\left(\frac{x+\Delta}{2}\right)} dc(x). \tag{3}$$

Then, F_{Δ^*} maximizes consumer surplus among all distributions over $[-1, 1]$.

This result establishes only the maximal consumer surplus $CS(F_{\Delta^*})$ in the monopoly model with product design. Given the analysis above, it is clear that any level in $[0, CS(F_{\Delta^*})]$ is attainable; formally, for any $CS \in [0, CS_{\Delta^*}]$, there is a distribution under which consumer surplus is equal to CS. This follows immediately from the fact that $CS(F_\Delta)$ is continuous in Δ and $CS(F_\Delta) = 0$ if $\Delta = 0$.

Taking into account product design fundamentally alters key predictions about consumer-optimal distributions. In the model with a fixed location, or equivalently without design, consumer-optimal distributions ensure that the seller is indifferent between p_{Δ_0} and any price above it. With product design, this is no longer the case: The consumer-optimal distribution F_{Δ^*} is such that

[17] The case when $\Delta' < \Delta/2$ requires an independent proof, which can be found in the appendix.

[18] Strictly speaking, we cover only those distributions under which $\ell = 0$ is the seller's optimal design associated with reach Δ. If the seller's optimal location ℓ is, e.g., positive then the bound in Lemma 1 can be adjusted as follows: for any $y \in [0, \Delta]$,

$$F\left([\ell - y, \ell + y]\right) \leq 1\frac{1 - c(\Delta)}{1 - c(\frac{y+\Delta}{2})} - 1.$$

Note that (ℓ, Δ) can be optimal only when $\ell + \Delta \leq 1$.

the seller is indifferent only over prices (associated with different optimal designs) in the interval $[1-c(\Delta^*), 1-c(\Delta^*/2)]$; and she strictly prefers them to all prices above $1-c(\Delta^*/2)$. In fact, if one were to consider a distribution that makes the seller indifferent between offering $(\ell, \Delta) = (0, \Delta^*)$ and $(\ell, \Delta) = (0, 0)$ (design 0 and price 1), the seller would prefer $(\ell, \Delta) = (\Delta_{p^*}/2, \Delta_{p^*}/2)$ or $(\ell, \Delta) = (-\Delta_p/2, \Delta_p/2)$ to $(\ell, \Delta) = (0, \Delta^*)$.

To further illuminate Theorem 2, we consider two canonical disutility functions, linear and quadratic, that are commonly adopted in more applied studies. We provide a more detailed characterization of the consumer-optimal distributions and the resulting maximal consumer surplus for each case.

Linear Disutility. Suppose $c(y) = ty$ for all $y \geq 0$ and some $t > 0$. Then, $c'(x) = c'\left(\frac{x+\Delta}{2}\right) = t$, so $CS(F_\Delta)$ can be explicitly solved:

$$CS(F_\Delta) = -t\Delta + 4(1 - t\Delta) \ln \left(\frac{1 - t\frac{\Delta}{2}}{1 - t\Delta} \right).$$

Observe that this expression depends only on $\eta := t\Delta$. The optimal value of η, denoted η^*, satisfies the following first order condition:

$$3 - 4 \ln \left(\frac{1 - \frac{\eta^*}{2}}{1 - \eta^*} \right) - \frac{2(1 - \eta^*)}{1 - \frac{\eta^*}{2}} = 0.$$

The solution is $\eta^* \approx 0.5123$, and the resulting maximal consumer surplus is

$$\begin{aligned} \overline{CS} &:= -\eta^* + 4(1 - \eta^*) \ln \left(\frac{1 - \frac{\eta^*}{2}}{1 - \eta^*} \right), \\ &\approx 0.3113. \end{aligned}$$

Since $\Delta \leq 1$, the optimal reach is given by $\Delta^* = \min\{\eta^*/t, 1\}$. Consequently, $CS(F_{\Delta^*}) = \overline{CS}$ if $\Delta^* < 1$, while $CS(F_{\Delta^*}) < \overline{CS}$ if $\Delta^* = 1$.

The following two facts are of particular interest. First, for $t > \eta^*$ the price induced by the seller-optimal distribution is strictly larger than $1 - c(\overline{\Delta})$—the minimal rationalizable price. Second, for $t \geq \eta^*$, the maximal consumer surplus does not depend on the cost parameter t; it is equal to $\overline{CS}$ regardless of $t (\geq \eta^*)$. If t rises then the distribution F_{Δ^*} becomes proportionally contracted, so the resulting profit and consumer surplus stay constant.

Quadratic Disutility. Suppose $c(y) = ty^2$ for all $y \geq 0$ and some $t > 0$. Unlike in the linear case, the integral in (3) cannot be solved in closed form. Numerically, it can be shown that $\Delta^* = \min\{\sqrt{0.4919/t}, 1\}(\leq \overline{\Delta} = \min\{\sqrt{4/(7t)}, 1\})$, $CS(F_{\Delta^*}) \approx 0.2908$ if $\Delta^* < 1$, and $CS(F_{\Delta^*}) < 0.2908$ if $\Delta^* = 1$.

Consumer Surplus and Disutility. Note that the maximal consumer surplus is strictly higher with linear disutility than with quadratic disutility. In fact, a stronger result holds: $\overline{CS} \approx 0.3113$ (the highest consumer surplus obtained under linear disutility) is a tight upper bound for consumer surplus.

Proposition 4 *For any increasing and convex disutility* c*,* $CS(F_\Delta) \leq \overline{CS} \approx 0.3113$*.*

Proof. Since c is convex, we have $c\left(\frac{x+\Delta}{2}\right) \leq \frac{c(x)+c(\Delta)}{2}$ for any $x \in [0, \Delta]$. This implies that

$$\begin{aligned} CS(F_\Delta) &= -c(\Delta) + 2(1-c(\Delta)) \int_0^\Delta \frac{1}{1-c\left(\frac{x+\Delta}{2}\right)} dc(x) \\ &\leq -c(\Delta) + 2(1-c(\Delta)) \int_0^\Delta \frac{1}{1-\frac{c(x)+c(\Delta)}{2}} dc(x) \\ &= -c(\Delta) + 4(1-c(\Delta)) \ln\left(\frac{1-\frac{c(\Delta)}{2}}{1-c(\Delta)}\right), \end{aligned}$$

where the last equality is through direct calculus. As shown in the case of linear disutility, the last expression is maximized when $c(\Delta) = \eta^*$, and the maximized value is equal to $\overline{CS}$. ■

The above established upper bound $\overline{CS}$ on consumer surplus is strictly below $1/e \approx 0.3679$—the maximal consumer surplus in the model without design (see Theorem 1)—implying that the seller's ability to design the product, while necessarily beneficial to the seller, can be costly to consumers. In particular, it prevents consumers from obtaining more than $\overline{CS} \approx 0.3113$ share of total feasible surplus, when the corresponding share in the absence of product design is $1/e \approx 0.3679$. To put it differently, by neglecting the impact of product design, the classical monopoly model overestimates consumer surplus at the top.

5 Profits

In this section, we characterize the range of profits that the monopoly model with product design can account for. Clearly, the maximal profit attains when all consumers prefer the same design (i.e., F is degenerate), allowing the seller to extract full potential surplus. We characterize the lowest possible profit in our model and then show that any profit between the lowest and the highest levels can be achieved with some distribution of tastes.

