The Impact of Bequest Taxation on Wealth Inequality - Theory and Evidence

Berenice Anne Neumann* Niklas Scheuer†

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We study the effect of bequests and their taxation on wealth inequality. We allow for random death and birth in a continuous-time, dynastic framework. Individuals behave optimally and accumulate wealth over their lifetime. Bequests above a tax exemption threshold are taxed according to a fixed rate. We derive a stochastic differential equation modeling dynastic wealth and obtain an analytical expression for the coefficient of variation. By calibrating our model to German wealth data, we utilize these analytical results to project empirical wealth inequality across various bequest tax rates and tax exemption thresholds. Most notably, our results indicate that a combination of a high tax exemption threshold paired with a high bequest tax rate reduces wealth inequality strongest when considering revenue-neutral alterations.

Keywords: wealth, bequest, taxation, wealth inequality, analytical solution

JEL Codes: D31, E21, H24

1. Introduction

Wealth levels are more extreme than ever (Freund and Oliver, 2016). Although many wealth owners are self-made, a substantial amount inherited their fortune, e.g., around 50% in Sweden (Ohlsson et al., 2020) and about 60-70% in France (Piketty and Zucman, 2015). In light of the large estimated bequest volumes (e.g., in Germany wealth around EUR 2.1 billion is expected to be bequeathed across generations between 2015 and 2024 (Braun, 2015)), bequest taxation, especially in the absence of a wealth tax[1], could offer a tool for alleviating wealth inequality. This has sparked debates on the role of bequest taxation, especially in Germany (Beckert and Arndt, 2017; Fleurbaey et al., 2022).

*Trier University, Department IV - Mathematics

†Trier University, Department IV - Economics, corresponding author

[1]Please see Drometer et al. (2018) for an overview of countries in Europe without a wealth tax.

Naturally, the impact of bequests on wealth inequality is covered in numerous papers[2] focusing on theory (Bossmann et al., 2007; Wan and Zhu, 2019), empirical analysis (Boserup et al., 2016; Elinder et al., 2018; Beznoska et al., 2020; Ohlsson et al., 2020) or a combination of both (De Nardi, 2004; Yang, 2013; Benhabib et al., 2016; De Nardi and Yang, 2016; Kaymak and Poschke, 2016; De Nardi and Fella, 2017; Nekoei and Seim, 2023; Escobar et al., 2023). These papers, if not purely empirical, concentrate mainly on the effect of intentional bequests on wealth inequality. Bequest taxation, if considered, is usually a minor aspect. In contrast, our model systematically analyzes how bequest *taxation* affects wealth inequality.

We consider a continuum of dynasties, where at each point in time one individual from each dynasty is alive. During her lifetime the individual chooses consumption optimally and accumulates wealth accordingly. Upon death, which occurs randomly with a constant death rate, a new representative of the dynasty is born and receives the dynasty wealth minus bequest taxes as initial endowment. The bequest taxes are charged as a flat tax on all wealth exceeding the tax exemption threshold. Ultimately, we obtain that dynasty wealth is given as the solution of a stochastic differential equation driven by a Poisson process.

Our theoretical analysis focuses on the explicit computation of the coefficient of variation as a measure of wealth inequality. For this, we proceed in several steps: First, we solve the stochastic differential equation and obtain an analytical expression for dynasty wealth. In light of this, we compute the first and second moment of the wealth distribution given fixed initial wealth levels. Then we derive the first and second moment for arbitrary initial wealth distributions. This culminates in an analytic expression for the coefficient of variation. This explicit representation allows us to analyze the effect of the bequest taxation scheme on wealth inequality.

Concentrating on Germany, we perform a comparative statics analysis using wealth data from the German Socio-Economic Panel (Goebel et al., 2019). First, we investigate the impact of different bequest tax rates and tax exemption thresholds across various time periods on the coefficient of variation. In a second step, we focus on revenue-neutral policy experiments. We observe that a higher bequest tax rate, which coincides with a higher tax exemption threshold under revenue neutrality, leads to lower wealth inequality.

To conclude the introduction, we outline the related literature: Bequests can theoretically be modeled as accidental, intentional or both. Accidental bequests as considered in Abel (1985) are motivated by empirical evidence (Wilhelm, 1996) and the fact that individuals dissave towards older ages (Hurd, 1987). The concept of warm-glow (Andreoni, 1989) formed the basis for the analysis of intentional bequests. Regarding the latter, many papers work with an overlapping-generation framework, e.g., De Nardi (2004), Bossmann et al. (2007), Yang (2013), De Nardi and Yang (2016), De Nardi and Fella (2017), and Wan and Zhu (2019). As explained above, bequest taxation, if considered, is usually a minor aspect. Most relevant regarding the influence of bequest taxation on wealth inequality are De Nardi and Yang (2016), Bossmann et al. (2007), Wan and Zhu (2019) and Zhu (2019): De Nardi and Yang (2016) numerically study the influence of changes in either the tax rate or the tax exemption threshold on wealth inequality. Bossmann et al. (2007) analytically analyze the influence of changes in the tax rate on the coefficient of variation and (under stronger assumptions) on the Gini coefficient. Wan and Zhu (2019) focus on the effect of changes in the tax rate on

[2] We discuss the related literature and our contribution to it in more detail below.

long-run wealth inequality. Similarly, Zhu (2019) studies the effect of flat estate taxation (without exemption thresholds) on long-run wealth inequality.

Another branch of the literature considers the evolution of wealth distributions in a dynastic setup. Benhabib et al. (2016) and Kaymak and Poschke (2016) analyze intentional bequests. Whereas the first authors do not consider bequest taxation, the latter authors numerically assess the impact of changes in the taxation scheme. Kasa and Lei (2018) and Birkner et al. (2023) consider the case where no wealth is bequeathed (i.e., taxed with a rate of 100%) and instead every newborn with a fixed initial amount of wealth[3].