5.1 A Profit Lower Bound

Consider any distribution of tastes F. The seller can always cover the whole market by choosing $(\ell, \Delta) = (0, 1)$ and earn $1 - c(1)$. Therefore, $\pi(F)$—the seller's maximized profit given F—should be at least $1 - c(1)$.[19] The seller can refine this strategy and target consumers in either $[-1, 0]$ or $[0, 1]$, depending on which subinterval is more populous. This is achieved by choosing the design $-1/2$ or $1/2$ and charging the price $1 - c(1/2)$. Since one of the two intervals must have at least half of the consumers, the strategy guarantees the profit of at least $1/2(1 - c(1/2))$ to the seller. Therefore, $\pi(F) \geq 1/2(1 - c(1/2))$ for any $F \in \mathcal{F}$.

The above logic can be extended to an arbitrary integer n: Consider the strategy of partitioning the interval $[-1, 1]$ into n subintervals of equal width and serving only those consumers on the most densely-populated subinterval (by positioning at its center and charging $p = 1 - c(1/n)$). Since the most populous subinterval should contain at least $1/n$ of consumers, the seller can achieve at least $1/n(1 - c(1/n))$. Again, this means that $\pi(F) \geq 1/n(1 - c(1/n))$, regardless of $F \in \mathcal{F}$. The following result ensues.

Lemma 3 *For any distribution of tastes $F \in \mathcal{F}$,*

$$\pi(F) \geq \underline{\pi} := \sup_{\Delta \in \left\{1, \frac{1}{2}, \frac{1}{3} \cdots\right\}} \Delta(1 - c(\Delta)).$$

One might think that the lower bound in Lemma 3 can be improved upon by considering an arbitrary reach $\Delta \in [0, 1]$ and serving some subinterval $[\ell - \Delta, \ell + \Delta]$. It is tempting to think that, since the total mass of consumers is 1, at least one such interval would contain at least Δ consumers. This is, however, not the case unless $1/\Delta$ is an integer: For example, consider the binary distribution that assigns equal probability mass to -1 and 1. In this case, an interval can include at most mass $1/2$ of consumers, unless it encompasses the whole space $[-1, 1]$. Therefore, if $\Delta \in (1/2, 1)$ then there does not exist any ℓ such that $F([\ell - \Delta, \ell + \Delta]) \geq \Delta$.[20]

If Δ can take any value in $[0, 1]$ then the resulting profit coincides with the maximized profit

[19] The price required to cover the whole market, $1 - c(1)$, could be negative. If the consumers' disutility from the object is too large, the far away consumers may need to be subsidized to consume the product. This, nevertheless, imposes a lower bound on the profit.

[20] More generally, if there is probability mass of equal size $(1/n)$ at locations $\{-1, -1+2/(n-1), ..., 1\}$, the seller should cover at least portion $1/(n-1)$ of the interval to serve more than mass $1/n$ of consumers.

under the uniform distribution: If F is uniform over $[-1, 1]$ then it is always optimal to choose $\ell = 0$, and the seller's problem reduces to

$$\pi^U := \max_{\Delta \in [0,1]} \Delta \left(1 - c(\Delta)\right). \tag{4}$$

Together with concavity of $\Delta(1-c(\Delta))$, the relationship between $\underline{\pi}$ and π^U leads to the subsequent result.

Lemma 4 *Let Δ^U denote the solution to (4).*

(a) $\pi^U = \underline{\pi}$ if and only if $\Delta^U = 1/n$ for some $n \in \mathbb{N}$.

(b) Let $\widehat{n} = \lceil 1/\Delta^U \rceil$.[21] *Then,*

$$\underline{\pi} = \max\left\{ \frac{1}{\widehat{n}-1}\left(1 - c\left(\frac{1}{\widehat{n}-1}\right)\right), \frac{1}{\widehat{n}}\left(1 - c\left(\frac{1}{\widehat{n}}\right)\right) \right\}.$$

Proof. Part (a) is trivial. Part (b) holds because, by the definition of $\widehat{n}$ and concavity of $\Delta(1-c(\Delta))$, $1/n(1 - c(1/n))$ is strictly increasing if $n < \widehat{n} - 1$ and decreasing if $n > \widehat{n}$. ■

Lemma 4 provides a tractable way to solve the discrete maximization problem in Lemma 3. One can first consider the unrestricted problem (4) where Δ is permitted to take any value in $[0, 1]$ or, equivalently, find the optimal reach under the uniform distribution. Since the problem is concave, it suffices to consider the two reaches of the form $1/n$ that are closest to the solution of the unrestricted problem.

Linear Disutility. Suppose $c(y) = ty$ for some $t > 0$. Then, $\Delta^U = \min\{1/(2t), 1\}$. For any t, $\widehat{n}$ in Lemma 4.(b) is the smallest integer such that $\widehat{n} \geq 2t$. Let t_n be the value of t such that the seller is indifferent between $\Delta = 1/(n-1)$ and $\Delta = 1/n$:

$$t_n = \frac{n(n-1)}{2n-1} \in \left(\frac{n-1}{2}, \frac{n}{2}\right).$$

As a consequence, if $t \in [t_n, t_{n+1}]$ then

$$\underline{\pi} = \frac{1}{n}\left(1 - c\left(\frac{1}{n}\right)\right).$$

See Figure 4 for a graphical illustration of this argument.

[21] W use $\lceil x \rceil$ to denote the smallest integer greater or equal to x.

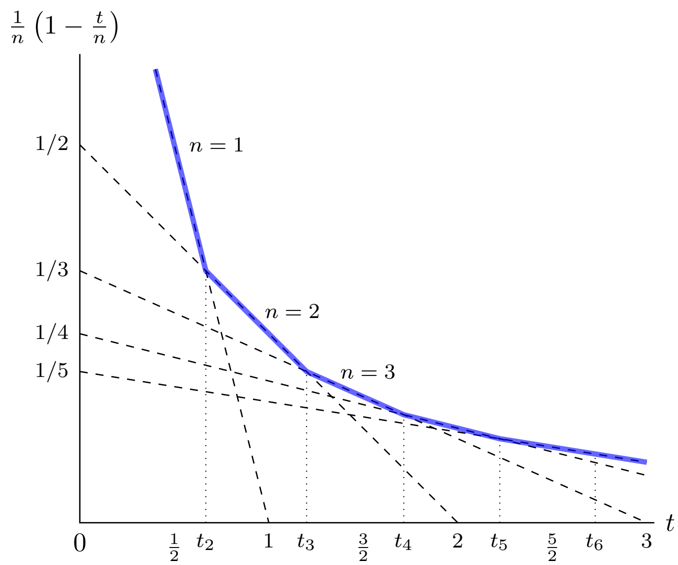

Figure 4: The optimal value of n that yields $\underline{\pi}$ depending on t in the linear case where $c(y) = ty$. Each dashed line depicts $\frac{1}{n}\left(1-\frac{t}{n}\right)$ for some n, and the blue translucent curve shows its upper envelope, which coincides with $\underline{\pi}$.

5.2 Achieving the Profit Lower Bound

The main result of this section is that $\underline{\pi}$ in Lemma 3 is the tight lower bound for the seller's profit. In other words, $\underline{\pi}$ is the seller's lowest possible profit in our monopoly model with product design.