The remainder of this paper is organized as follows: In Section 2, we introduce the model. We establish important properties in Section 3 allowing us to determine the coefficient of variation analytically in Section 4. Building upon our theoretical framework, Section 5 presents a numerical analysis of the effects of the current tax regime and proposed alterations. Finally, Section 6 provides our concluding remarks. Appendix A contains the proofs of all results.

2. Model

We consider the same model as in Birkner et al. (2023) but introduce a different bequest taxation scheme. More precisely, whereas in Birkner et al. (2023) no wealth is bequeathed within a dynasty, but every newborn starts with a fixed initial wealth level, here we consider a taxation scheme used in many European countries (Drometer et al., 2018), where wealth is bequeathed to the newborn individual subject to bequest taxation with a tax exemption threshold. Hence, we only describe the individual optimization problem in Section 2.1 and refer the reader for additional details as well as proofs to Birkner et al. (2023, Section A.1). In Section 2.2 we then describe the dynasty, our bequest taxation scheme and the resulting dynamics of dynasty wealth.

2.1. The individual

An individual is born at t_0, shows a constant aversion for risk $\sigma > 0$, also satisfying $\sigma \neq 1$, and derives instantaneous utility from consuming c according to a constant relative risk aversion (CRRA) utility function

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}$$

over her lifetime. Death approaches randomly according to a constant death rate δ, implying that the lifetime $T - t_0$ is exponentially distributed with parameter δ. Bequests will play a role later when moving to dynasty wealth but occur purely accidentally as in Abel (1985). This translates to an exclusion of warm-glow on an individual stage, i.e., bequests are not contributing to an individual's utility. Thus, the maximization problem features expected lifetime utility

$$U(t_0) = \mathbb{E}_{t_0}\left[\int_{t_0}^{T} e^{-\hat{\rho}[s-t_0]} u(c(s))\,\mathrm{d}s\right]$$

[3]Please see Birkner et al. (2023) for a detailed review regarding this branch of the literature.

as our objective function, and

$$\dot{a}\left(t\right)=\left(r-\tau_{a}\right)a\left(t\right)+\left(1-\tau_{w}\right)w-c\left(t\right) \tag{1}$$

as the budget constraint. Equation (1) captures the law of motion for *individual* wealth over time, which is characterized by capital income net of wealth tax $(r-\tau_a)a(t)$, net labor income $(1-\tau_w)w$ and consumption expenditure $c(t)$. We impose a natural borrowing limit, which requires the present value of net labor income to exceed a potential initial debt $a_i<0$.

We solve the individual optimization problem and obtain the following expression for optimal consumption

$$c(t)=\frac{\rho-(1-\sigma)(r-\tau_a)}{\sigma}\left(a(t)+W\right),$$

where the rate $\rho=\hat{\rho}+\delta$ is the sum of the time preference rate $\hat{\rho}$ and the death rate δ, and

$$W=\frac{(1-\tau_w)w}{r-\tau_a}$$

describes the present value of net labor income. Based on the optimal consumption path, individual wealth $a(t)$ evolves according to the ordinary differential equation

$$\dot{a}(t)=z(a(t)+W) \tag{2}$$

where

$$z=\frac{r-\tau_a-\rho}{\sigma} \tag{3}$$

describes the optimal wealth growth rate as an outcome of our maximization problem. Solving the ordinary differential equation (2), we obtain that individual wealth is given by

$$a(t)=(a(t_0)+W))e^{z(t-t_0)}-W. \tag{4}$$

Concentrating on realistic scenarios, we henceforth focus on a positive wealth growth z and emphasize that all further results are contingent upon that.

2.2. The dynasty

We consider a continuum of dynasties, where at each point in time one individual from each dynasty is alive. Once an individual dies within a dynasty, a new one is born implying everlasting dynasties and a constant population (cf. Blanchard (1985)). We denote by $T_0=0$, T_1, T_2, $\ldots$ the death/birth dates of the dynasty representatives. As explained above, the lifetime T_k-T_{k-1} of each representative is exponentially distributed with death rate δ. It is well known that the death/birth dates $T_1,T_2,\ldots$ are the jump times of a Poisson process $N_i(t)$, which we will use in the following as a central modeling tool.

In the present paper, dynasty wealth is the main object of interest. By $A_i(t)$ we denote the wealth level of the dynasty i at time t. The initial wealth level is given by a fixed constant a_i. During the lifetime of an individual, dynasty wealth is the individual's wealth. Hence, as explained in Section 2.1, dynasty wealth evolves as described in (2).

If at time T_k the representative of the dynasty dies, her wealth level upon death is given by $A_i(T_k-)$.[4] This wealth is bequeathed to the newborn representative subject to bequest taxation. This taxation scheme consists of a tax exemption threshold Φ, which describes the amount of wealth free of tax, as well as a constant tax rate τ_B, which describes the rate, with which wealth exceeding the threshold is taxed. Hence, the newborn representative receives

$$A_i(T_k) = \begin{cases} A_i(T_k-) & \text{if } A_i(T_k-) \leq \Phi \\ \Phi + (1-\tau_B)(A_i(T_k-) - \Phi) & \text{if } A_i(T_k-) > \Phi \end{cases} \tag{5}$$

as initial wealth. Equivalently, we can state that the newborn representative's initial wealth is the predecessor's wealth minus the bequest taxes, which are given by $\mathbb{I}_{\{A_i(T_k-)>\Phi\}}\tau_B(A_i(T_k-)-\Phi)$.[5]

In mathematical terms, we can now describe the evolution of dynasty wealth through a stochastic differential equation driven by the Poisson process $N_i(t)$. Namely, the wealth of the dynasty satisfies

$$\mathrm{d}A_i(t) = z\,(A_i(t) + W)\,\mathrm{d}t - \mathbb{I}_{\{A_i(t-)>\Phi\}}\tau_B(A_i(t-) - \Phi)\mathrm{d}N_i(t). \tag{6}$$

The drift $z(A_i(t) + W)$ of this process exactly describes the deterministic evolution during the lifetime of an individual as described in Section 2.1. The process $N_i(t)$ jumps if and only if the current dynasty representative dies. At this time the dynasty's wealth is reduced by the bequest taxes derived before.