Theorem 3 *There exists a distribution F such that $\pi(F) = \underline{\pi}$.*

We explicitly construct a distribution F that delivers Theorem 3. The distribution we present below has a particularly simple structure: its density function is a step function that takes only two values. The distribution function we construct may not be the only distribution function that minimizes the seller's payoff. However, any such distribution should satisfy certain properties we derive. In what follows, we let n^* refer to the natural number such that $\underline{\pi} = 1/n^* \left(1 - c\left(1/n^*\right)\right)$.

A *necessary* condition for F to satisfy $\pi(F) = \underline{\pi}$ is that it never assigns strictly more probability than $1/n^*$ to any interval with length $2/n^*$, that is,

$$F\left(\left[\ell - \frac{1}{n^*}, \ell + \frac{1}{n^*}\right]\right) \leq \frac{1}{n^*} \text{ for all } \ell \in [-1, 1]. \tag{5}$$

Otherwise, the seller could obtain strictly more than $\underline{\pi}$ by choosing that particular ℓ along with

$\Delta = 1/n^*$. On the other hand, since the total measure of consumers is 1, it must be that

$$\sum_{k=1}^{n^*} F\left(\left[-1 + \frac{2(k-1)}{n^*}, -1 + \frac{2k}{n^*}\right]\right) \geq F([-1,1]) = 1.$$

The inequality (instead of equality) is because of potential probability mass at $-1 + 2k/n^*$ for $k = 1, ..., n^* - 1$. Combining these inequalities, it follows that F should assign probability mass $1/n^*$ to all $\left[-1, -1 + \frac{2}{n^*}\right]$,...,$\left[1 - \frac{2}{n^*}, 1\right]$, that is,

$$F\left(\left[-1 + \frac{2(k-1)}{n^*}, -1 + \frac{2k}{n^*}\right]\right) = \frac{1}{n^*} \text{ for all } k = 1, ..., n^*. \tag{6}$$

Given (5) and the requirement that $F([-1,1]) = 1$, one might think that only the uniform distribution satisfies the above necessary properties. This is not the case—there are other distributions that satisfy both (5) and $F([-1,1]) = 1$. In fact, as demonstrated in Lemma 4, the uniform distribution is usually not profit-minimizing.

Another necessary property for F to yield $\underline{\pi}$ is that serving the consumers on $[-1 + 2(k-1)/n^*, -1 + 2k/n^*]$—by positioning at its center and charging the price $1 - c(1/n^*)$—must be locally optimal for the seller; global optimality is addressed later. Consider the subinterval $\left[-1, -1 + \frac{2}{n^*}\right]$. For the seller to obtain (no more than) $\underline{\pi}$, $1/n^*$ should be the reach maximizing $(1 - c(\Delta))F([-1, -1 + 2\Delta])$. Let f denote the density function of F. Evaluating the first-order condition at $\Delta = 1/n^*$ and invoking (6), we obtain

$$f\left(-1 + \frac{2}{n^*}\right) = \frac{1}{2n^*} \frac{c'(1/n^*)}{1 - c(1/n^*)}.$$

Applying the same argument to all other subintervals results in:

$$f\left(-1 + \frac{2}{n^*}\right) = ... = f\left(1 - \frac{2}{n^*}\right) = \frac{1}{2n^*} \frac{c'(1/n^*)}{1 - c(1/n^*)}. \tag{7}$$

The following distribution combines Lemma 4.(b) with the two necessary conditions, (6) and (7), in a simple manner.

Definition 2 *Let $\widehat{n}$ be the value defined in Lemma 4 and $f_n := \frac{1}{2n} \frac{c'(1/n)}{1 - c(1/n)}$ for each $n \in \mathbb{N}$. We*

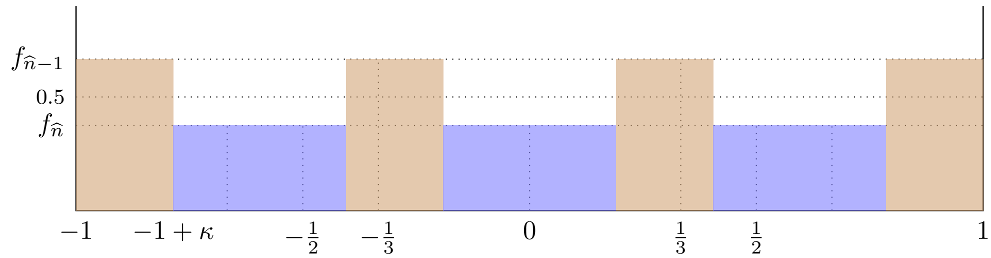

Figure 5: The density of distribution F^* defined in Definition 2. The cost function used for this figure is $c(y) = \frac{12}{7}y$. In this case, $\widehat{n} = 4$, and $\underline{\pi} = \frac{1}{3}\left(1 - c\left(\frac{1}{3}\right)\right) = \frac{1}{4}\left(1 - c\left(\frac{1}{4}\right)\right)$.

define F^* to be a piecewise linear distribution function with density

$$f^*(x) = \begin{cases} f_{\widehat{n}-1} & \text{if } x \in \left[-1 + \frac{2(k-1)}{\widehat{n}-1} - \kappa\frac{k-1}{\widehat{n}-1}, -1 + \frac{2(k-1)}{\widehat{n}-1} + \kappa\frac{n-k}{\widehat{n}-1}\right) \\ f_{\widehat{n}} & \text{otherwise,} \end{cases}$$

where $\kappa := \frac{1-2f_{\widehat{n}}}{\widehat{n}(f_{\widehat{n}-1}-f_{\widehat{n}})}$.[22]

Figure 5 shows a representative structure of the density function f^*. It alternates between two density levels, $f_{\widehat{n}-1}$ (high) and $f_{\widehat{n}}$ (low). By construction, f coincides with $f_{\widehat{n}}$ around $-1 + \frac{2}{\widehat{n}}, ..., 1 - \frac{2}{\widehat{n}}$ and with $f_{\widehat{n}-1}$ around $-1 + \frac{2}{\widehat{n}-1}, ..., 1 - \frac{2}{\widehat{n}-1}$, thus ensuring that f^* satisfies the necessary condition (7), whether the optimal n^* is $\widehat{n} - 1$ or $\widehat{n}$. In addition, the lengths of the subintervals are chosen so that F assigns probability $1/\widehat{n}$ to all intervals $\left[-1, -1 + \frac{2}{\widehat{n}}\right], ..., \left[1 - \frac{2}{\widehat{n}}, 1\right]$ and $1/(\widehat{n} - 1)$ to all intervals $\left[-1, -1 + \frac{2}{\widehat{n}-1}\right], ..., \left[1 - \frac{2}{\widehat{n}-1}, 1\right]$; these guarantee condition (6), whether the optimal n^* is $\widehat{n} - 1$ or $\widehat{n}$. Note that κ is the width of each high-density interval.

Given F^*, the seller can achieve $\underline{\pi}$ by serving $\left[-1, -1 + \frac{2}{\widehat{n}}\right]$ or $\left[-1, -1 + \frac{2}{\widehat{n}-1}\right]$. It suffices to show that the seller's profit *cannot* exceed $\underline{\pi}$. We establish this result in two steps. First, we show that for any $\Delta > 0$ and $\ell \in [-1, 1]$,

$$F^*([-1, -1 + 2\Delta]) \geq F^*([\ell - \Delta, \ell + \Delta]).$$

In other words, given F^*, for any reach Δ, it is weakly better for the seller to serve $[-1, -1 + 2\Delta]$ than any other $[\ell - \Delta, \ell + \Delta]$. This is because f^* has a periodic structure and each subinterval with high density $f_{\widehat{n}-1}$ has the same length; if ℓ increases from $-1 + \Delta$ then $F([\ell - \Delta, \ell + \Delta])$ may become smaller than, but cannot exceed, $F^*([-1, -1 + 2\Delta])$. Second, restricting attention to

[22] The value κ is such that $\widehat{n}\kappa f_{\widehat{n}-1} + (2 - \widehat{n}\kappa)f_{\widehat{n}} = 1$. In other words, κ is defined to be the common width of high-density regions that makes total probability equal to 1. For any convex $c(\cdot)$, f_n is strictly decreasing in n, and $\kappa \in [0, 1/(2\widehat{n})]$.

the intervals of the form $[-1,-1+2\Delta]$, the seller's profit $(1-c(\Delta))F^*\left([-1,-1+2\Delta]\right)$ has only two local maximizers, $\frac{1}{\widehat{n}}$ and $\frac{1}{\widehat{n}-1}$. These together imply that the seller cannot do strictly better than serving either $\left[-1,-1+\frac{2}{\widehat{n}}\right]$ or $\left[-1,-1+\frac{2}{\widehat{n}-1}\right]$. All remaining details can be found in the appendix.

5.3 Discussion

We conclude this section by providing a few relevant discussions.

Attainable profits. The highest profit the seller can achieve in our model is 1, which is obtained if (and only if) the distribution of tastes is degenerate. The following result shows that any profit between $\underline{\pi}$ and 1 is rationalizable.

Proposition 5 *There exists a distribution $F \in \Delta([-1,1])$ such that $\pi(F) = \pi$ if and only if $\pi \in [\underline{\pi}, 1]$.*

Proof. We provide a sketch of the proof here, relegating a comprehensive proof to the appendix. For each $\eta \in [0,1]$, we let $\mathcal{F}^\eta$ denote the set of all distributions over $[-\eta,\eta]$ and then find a profit-minimizing distribution in $\mathcal{F}^\eta$. If F coincides on its support with some distribution in $\mathcal{F}^\eta$, the seller has no incentive to cover $[-1,-\eta)$ and $(\eta,1]$. Combining this with the logic used to derive the profit lower bound in Section 5.1, it follows that the seller's profit cannot be lower than

$$\underline{\pi}^\eta := \max_{\Delta\in\left\{1,\frac{1}{2},\frac{1}{3}\cdots\right\}} \Delta(1-c(\eta\Delta)).$$

Then, following the same steps as in Section 5.2, one can construct a distribution under which the seller's maximized profit is $\underline{\pi}^\eta$. The desired result follows from the fact that $\underline{\pi}^\eta$ continuously increases from $\underline{\pi}^1 = \underline{\pi}$ to $\underline{\pi}^0 = 1$ as η decreases from 1 to 0. ∎

Comparison to the classical monopoly model without design. This result is in stark contrast to the corresponding result in the monopoly model without product design where the seller can obtain any profit in $[0,1]$. The classical monopoly model, thus, underestimates the seller's profit at the lower end by neglecting the fact that the seller designs the product to appeal to a wider public.

Uniform vs Profit-Minimizing Distributions. As shown in Section 5.1, the seller's problem under the uniform distribution and that under her worst distribution F^* differ only in that the former is an integer version of the latter. Lemma 4 established that the uniform distribution is a profit-minimizing distribution whenever $\Delta^U = 1/n$ for some $n \in \mathbb{N}$.[23] The following result suggests that the uniform distribution generally gives low profits to the seller when the disutility function is linear.

Proposition 6 *Suppose $c(y) = ty$ for some $t > 0$. Then $\pi^U/\underline{\pi} \leq 9/8$.*

Proof. See the appendix. ∎

When the disutility function is linear, the seller's profit under the uniform distribution is at most 12.5% above her lowest profit. In fact, if $t > 1$ then the ratio reduces to $25/24 \approx 1.0417$, so the maximum difference becomes around 4%. This result highlights the need for caution when adopting the standard practice of assuming a uniform distribution in the Hotelling framework.

6 Pareto Frontier

Our results in Section 4 revealed that the set of attainable consumer surplus in our model is $[0, CS(F_{\Delta^*})]$, where Δ^* is as given in Theorem 2 and F_Δ is as defined in Section 4.2. Meanwhile, those in Section 5 showed that the set of rationalizable profits is $[\underline{\pi}, 1]$, where $\underline{\pi}$ is as defined in Lemma 3. Clearly, the maximal consumer surplus, $CS(F_{\Delta^*})$, and the maximal producer surplus, 1, cannot be jointly achieved, because the latter requires the distribution of tastes F to be degenerate, in which case the resulting consumer surplus is 0. This section characterizes the Pareto payoff frontier—the upper envelope of payoff vectors attainable in our model.[24]

Recall that for each $p \in [\underline{p}, 1]$, F_{Δ_p} maximizes consumer surplus among those distributions that rationalize p. In addition, under F_{Δ_p}, the seller serves all consumers, resulting in a profit of p. Since the seller's profit cannot exceed p when charging this price, F_{Δ_p} Pareto dominates all distributions that rationalize p. This observation leads to the following result.

Proposition 7 *The Pareto payoff frontier across all distributions is given by*

$$\left\{(p, \overline{CS}(p)) : p \in [1 - c(\Delta^*), 1]\right\},$$

[23] While it may seem that the uniform distribution is generically not profit-minimizing, this is not quite the case, due to the boundary case where $\Delta_u = 1$, which obtains for a non-negligible set of cost functions.

[24] Our characterization of the frontier is similar in spirit to Bergemann et al. (2015) and Roesler and Szentes (2017), although the set of distributions here is not restricted by Bayes plausibility.

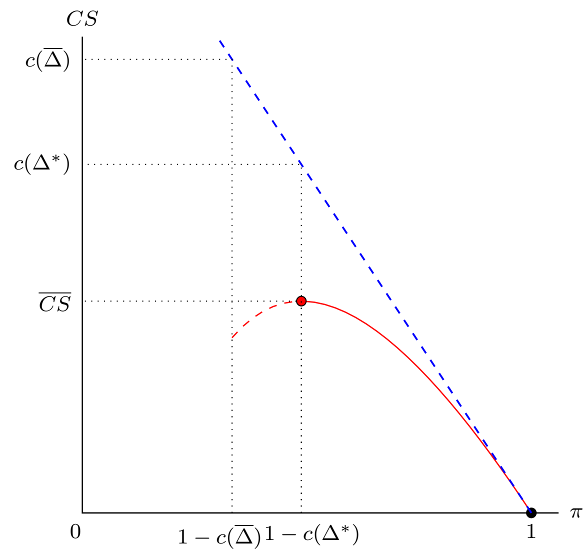

Figure 6: The maximal consumer surplus (red dot) and the Pareto frontier (red solid) when $c(y) = 0.8y$. The blue dashed line represents the case where full surplus of 1 is realized (i.e., $CS+\pi = 1$).

where $\overline{CS}(p)$ *is defined as*

$$\overline{CS}(p) := \max\{CS(F_\Delta) : 1 - c(\Delta) \geq p\}.$$

Figure 6 illustrates Proposition 7. For the linear disutility function, $CS(F_\Delta)$ is quasi-concave in Δ, increasing until Δ^* and then decreasing. In this case, the Pareto payoff frontier is spanned by $\{F_\Delta : \Delta \in [0, \Delta^*]\}$. If $CS(F_\Delta)$ is not quasi-concave, then the Pareto frontier is spanned by a subset of $\{F_\Delta : \Delta \in [0, \Delta^*]\}$. In that case, for each $p \in [1 - c(\Delta^*), 1]$ it suffices to identify the highest attainable consumer surplus with a weakly higher price; note that $\overline{CS}(p)$ is necessarily quasi-concave.