3. Dynasty wealth: Important properties

In this section, we lay the foundations for analyzing wealth inequality. This includes two preliminary results describing relevant properties of the dynasty wealth process as well as an explicit characterization of the solution shedding light on the different effects of the bequest taxation scheme.

As a first step, we show that the stochastic differential equation admits a unique solution for which the second and, as a result, also the first moment are finite. Whereas the existence and uniqueness result shows that our model is mathematically sound, the existence of the first and second moment are crucial for the further analysis of wealth inequality as it implies that the coefficient of variation is well-defined. Moreover, we prove that the process is a homogeneous strong Markov process, which means that the future evolution of dynasty wealth only depends on the current wealth level and not on the past evolution. Finally, we also obtain that our process has a certain structure, namely it is a piecewise-deterministic Markov process. This means that in between jump times, it evolves deterministically according to an ordinary differential equation and at random times it jumps to a certain wealth level.

[4] We use the notation T_k- to highlight the point in time *an instant* before a jump of the Poisson process. Accordingly, we here refer to the wealth level an instant before the individual's death.

[5] We introduce the indicator function $\mathbb{I}_{\{A_i(T_k-)>\Phi\}}$ which equals 1 if the condition attached, i.e., $A_i(T_k-) > \Phi$, is fulfilled and 0 otherwise.

Lemma 3.1. *The stochastic differential equation* (6) *admits a unique solution* $(A_i(t))_{t\geq 0}$*, which satisfies*

$$\mathbb{E}\left[\sup_{t\in[0,T]} A_i(t)^2\right] < \infty \quad \textit{for all } T > 0.$$

Moreover, $(A_i(t))_{t\geq 0}$ *is a homogeneous piecewise-deterministic Markov process that satisfies the strong Markov property.*

As a second preliminary result we collect the observation that if dynasty wealth exceeded the tax exemption threshold at some point in time, it will do so for all future times. The intuition behind this result is that during the lifetime of an individual wealth is increasing, and, upon death the tax exemption threshold ensures that the individual's wealth will be at least Φ (see (5)).

Lemma 3.2. *If* $A_i(t) \geq \Phi$*, then the wealth level* $A_i(s)$ *will never fall below* Φ *for all* $s > t$*.*

As long as $A_i(t) \leq \Phi$, then the jump height in the stochastic differential equation (6) is 0, which means that in this case (6) is indeed an ordinary differential equation that reads

$$\mathrm{d}A_i(t) = z(A_i(t) + W).$$

Its solution is given by $A_i(t) = (a_i + W)e^{zt} - W$ as long as t is given such that $A_i(t) \leq \Phi$. As this solution is strictly increasing, we compute the time at which $A_i(t) = \Phi$ as

$$t_\Phi(a_i) = \frac{1}{z}\ln\left(\frac{\Phi + W}{a_i + W}\right).$$

Hence, we obtain that for $a_i \leq \Phi$ the process $(A_i(t))_{t\geq 0}$ satisfies

$$A_i(t) = (a_i + W)e^{zt} - W, \quad \text{for all } t \in [0, t_\Phi(a_i)]. \tag{7}$$

Once the process reaches Φ, i.e., at the time t such that $A_i(t) = \Phi$, the process becomes stochastic. However, due to the Markov property, we know that the evolution of this process starting from time t with $A_i(t) = \Phi$ is stochastically the same as the evolution of the process started in time 0 with initial wealth level Φ. Hence, in the following we can separate the analysis for the process lying below Φ and then for the process with initial wealth level $a_i \geq \Phi$.

So far, we described the explicit solution for (6) whenever $a_i < \Phi$ and $t \in [0, t_\Phi(a_i)]$. The next result describes the solution for initial wealth levels $a_i \geq \Phi$:

Theorem 3.3 (explicit solution)**.** *Assume that* $a_i \geq \Phi$*. Then the wealth process of the dynasty satisfies*

$$A_i(t) = (a_i + W)e^{zt}(1-\tau_B)^{N_i(t)} - W + \tau_B(\Phi + W)\sum_{k=1}^{N_i(t)} e^{z(t-T_k)}(1-\tau_B)^{N_i(t)-k}. \tag{8}$$

Given an initial wealth above the tax exemption threshold, our explicit solution for dynasty wealth at a point in time t can be characterized by two key parts. The first part describes wealth in a hypothetical regime without a tax exemption threshold. In this setting, initial wealth, added the present value of net labor income, grows at rate z and is reduced to a share $1-\tau_B$ of previous wealth for every of the $N_i(t)$ deaths. The second part represents the effect of the tax exemption threshold. This can be interpreted as a hypothetical refund on the taxation of the $N_i(t)$ deaths from the first part. This refund, i.e., a share τ_B of $\Phi + W$ is again subject to a) wealth growth between the death date of every individual and today t and b) taxation such that a share $1-\tau_B$ remains with the dynasty after every individual death. Hence, not only the number of deaths is relevant (first part) for the dynastic wealth, but also the exact times at which a dynasty representative dies (second part).

4. Wealth inequality

In this section, we turn to the analysis of wealth inequality. For this, we use the coefficient of variation as a relative measure of inequality for two reasons: First, it allows to compare inequality across different populations (cf. Yntema (1933)), which in turn enables us to assess the impact of different bequest taxation schemes on wealth inequality. Second, it is commonly used when analyzing inheritance-driven wealth inequality (Bönke et al., 2017; Bossmann et al., 2007; Wolff, 2002; Wolff and Gittleman, 2014).

The central result in this section is an explicit representation of the coefficient of variation given a continuous distribution of initial wealth levels (cf. Theorem 4.3). For this, we recall that the coefficient of variation is defined as

$$\mathrm{CV}(A_i(t)) = \frac{\sqrt{\mathrm{Var}(A_i(t))}}{\mathbb{E}[A_i(t)]}.$$

Hence, our derivation of the coefficient will consist of two steps: First, we compute the first and second moment of $A_i(t)$ for all $t \geq 0$. Then we will use these expressions to arrive at the desired explicit representation.