7 Conclusion

We study how product design impacts the predictions for the monopoly market. It gives the seller a strategic edge, rendering too low prices suboptimal, ensuring strictly positive profits for the seller, and reducing the set of consumer surplus that can arise. Notably, the maximal consumer surplus in the model with design is lower than in the model without design. Perhaps more surprising are the features of consumer-optimal and profit-minimizing distributions. The consumer-optimal distribution makes the seller strictly prefer the equilibrium price (and those just above it) to those prices at the top (close to 1), which is in stark contrast to the property of the consumer-optimal distribution in the model without design, namely, that the seller is indifferent between the equilibrium price and all prices above it. Furthermore, any profit-minimizing distribution exhibits a type of uniformity property, allocating equal probability across a finite number of partition intervals of identical width.

In this paper, we examine the canonical Hotelling model, where all consumers share the same valuation for their preferred design, and disutility depends solely on the distance between the preferred and actual designs. In oligopolistic settings, the complexity of the environment often necessitates additional assumptions, such as specific functional forms for disutility (e.g., linear or quadratic) and a uniform distribution of consumer tastes; e.g., Thisse and Vives (1988), Anderson et al. (1997). The monopolistic setting, however, allows for a more in-depth analysis. Future research could explore generalizations of the model, such as cases where consumers have private information about both the location and value of their preferred design or where disutility functions are more general. In addition, the Hotelling model has recently gained renewed attention due to its suitability for modeling the information required to match consumer preferences with potential products by a platform; e.g. Hidir and Vellodi (2021) investigate such a setting using a uniform distribution of tastes. The broader impact of information on the outcomes of the Hotelling model, however, remains elusive. Future work could address questions such as the seller's incentive to acquire information about the distribution of consumer preferences or how the type of information consumers possess influences their willingness to disclose it; Kim and Kos (2023) examine the extreme case where the seller has no information about the distribution of tastes.[25]

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Appendix: Omitted Proofs

Proof of Proposition 2. Define $h(\Delta) := 2c(\Delta) - c(\Delta/2)$. Then, $\overline{\Delta}$ is the maximal value of $\Delta \in [0,1]$ such that $h(\Delta) \leq 1$. We show that h is continuous and strictly increasing; this implies not only that $\overline{\Delta}$ is well defined, but also that $1 - c(\Delta) \geq 1/2(1 - c(\Delta/2))$ holds if and only if $\Delta \leq \overline{\Delta}$.

Continuity of h follows from that of c. For monotonicity, consider any $0 \leq \Delta < \Delta'$. Then,

$$\begin{aligned} h(\Delta') - h(\Delta) &= 2(c(\Delta') - c(\Delta)) - (c(\Delta'/2) - c(\Delta/2)) \\ &= c(\Delta') - c(\Delta) + [(c(\Delta') - c(\Delta)) - (c(\Delta'/2) - c(\Delta/2))] \\ &> c(\Delta') - c(\Delta) > 0, \end{aligned}$$

where the first inequality holds because c is convex. ■

Proof of Proposition 3. We prove a slightly more general result than stated in Proposition 3, namely, that $\ell = 0$ is not optimal for any distribution of tastes $F^0_\Delta \in \mathcal{F}^0_\Delta$, provided that $\Delta > 0$ (not just when $\Delta = \Delta^0$). Given F^0_Δ, the seller is indifferent between $(0, \Delta)$ and $(0, \Delta/2)$. Therefore, we have

$$\begin{aligned} \pi(0, \Delta) &= (1 - c(\Delta))F^0_\Delta([-\Delta, \Delta]) \\ &= 1 - c(\Delta) \\ &= \pi(0, \Delta/2) \\ &= (1 - c(\Delta/2))F^0_\Delta([-\Delta/2, \Delta/2]). \end{aligned}$$

Similarly, the seller is indifferent between $(0, \Delta)$ and $(0, 0)$, so we also have

$$\begin{aligned}\pi(0, \Delta) &= 1 - c(\Delta) \\ &= \pi(0, 0) \\ &= F^0_\Delta([0, 0]).\end{aligned}$$

Claim 1 *The following holds:*

$$\max\left\{F^0_\Delta\left([0, \Delta]\right), F^0_\Delta\left([-\Delta, 0]\right)\right\} \geq 1 - \frac{c(\Delta)}{2}.$$

Proof. If the inequality fails then the following contradiction emerges:

$$\begin{aligned}1 &= F^0_\Delta\left([-\Delta, \Delta]\right) \\ &= F^0_\Delta\left([\Delta, 0]\right) + F^0_\Delta\left([0, \Delta]\right) - F^0_\Delta([0, 0]) \\ &< 2 - c(\Delta) - (1 - c(\Delta)) = 1.\end{aligned}$$

■

Claim 2 *If c is weakly convex then for any $y > 0$ we have*

$$(1 - c(y/2))\left(1 - \frac{c(y)}{2}\right) > 1 - c(y).$$

Proof. For any convex c such that $c(0) = 0$, we have $c(y/2) \leq c(y)/2$. Therefore,

$$\begin{aligned}(1 - c(y/2))\left(1 - \frac{c(y)}{2}\right) &\geq \left(1 - \frac{c(y)}{2}\right)\left(1 - \frac{c(y)}{2}\right) \\ &= 1 - c(y) + \frac{c(y)^2}{4} \\ &> 1 - c(y).\end{aligned}$$

■

We proceed to show that either $(\Delta/2, \Delta/2)$ or $(-\Delta/2, \Delta/2)$—moving her design to either $-\Delta/2$ or $\Delta/2$ and charging $1 - c(\Delta/2)$—is a profitable deviation for the seller. In Claim 1, without loss of generality, suppose $F([0,\Delta]) \geq 1 - c(\Delta/2)$. Then, we have

$$\begin{aligned}\pi(\Delta/2, \Delta/2) &= (1 - c(\Delta/2))F([0,\Delta]) \\ &\geq (1 - c(\Delta/2))\left(1 - \frac{c(\Delta)}{2}\right) \\ &> 1 - c(\Delta) = \pi(0, \Delta),\end{aligned}$$

where the weak inequality is due to Claim 1, while the strict inequality is due to Claim 2.

■

Proof of Lemma 1. We use the fact that for each $x \in [0, \Delta]$, the seller's deviation to serve consumers on $[-x, \Delta]$ or $[-\Delta, x]$—by choosing design $(\Delta - x)/2$ or $(x - \Delta)/2$ with reach $(\Delta + x)/2$—should not be profitable. For the former, it must be that

$$\left(1 - c\left(\frac{x + \Delta}{2}\right)\right) F([-x, \Delta]) \leq (1 - c(\Delta))F([-\Delta, \Delta]).$$

Similarly, for the latter not to be profitable, it must be that

$$\left(1 - c\left(\frac{x + \Delta}{2}\right)\right) F([-\Delta, x]) \leq (1 - c(\Delta))F([-\Delta, \Delta]).$$

Combining the above two inequalities yields

$$\begin{aligned}&\left(1 - c\left(\frac{x + \Delta}{2}\right)\right)(F([-x, \Delta]) + F([-\Delta, x])) \\ = &\left(1 - c\left(\frac{x + \Delta}{2}\right)\right)(F([-\Delta, \Delta]) + F([-x, x])) \\ \leq &\ 2(1 - c(\Delta))F([-\Delta, \Delta]),\end{aligned}$$

which can be simplified to

$$\begin{aligned}F([-x, x]) &\leq \left(2\frac{1 - c(\Delta)}{1 - c(\frac{x+\Delta}{2})} - 1\right) F([-\Delta, \Delta]) \\ &\leq \left(2\frac{1 - c(\Delta)}{1 - c(\frac{x+\Delta}{2})} - 1\right).\end{aligned}$$

The last inequality holds because the term inside the parenthesis is non-negative whenever $\Delta \leq \overline{\Delta}$, that is, Proposition 2 implies

$$1-c(\Delta) \geq \frac{1}{2}\left(1-c\left(\frac{\Delta}{2}\right)\right)$$
$$\geq \frac{1}{2}\left(1-c\left(\frac{x+\Delta}{2}\right)\right).$$

■

Proof of Lemma 2. By construction, for each $x \in [0, \Delta]$, we have

$$(1-c(\Delta))F_{\Delta}\left([-\Delta,\Delta]\right) = F_{\Delta}([-x,\Delta])\left(1-c\left(\frac{x+\Delta}{2}\right)\right)$$
$$= F_{\Delta}([-\Delta,x))\left(1-c\left(\frac{x+\Delta}{2}\right)\right),$$

which leads to $F_{\Delta}([0,x]) = F_{\Delta}([-x,0])$. In particular, $F_{\Delta}([0,\Delta]) = F_{\Delta}([-\Delta,0])$, and thus $F_{\Delta}((0,\Delta]) = F_{\Delta}([-\Delta,0))$, which together with $F_{\Delta}([0,x]) = F_{\Delta}([-x,0])$ implies $F([-\Delta,x]) = F([-x,\Delta])$, that is, symmetry of F around 0.

By definition, $F_{\Delta}([-x,x])$ is strictly increasing in x, and $F_{\Delta}([-\Delta,\Delta]) = 1$. Therefore, F_{Δ} is well defined as long as

$$F\left([0,0]\right) = \lim_{x\downarrow 0} F([-x,x]) = 2\frac{1-c(\Delta)}{1-c(\frac{\Delta}{2})} - 1 \geq 0,$$

which can be rewritten as

$$1-c(\Delta) \geq \frac{1}{2}\left(1-c\left(\frac{\Delta}{2}\right)\right).$$

This inequality holds whenever $\Delta \leq \overline{\Delta}$ (see Proposition 2).

Next, we prove that $(0,\Delta)$ is an optimal strategy for the seller given F_{Δ}. Fix a $\Delta' \in \left[\frac{\Delta}{2},\Delta\right]$, and let $\ell' = \Delta - \Delta'$. By construction, $\pi(0,\Delta) = \pi(\ell',\Delta')$. We show that $\pi(\ell',\Delta') \geq \pi(\ell,\Delta')$ for any $\ell \in [-1,1]$, that is, ℓ' is the seller's optimal design for the reach Δ'. Since $supp(F_{\Delta}) = [-\Delta,\Delta]$, the inequality clearly holds if $\ell < -\ell'$ or $\ell > \ell'$. For $\ell \in [0, \ell')$ the result follows from

$$\begin{aligned} D(\ell, \Delta') - D(\ell', \Delta') &= F_{\Delta}\left([\ell - \Delta', \ell + \Delta']\right) - F_{\Delta}\left([\ell' - \Delta', \ell' + \Delta']\right) \\ &= F_{\Delta}\left([\ell - \Delta', \ell + \Delta']\right) - F_{\Delta}\left([\Delta - 2\Delta', \Delta]\right) \\ &= F_{\Delta}([\ell - \Delta', \Delta - 2\Delta')) - F_{\Delta}((\ell + \Delta', \Delta]) \\ &< 0, \end{aligned}$$

where the second equality is because $\ell' = \Delta - \Delta'$; the third equality is because $\ell < \ell' = \Delta - \Delta'$ and $\Delta - 2\Delta' \leq 0 < \Delta' \leq \ell + \Delta'$; and the inequality holds because the two intervals are of the same length, do not include $\ell = 0$ (where the mass point is), and the density function (conditional on $x \neq 0$) is symmetric around 0 and strictly increasing in $|x|$ (see Figure 3). The symmetric argument applies when $\ell \in (-\widehat{\ell}, 0)$.

Now, consider $\Delta' \in \left[0, \frac{\Delta}{2}\right]$. Given the shape of F_{Δ}, there are two cases to consider, one in which $\ell' = \Delta'$ (which ensures to include consumers at 0) and the other in which $\ell' = \Delta - \Delta'$ (which enables the seller to serve all consumers near Δ). In the former case, the inequality

$$\begin{aligned} \pi(\ell, \Delta') &= (1 - c(\Delta'))F_{\Delta}\left([0, 2\Delta']\right) \\ &\leq \pi(\ell', \Delta') \\ &= \pi(0, \Delta) \\ &= 1 - c(\Delta) \end{aligned}$$

can be rewritten as

$$\frac{1}{1 - c\left(\Delta' + \frac{\Delta}{2}\right)} + \frac{1}{1 - c\left(\frac{\Delta}{2}\right)} \leq \frac{1}{1 - c(\Delta)} + \frac{1}{1 - c(\Delta')}.$$

This inequality holds because convexity of $c(\cdot)$ implies that $\frac{1}{1-c(\cdot)}$ is strictly convex, $(\Delta' + \frac{\Delta}{2}) + \frac{\Delta}{2} = \Delta + \Delta'$, and $\max\{\frac{\Delta}{2}, \Delta' + \frac{\Delta}{2}\} \leq \Delta$.

When $\ell' = \Delta - \Delta'$, the inequality

$$\begin{aligned} \pi(\ell, \Delta') &= (1 - c(\Delta'))F_{\Delta}\left([\Delta - 2\Delta', \Delta]\right) \\ &\leq \pi(0, 1 - c(\Delta)) \\ &= 1 - c(\Delta) \end{aligned}$$

is equivalent to

$$\frac{1}{1-c(\Delta)} \leq \frac{1}{1-c(\Delta')} + \frac{1}{1-c(\Delta-\Delta')}.$$

Since $\frac{1}{1-c(\cdot)}$ is strictly convex, the right-hand side is minimized when $\Delta' = \frac{\Delta}{2}$. Therefore, the inequality holds for any $\Delta' \in \left[0, \frac{\Delta}{2}\right)$ if and only if

$$\frac{1}{1-c(\Delta)} \leq \frac{2}{1-c\left(\frac{\Delta}{2}\right)},$$

which holds due to $\Delta \leq \overline{\Delta}$ (see Proposition 2). ■

Proof of Theorem 3. We establish that F^*, given by Definition 2, satisfies $\pi(F^*) = \underline{\pi}$.