We introduce the following shorthand notation

$$\eta(a_i, t) = \mathbb{E}\left[A_i(t)|A_i(0) = a_i\right] \quad \text{and} \quad \zeta(a_i, t) = \mathbb{E}\left[A_i(t)^2|A_i(0) = a_i\right]$$

for the first and second moment of dynasty wealth given the fixed initial wealth level a_i. By Lemma 3.1 both quantities are finite. Moreover, given the explicit solution for $a_i < \Phi$ and $t \in [0, t_\Phi(a_i)]$ in (7) as well as using the Markov property, we obtain for $a_i < \Phi$ that

$$\eta(a_i, t) = \begin{cases} (a_i + W)e^{zt} - W & \text{if } t \leq t_\Phi(a_i) \\ \eta(\Phi, t - t_\Phi(a_i)) & \text{if } t > t_\Phi(a_i) \end{cases}$$

$$\zeta(a_i, t) = \begin{cases} ((a_i + W)e^{zt} - W)^2 & \text{if } t \leq t_\Phi(a_i) \\ \zeta(\Phi, t - t_\Phi(a_i)) & \text{if } t > t_\Phi(a_i). \end{cases}$$

Hence, it suffices to compute the first and second moment for an initial wealth level $a_i \geq \Phi$. As a first step we prove, using tools from stochastic analysis (e.g., Itô's Lemma, Fubini's

Theorem), that the first and second moment solve the following ordinary differential equations:

Lemma 4.1. *Assume that $a_i \geq \Phi$. The first and second moment satisfy the following ordinary differential equations*

$$\dot{\eta}(a_i,t) = (z - \tau_B\delta)\eta(a_i,t) + zW + \delta\tau_B\Phi \tag{9}$$

$$\dot{\zeta}(a_i,t) = (2z + \delta(\tau_B^2 - 2\tau_B))\zeta(a_i,t) + (2zW + 2\delta(1-\tau_B)\tau_B\Phi)\eta(a_i,t) + \delta\tau_B^2\Phi^2 \tag{10}$$

subject to the initial conditions $\eta(a_i,0) = a_i$ and $\zeta(a_i,0) = a_i^2$.

In a next step, we solve these ordinary differential equations and arrive at the following result.

Theorem 4.2. *Assume that $a_i \geq \Phi$ and $z \neq \delta\tau_B(1-\tau_B)$[6]. Let us write*

$$\begin{aligned}
\alpha &= z - \tau_B\delta \\
\beta &= 2zW + 2\delta(1-\tau_B)\tau_B\Phi \\
\gamma &= 2z + \delta(\tau_B^2 - 2\tau_B).
\end{aligned}$$

Then the first and second moment satisfy

$$\begin{aligned}
\eta(a_i,t) &= a_i e^{\alpha t} + (zW + \delta\tau_B\Phi)\left(\mathbb{I}_{\{\alpha\neq 0\}}\frac{1}{\alpha}(e^{\alpha t}-1) + \mathbb{I}_{\{\alpha=0\}}t\right) \\
\zeta(a_i,t) &= \frac{a_i\beta}{\alpha-\gamma}(e^{\alpha t} - e^{\gamma t}) + a_i^2 e^{\gamma t} + \frac{\beta(zW+\delta\tau_B\Phi)}{\alpha-\gamma}\left(\mathbb{I}_{\{\alpha\neq 0\}}\frac{1}{\alpha}(e^{\alpha t}-1) + \mathbb{I}_{\{\alpha=0\}}t\right) \\
&\quad + \left(\delta\tau_B^2\Phi^2 - \frac{\beta(zW+\delta\tau_B\Phi)}{\alpha-\gamma}\right)\left(\mathbb{I}_{\{\gamma\neq 0\}}\frac{1}{\gamma}(e^{\gamma t}-1) + \mathbb{I}_{\{\gamma=0\}}t\right)
\end{aligned}$$

respectively.

Finally, using the law of total expectation, the law of total variance as well as some results regarding conditional expectations, we arrive at the desired representation of the coefficient of variation:

Theorem 4.3. *For an initial wealth distribution with probability density function $f(a_i)$ such that*

$$\mathbb{E}\left[A_i(0)^2\right] = \int_0^\infty f(a_i)a_i^2 \mathrm{d}a_i < \infty$$

the coefficient of variation is given by

$$\mathrm{CV}(A_i(t)) = \frac{\sqrt{\int_0^\infty f(a_i)\zeta(a_i,t)\mathrm{d}a_i - \left(\int_0^\infty f(a_i)\eta(a_i,t)\mathrm{d}a_i\right)^2}}{\int_0^\infty f(a_i)\eta(a_i,t)\mathrm{d}a_i}.$$

[6] The assumption $z \neq \delta\tau_B(1-\tau_B)$ is only a technical condition. It is possible to obtain an explicit representation also for the case $z = \delta\tau_B(1-\tau_B)$ using the same method. However, this condition describes a special parameter constellation that cannot occur for realistic parameter choices in our numerical illustration. Hence, we do not cover this case here.

5. Numerics

In this section, we project the evolution of wealth inequality in Germany under various bequest tax rates and tax exemption thresholds. We use wealth data from the German Socio-Economic Panel (Goebel et al., 2019), henceforth GSOEP, first to calibrate the individual time preference rate and second, to have a suitable initial distribution of wealth levels for the projection.

The wealth data from the GSOEP is available for four waves[7] (2002, 2007, 2012 and 2017). Focusing on private[8] wealth data 'pwealth', we concentrate on individual net overall wealth 'w0111a' for individuals participating in all four waves. This leaves us with 7061 individuals. Two individuals violating the natural borrowing constraint are excluded. The net overall wealth is characterized as the addition of all assets minus all debts. We focus on net overall wealth, as this ensures consistent documentation of wealth changes. Due to false or non-observed responses, data are typically edited and imputed. We choose private wealth based on the first imputation method, which is indicated by the letter 'a' inside of 'w0111a'.