We first show that for any reach Δ, it is without loss of generality to consider design $\ell = -1+\Delta$, that is, for any (ℓ, Δ),

$$\pi(\ell, \Delta; F^*) \leq \pi(-1+\Delta, \Delta; F^*),$$

which is equivalent to

$$\begin{aligned} D(\ell, \Delta; F^*) &= F^*\left([\ell-\Delta, \ell+\Delta]\right) \\ &\leq F^*\left([-1, -1+2\Delta]\right) \\ &= D(-1+\Delta, \Delta; F^*). \end{aligned}$$

Note that given Δ, it is clearly not optimal for the seller to choose $\ell < -1+\Delta$ or $\ell > 1-\Delta$. Therefore, we restrict attention to $\ell \in [-1+\Delta, 1-\Delta]$.

First, consider the case where $f^*(\ell-\Delta) = f_{n^*-1}$ (higher density). Let ℓ' be the smallest location such that $f^*(x) = f_{n^*-1}$ for all $x \in [\ell'-\Delta, \ell-\Delta]$. In this case, the seller can increase her demand (profit) by moving to ℓ':

$$\begin{aligned} D(\ell', p; F^*) - D(\ell, p; F^*) &= (\ell-\ell')f_{n^*-1} - \int_{\ell'+\Delta}^{\ell+\Delta} f^*(x)dx \\ &= \int_{\ell'+\Delta}^{\ell+\Delta} \left(f_{n^*-1} - f^*(x)\right)dx \geq 0, \end{aligned}$$

where the inequality holds because f^* never exceeds f_{n^*-1}. The desired result then follows from the fact that f^* is periodic, so $F^*\left([-1, -1+2\Delta]\right) = F^*\left([\ell'-\Delta, \ell'+\Delta]\right)$.

Next, consider the case where $f^*(\ell + \Delta) = f_{n^*-1}$. In this case, for the same reason as in the previous case, the seller can increase her demand by moving to the *right* (and then to $1 - \Delta$), which yields $D(1 - \Delta, \Delta; F^*) \geq D(\ell, \Delta; F^*)$. Combining this with the symmetry of F^*, we have $D(-1 + \Delta, \Delta; F^*) = D(1 - \Delta, \Delta; F^*) \geq D(\ell, \Delta; F^*)$.

Finally, consider the case where $f^*(\ell - \Delta) = f^*(\ell + \Delta) = f_{n^*}$ (lower density). Let ℓ' be the largest location below ℓ such that $f^*(\ell' - \Delta) = f_{n^*-1}$ or $f^*(\ell' + \Delta) = f_{n^*-1}$ (i.e., ℓ^* is the location at which one of the edges of $[\ell' - \Delta, \ell' + \Delta]$ meets the higher density region). By construction, $D(\ell', \Delta; F^*) = D(\ell, \Delta; F^*)$. But, now $f^*(\ell' - \Delta) = f_{n^*-1}$ or $f^*(\ell' + \Delta) = f_{n^*-1}$, so one of the above cases applies to ℓ'. This leads to $D(-1 + \Delta, \Delta; F^*) \geq D(\ell', \Delta; F^*) = D(\ell, p; F^*)$.

The above analysis implies that the problem reduces to

$$\max_{\Delta} \widehat{\pi}(\Delta) := (1 - c(\Delta))F^* \left([-1, -1 + 2\Delta]\right).$$

We show $\widehat{\pi}(\Delta)$ has only two local maximizers, $\Delta_1 := \frac{1}{n^*}$ and $\Delta_2 := \frac{1}{n^*-1}$. Define $\Delta_3 := \frac{2-\kappa}{n^*-1}$, so that

$$f^*(x) = \begin{cases} f_{n^*-1} & \text{if } x \in [-1, -1 + \kappa) \text{ or } x \in [-1 + \Delta_3, -1 + \Delta_3 + \kappa) \\ f_{n^*} & \text{if } x \in [-1 + \kappa, -1 + \Delta_3). \end{cases}$$

The desired result follows from the following four results.

(i) $\widehat{\pi}(\Delta)$ is increasing if $\Delta < \Delta_1$.

Since $c(\cdot)$ is increasing and convex, $c(\Delta) < c(\Delta_1)$ and $c'(\Delta) \leq c'(\Delta_1)$. Combining this with the fact that $f^*(-1 + 2\Delta_1) = f_{n^*} \leq f^*(-1 + 2\Delta)$ for any Δ, we get

$$\begin{aligned} \widehat{\pi}'(\Delta) &= -c'(\Delta)F^*(-1 + 2\Delta) + 2(1 - c(\Delta))f^*(-1 + 2\Delta) \\ &> -c'(\Delta_1)F^*(-1 + 2\Delta_1) + 2(1 - c(\Delta_1))f^*(-1 + 2\Delta_1) \\ &= \widehat{\pi}'(\Delta_1) \\ &= 0. \end{aligned}$$

(ii) $\widehat{\pi}(\Delta)$ is decreasing if $\Delta \in (\Delta_1, \Delta_3]$.

In this case, $f^*(-1+2\Delta) = f_{n^*}$ (low density). Combining this with $\Delta > \Delta_1$ (so $c(\Delta) > c(\Delta_1)$

and $c'(\Delta) \geq c'(\Delta_1)$) leads to

$$\begin{aligned}
\widehat{\pi}'(\Delta) &= -c'(\Delta)F^*(-1+2\Delta) + 2(1-c(\Delta))f^*(-1+2\Delta) \\
&= -c'(\Delta)F^*(-1+2\Delta) + 2(1-c(\Delta))f_{n^*} \\
&< -c'(\Delta_1)F^*(-1+2\Delta_1) + 2(1-c(\Delta_1))f^*(-1+2\Delta_1) \\
&= \widehat{\pi}'(\Delta_1) \\
&= 0.
\end{aligned}$$

(iii) $\widehat{\pi}(\Delta)$ is increasing if $\Delta \in (\Delta_3, \Delta_2)$.

In this case, $f^*(-1+2\Delta) = f_{n^*-1}$ (high density). Then, similarly to (i),

$$\begin{aligned}
\widehat{\pi}'(\Delta) &= -c'(\Delta)F^*(-1+2\Delta) + 2(1-c(\Delta))f^*(-1+2\Delta) \\
&> -c'(\Delta_2)F^*(-1+2\Delta_2) + 2(1-c(\Delta_2))f^*(-1+2\Delta_2) \\
&= \widehat{\pi}'(\Delta_2) \\
&= 0.
\end{aligned}$$

(iv) $\widehat{\pi}(\Delta)$ is decreasing if $\Delta > \Delta_2$.