We use the following values for our exogenous parameters which are reported in Table 1: We assume an interest rate of 5% and take the average life expectancy of Germany, i.e., roughly 81 years (The World Bank, 2024). Since Germany abolished wealth taxation in 1996 (Drometer et al., 2018), we set wealth tax τ_a equal to 0. According to the German Statistical Office (2023b) the average income in 2017 is approximately EUR 34963 accompanied by an average income taxation of approximately 21%. Lastly, we set risk aversion to 0.5. However, as risk aversion acts as a scaling parameter on $\hat{\rho}$ and z, respectively, this choice changes the calibrated values, but not the results.

Parameters	r	δ	τ_a	w	τ_w	σ
Value	0.05	1/81	0	34963	0.21	0.5

Table 1: Exogenous parameters

For the calibration, we compute for each remaining individual the individual time preference rate using (4). Averaging we arrive at the value in Table 2. Using this, we compute z according to (3).

Parameters	$\hat{\rho}$	z
Value	0.03577	0.00376

Table 2: Calibrated parameters

For analyzing different tax policies, we first introduce a baseline scenario. Considering 2017, the tax exemption threshold is based on judicial law (Bundesministerium der Justiz, 1997, §§16, 19) and equals EUR 400000. Moreover, taxation of bequests above the threshold is

[7] Unfortunately, we do not have yearly data as wealth is not a mandatory part of the household questionnaire.

[8] The difference to a household is that private data is individual-specific as opposed to the former, which accounts for all members of a household.

subject to a progressive scheme. For our analysis, we compute a flat tax rate by calculating the rate of paid bequest taxes of children in Germany for 2017 to the total amount of taxable bequests. It equals approximately 18.5% (German Statistical Office, 2023a). We summarize the baseline case in Table 3.

Parameters	τ_B	Φ
Value	0.185	400000

Table 3: Baseline scenario parameters

Based on Tables 1, 2 and 3, we, ceteris paribus, project German wealth inequality for a) varying tax exemption thresholds and assuming a fixed bequest tax rate of 18.5% and b) varying bequest tax rates and assuming a fixed tax exemption threshold of EUR 400000. The results are illustrated in Figures 1 and 2, respectively.

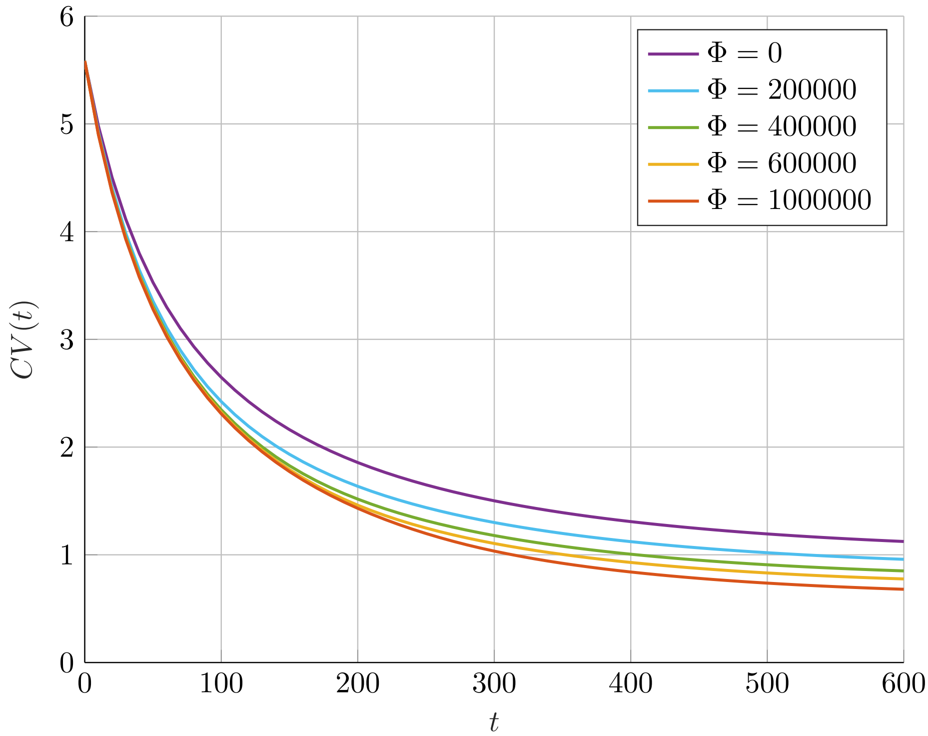

Figure 1: CV over time for different tax exemption thresholds Φ

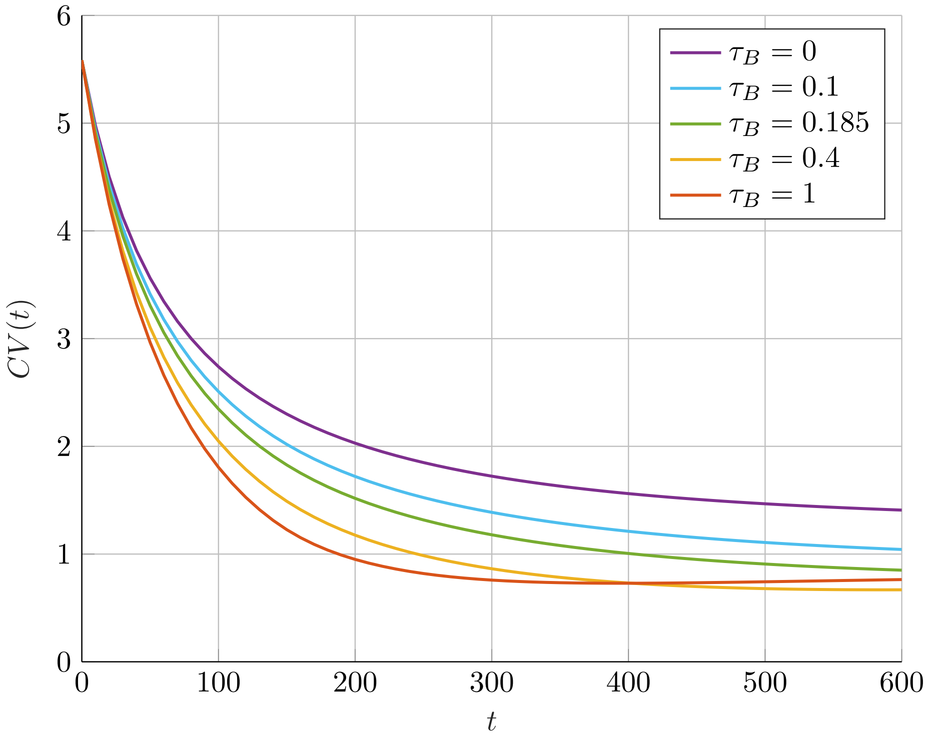

Figure 2: CV over time for different bequest tax rates τ_B

We observe in Figures 1 and 2 that initially wealth inequality is decreasing over time for all choices of bequest tax rates and tax exemption thresholds. Maintaining a fixed bequest tax rate, Figure 1 illustrates that a higher tax exemption threshold consistently corresponds to a reduced wealth inequality across all periods. This is because a higher tax exemption threshold, allows less affluent dynasties to transfer more wealth tax-free across generations upon death increasing expected dynasty wealth.

Focusing on Figure 2, if we keep the tax exemption threshold fixed, a higher bequest tax rate leads to a pronounced initial decline in wealth inequality. This is because higher wealth at death is reduced by a larger amount for higher bequest tax rates. In the long run, however, higher bequest taxation yields higher wealth inequality compared to some of the lower bequest tax rates. In particular, we observe that inequality is increasing in the long run for the extreme case of full taxation of wealth above the threshold. Here, it becomes evident that inequality is also driven by the differences in lifetime of the individuals.

As a robustness check, we determine the evolution of the Gini coefficients for different tax exemption thresholds and different bequest tax rates to compare them to the CV. The Gini coefficient reads

$$G(A_i(t)) = \frac{\mathbb{E}[|A_i^1(t) - A_i^2(t)|]}{2\mathbb{E}[A^1(t)]}$$

where $A_i^1(t)$ and $A_i^2(t)$ are independent copies of the dynasty wealth process $A_i(t)$ (Yitzhaki and Schechtman, 2013, Chapter 2). Whereas the expectation in the denominator can be computed as in Theorem 4.3, we do not have a closed-form expression for the expectation in the numerator. To compute this expectation, we employ the Monte Carlo method. By virtue of the explicit solution from Theorem 3.3, it suffices to simulate the underlying homogeneous Poisson process. One major drawback of this approach are the high computational costs to reduce the inherent simulation error. Accordingly, we simulate five million paths of the Poisson process. The results are illustrated in Figure 3.

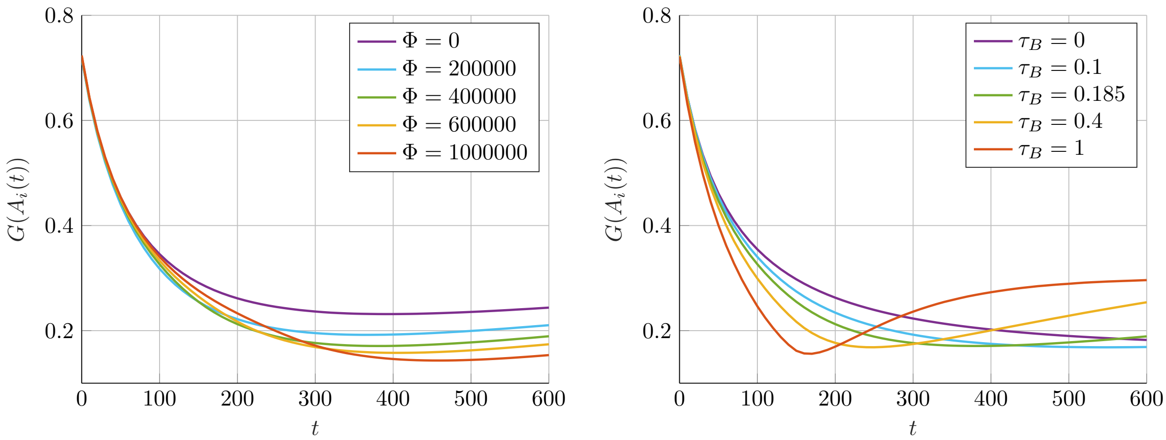

Figure 3: Gini over time for different tax exemption thresholds Φ (left) and different bequest tax rates τ_B (right)

The central tendencies present for the CV are also observed for the Gini coefficient in Figure 3. Namely, we see that inequality is initially decreasing and that higher bequest tax rates lead to larger inequality in the long run. The only difference is that higher tax exemption thresholds do not lead to a consistent reduction of wealth inequality in all periods. Instead, the effect only holds in the long run. The central reason is that the Gini coefficient is sensitive to small changes in the middle of the distribution (Davies et al., 2017) and in the beginning of the time period the modification of the tax exemption threshold mainly affects wealth levels in the middle of the distribution.