Since $f^*(-1+2\Delta_2) = f_{n^*-1}$ (high density), similarly to (ii),

$$\begin{aligned}
\widehat{\pi}'(\Delta) &= -c'(\Delta)F^*(-1+2\Delta) + 2(1-c(\Delta))f^*(-1+2\Delta) \\
&\leq -c'(\Delta)F^*(-1+2\Delta) + 2(1-c(\Delta))f_{n^*-1} \\
&< -c'(\Delta_2)F^*(-1+2\Delta_2) + 2(1-c(\Delta_2))f^*(-1+2\Delta_2) \\
&= \widehat{\pi}'(\Delta_2) \\
&= 0.
\end{aligned}$$

Finally, since $F([-1,-1+\Delta_1]) = 1/n^*$ and $F([-1,-1+\Delta_2]) = 1/(n^*-1)$, and $\underline{\pi}$ is given by part (b) of Lemma 4, the result follows. ■

Proof of Proposition 5. Given Theorem 3, it suffices to show that for each $\pi \in [\underline{\pi}, 1]$, there exists a distribution that gives π to the seller. Recall that we defined $\mathcal{F}^\eta$ as the set of all distributions over $[-\eta, \eta]$ and $\underline{\pi}^\eta$ as

$$\underline{\pi}^\eta := \max_{\Delta \in \left\{1, \frac{1}{2}, \frac{1}{3} \dots\right\}} \Delta(1 - c(\eta\Delta)).$$

We first show that for any $\eta \in (0,1]$, there exists a distribution in $\mathcal{F}^\eta$ under which the seller's maximized profit is $\underline{\pi}^\eta$. Let $\widehat{c}$ denote the function such that $\widehat{c}(y) = c(\eta y)$ for any $y \geq 0$. By Theorem 3 and the definition of $\widehat{c}$, there exists a distribution F under which the seller's maximized profit is equal to

$$\begin{aligned}\underline{\pi}^\eta &= \max_{\Delta \in \left\{1, \frac{1}{2}, \frac{1}{3} \ldots\right\}} \Delta(1 - \widehat{c}(\Delta)) \\ &= \max_{\Delta \in \left\{1, \frac{1}{2}, \frac{1}{3} \ldots\right\}} \Delta(1 - c(\eta\Delta)).\end{aligned}$$

Consider the distribution F^η such that $F^\eta(x) = F(x/\eta)$. By construction, the support of F^η belongs to $[-\eta, \eta]$, so $F^\eta \in \mathcal{F}^\eta$. It suffices to show that the seller's profit cannot exceed $\underline{\pi}^\eta$ under F^η. Consider any strategy $(\ell, p) = (\ell, 1 - c(\Delta))$, and let $\ell' := \ell/\eta$ and $\Delta' := \Delta/\eta$. Then, we have

$$\begin{aligned}\pi(\ell, 1 - c(\Delta); F^\eta) &= \left(F^\eta(\ell + \Delta) - F^\eta_-(\ell - \Delta)\right)(1 - c(\Delta)) \\ &= \left(F\left(\frac{\ell + \Delta}{\eta}\right) - F_-\left(\frac{\ell - \Delta}{\eta}\right)\right)(1 - \widehat{c}(\Delta/\eta)) \\ &= \left(F(\ell' + \Delta') - F_-(\ell' + \Delta')\right)(1 - \widehat{c}(\Delta')) \\ &= \pi(\ell', 1 - \widehat{c}(\Delta'); F) \\ &\leq \underline{\pi}^\eta,\end{aligned}$$

where the second equality is because of the definitions of F^η and $\widehat{c}$, the third equality is because of the definitions of ℓ' and Δ', and the inequality is due to the fact that $\underline{\pi}^\eta$ is the seller's maximized profit under F.

It remains to show that $\underline{\pi}^\eta$ continuously decreases in η; since $\underline{\pi}^1 = \underline{\pi}$ and $\underline{\pi}^0 = 1$, the intermediate value theorem ensures that for any $\pi \in [\underline{\pi}, 1]$ there exists $\eta \in [0,1]$ such that $\underline{\pi}^\eta = \pi$. Consider the following relaxed problem where Δ is allowed to be 0 as well:

$$\underline{\pi}_0^\eta := \max_{\Delta \in \left\{1, \frac{1}{2}, \frac{1}{3} \ldots\right\} \cup \{0\}} \Delta(1 - c(\eta\Delta)).$$

For each Δ, the function $\Delta(1 - c(\eta\Delta))$ is continuous in η. In addition, the domain $\left\{1, \frac{1}{2}, \frac{1}{3} \ldots\right\} \cup \{0\}$ is compact. The theorem of the maximum then implies that $\underline{\pi}_0^\eta$ is always well-defined and continuous in η. In fact, since $\Delta(1 - c(\eta\Delta))$ is decreasing in η for any Δ, the maximum $\underline{\pi}_0^\eta$ is also decreasing in η. The desired result then follows from the fact that $\Delta = 0$ can never be the solution to the above maximization problem, so $\underline{\pi}^\eta = \underline{\pi}_0^\eta$ always holds. ■

Proof of Proposition 6. If $t \leq \frac{1}{2}$, then the result is immediate, because $\pi^U = \underline{\pi} = 1 - t$. From now on, we restrict attention to $t > 1/2$ by redefining $t_1 = \frac{1}{2}$. Note that for any $t > t_1$, the seller's profit under the uniform distribution is given by $\pi^U = \frac{1}{4t}$.

Take any $t \in (t_n, t_{n+1}]$. Then,

$$\frac{\pi^U}{\underline{\pi}} = \frac{\frac{1}{4t}}{\frac{1}{n}\left(1 - \frac{t}{n}\right)} = \frac{1}{4}\frac{1}{\frac{t}{n}\left(1 - \frac{t}{n}\right)}.$$

Define a function $\phi : [t_n, t_{n+1}] \to \mathcal{R}_+$ as $\phi_n(t) := \frac{t}{n}\left(1 - \frac{t}{n}\right)$. If $n = 1$ (i.e., $t \in \left[\frac{1}{2}, \frac{2}{3}\right]$) then its maximum is given by $\phi_n(1/2) = 1/4$, while its minimum is given by $\phi_n(t_2) = \phi_n(2/3) = 2/9$. It follows that

$$\frac{\pi^U}{\underline{\pi}} \in \left[\frac{1}{4\phi_1(1/2)}, \frac{1}{4\phi_1(2/3)}\right] = \left[1, \frac{9}{8}\right] \text{ for } t \in [t_1, t_2].$$

For $n > 1$, the maximum of ϕ_n is given by $\phi_n(n/2) = 1/4$, while its minimum is given by $\phi_n(t_n) = \frac{n(n-1)}{(2n-1)^2}$. This implies that

$$\frac{\pi^U}{\underline{\pi}} \in \left[\frac{1}{4\phi_n(n/2)}, \frac{1}{4\phi_n(t_n)}\right] = \left[1, \frac{(2n-1)^2}{4n(n-1)}\right] \text{ for } t \in [t_n, t_{n+1}].$$

Since $\phi_n(t_n) = \frac{n(n-1)}{(2n-1)^2} = \frac{1}{1+1/(n(n-1))}$ is strictly increasing in n, the global maximum of $\pi^U/\underline{\pi}$ is given by $9/8$, which is achieved when $t = \frac{2}{3}$. ■