In the next step, we compare variations on the tax rate and the tax exemption threshold that are revenue-neutral (i.e., they yield the same average tax revenue over a hundred years). Analogously to above[9], we obtain that the average taxes paid until time t are given by $\mathbb{E}[S_i(t)] = \int s(a_i,t) f(a_i) \mathrm{d}a_i$, where for $a_i < \Phi$ we have

$$s(a_i,t) = \begin{cases} 0 & \text{if } t \leq t_\Phi(a_i) \\ s(\Phi, t - t_\Phi(a_i)) & \text{if } t > t_\Phi(a_i) \end{cases} \tag{11}$$

and for $a_i \geq \Phi$ we have

$$s(a_i,t) = \begin{cases} \delta\tau_B \left(\left(\frac{a_i}{\alpha} + \frac{zW+\delta\tau_B\Phi}{\alpha^2} \right) (e^{\alpha t} - 1) - \left(\frac{zW+\delta\tau_B\Phi}{\alpha} + \Phi \right) t \right) & \text{if } \alpha \neq 0 \\ \delta\tau_B \left((a_i - \Phi)t + \frac{zW+\delta\tau_B\Phi}{2} t^2 \right) & \text{if } \alpha = 0. \end{cases} \tag{12}$$

Firstly, we determine the average tax revenue based on our baseline scenario from above (cf. Table 3) using Equations (11) and (12). We now consider four realistic alternative bequest tax

[9]For details, we refer to Appendix A.7.

rates. Utilizing Equations (11) and (12), we determine the corresponding revenue-neutral tax exemption threshold. The resulting five bequest taxation policies are described in Table 4. We observe that a lower bequest tax rate coincides with a lower tax exemption threshold and vice versa. This is because to ensure revenue neutrality, a lower tax rate requires more taxpayers, which necessitates a lower tax exemption threshold, and vice versa.

τ_B	0.085	0.135	0.185	0.235	0.285
Φ	147298	282609	400000	510830	621023

Table 4: Five revenue-neutral bequest taxation schemes

Based on Tables 1 and 2, we, ceteris paribus, project tax revenue and wealth inequality over a hundred years for the bequest taxation policies from Table 4. The results are illustrated in Figures 4 and 5.

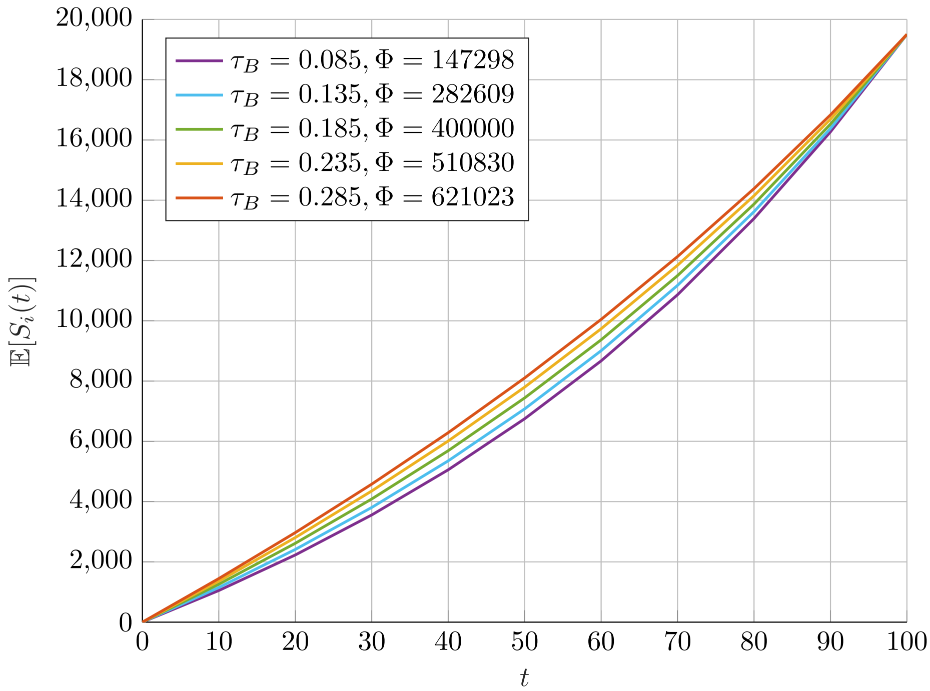

Figure 4: Average tax revenue over time for the bequest taxation schemes from Table 4

Figure 4 demonstrates that a lower bequest tax rate, coupled with a lower tax exemption threshold, initially results in lower tax revenues. However, over time, we observe a more emphasized surge in tax revenues compared to scenarios with higher bequest tax rates and tax exemption thresholds. The initially low tax revenue can be attributed to the reduced tax rate, which cannot be compensated by the larger tax base resulting from a lower tax exemption threshold. However, the low tax rate leads to larger wealth levels and, due to the exponential growth of wealth, to larger tax revenues later.

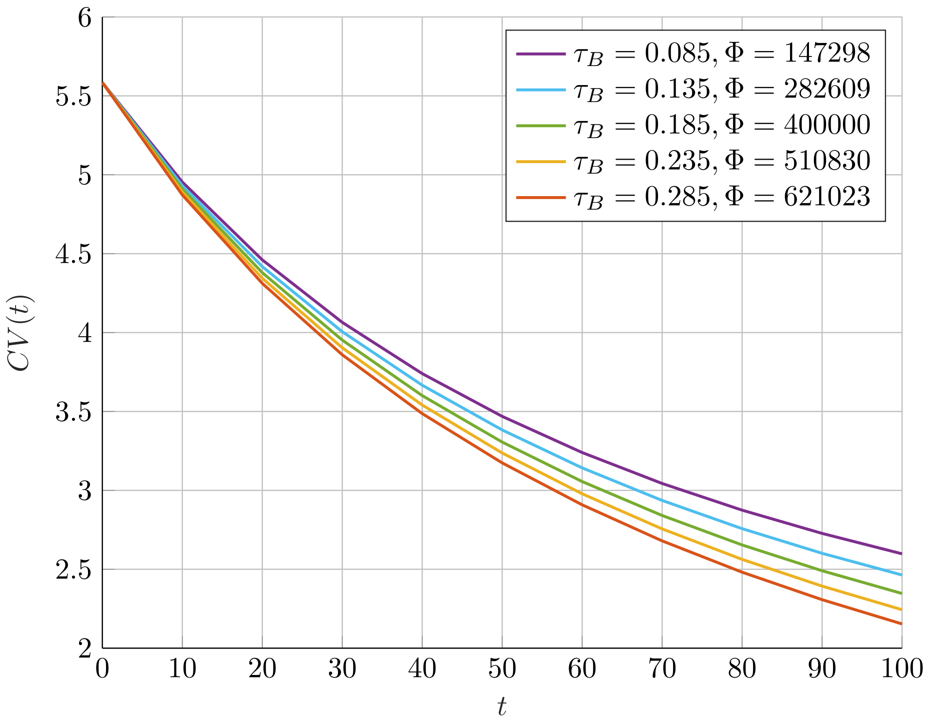

Figure 5: CV over time for the bequest taxation schemes from Table 4

Focusing on Figure 5, we observe that a high bequest tax rate paired with a high tax exemption threshold leads to lower wealth inequality over the entire considered time period. The inequality reduction is due to two effects: On the one hand, poor dynasties can accumulate wealth longer until they reach the tax exemption threshold. On the other hand, rich dynasties have to pay larger amounts of taxes and hence are subject to a larger reduction in wealth upon death. Both effects compress the wealth distribution resulting in lower wealth inequality.

6. Conclusion

This paper analyzes the effect of the bequest taxation on wealth inequality. We use a dynastic setting and model the taxation scheme explicitly, which leads to a stochastic differential equation for dynasty wealth. We solve the latter analytically and derive the coefficient of variation as a measure of wealth inequality in closed form. We project empirical wealth inequality in Germany using data from the GSOEP. Most notably, our results indicate that among all revenue-neutral taxation policies, a high tax exemption threshold paired with a high bequest tax rate achieves the lowest wealth inequality.

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References

Abel, Andrew B. (1985). “Precautionary Saving and Accidental Bequests”. In: *American Economic Review* 75.4, pp. 777–791.

Andreoni, James (1989). “Giving with Impure Altruism: Applications to Charity and Ricardian Equivalence”. In: *Journal of Political Economy* 97.6, pp. 1447–1458.

Beckert, Jens and H. Lukas R. Arndt (2017). “Verdient-Unverdient. Der öffentliche Diskurs um die Erbschaftssteuer in Deutschland und Österreich”. In: *Berliner Journal für Soziologie* 27.2, pp. 271–291.

Benhabib, Jess, Alberto Bisin, and Shenghao Zhu (2016). “The Distribution of Wealth in The Blanchard-Yaari Model”. In: *Macroeconomic Dynamics* 20.2, pp. 466–481.

Beznoska, Martin, Tobias Hentze, and Maximilian Stockhausen (2020). “The inheritance and gift tax in Germany: Reform potentials for tax revenue, efficiency and distribution”. In: *Public Sector Economics* 44.3, pp. 385–417.

Birkner, Matthias, Niklas Scheuer, and Klaus Wälde (2023). “The dynamics of Pareto distributed wealth in a small open economy”. In: *Economic Theory* 76.2, pp. 607–644.

Blanchard, Olivier J. (1985). “Debt, Deficits, and Finite Horizons”. In: *Journal of Political Economy* 93.2, pp. 223–247.

Bönke, Timm, Marten v. Werder, and Christian Westermeier (2017). “How inheritances shape wealth distributions: An international comparison”. In: *Economics Letters* 159, pp. 217–220.

Boserup, Simon H., Wojciech Kopczuk, and Claus T. Kreiner (2016). “The Role of Bequests in Shaping Wealth Inequality: Evidence from Danish Wealth Records”. In: *American Economic Review* 106.5, pp. 656–661.

Bossmann, Martin, Christian Kleiber, and Klaus Wälde (2007). “Bequests, taxation and the distribution of wealth in a general equilibrium model”. In: *Journal of Public Economics* 91, pp. 1247–1271.

Braun, Reiner (2015). *Erben in Deutschland 2015-2024: Volumen, Verteilung und Verwendung.* Studie des Deutschen Institut für Altersvorsorge.

Bundesministerium der Justiz (1997). “Erbschaftsteuer- und Schenkungsteuergesetz (ErbStG)”. In: *Bundesgesetzblatt.* URL: `https://www.gesetze-im-internet.de/erbstg_1974/`.

Davies, James B., Nicole M. Fortin, and Thomas Lemieux (2017). “Wealth inequality: Theory, measurement and decomposition”. In: *Canadian Journal of Economics/Revue canadienne d'économique* 50.5, pp. 1224–1261.

Davis, M.H.A. (1993). *Markov Models and Optimization.* Vol. 49. Monographs on Statistics and Applied Probability. Chapman & Hall.

De Nardi, Mariacristina (2004). “Wealth Inequality and Intergenerational Links”. In: *The Review of Economic Studies* 71.3, pp. 743–768.

De Nardi, Mariacristina and Giulio Fella (2017). “Saving and wealth inequality”. In: *Review of Economic Dynamics* 26, pp. 280–300.

De Nardi, Mariacristina and Fang Yang (2016). “Wealth inequality, family background, and estate taxation”. In: *Journal of Monetary Economics* 77, pp. 130–145.

Drometer, Marcus et al. (2018). “Wealth and Inheritance Taxation: An Overview and Country Comparison”. In: *ifo DICE Report* 16.2, pp. 45–54.

Dym, Harry (2013). *Linear Algebra in Action*. 2nd. Vol. 78. Graduate Studies in Mathematics. American Mathematical Society.

Elinder, Mikael, Oscar Erixson, and Daniel Waldenström (2018). “Inheritance and wealth inequality: Evidence from population registers”. In: *Journal of Public Economics* 165, pp. 17–30.

Escobar, Sebastian, Henry Ohlsson, and Håkan Selin (2023). “Giving to the children or the taxman?: Lessons from a Swedish inheritance tax loophole”. In: *European Economic Review* 153, p. 104382.

Fleurbaey, Marc et al. (2022). “Premature deaths, accidental bequests, and fairness”. In: *The Scandinavian Journal of Economics* 124.3, pp. 709–743.

Freund, Caroline and Sarah Oliver (2016). “The Origins of the Superrich: The Billionaire Characteristics Database”. In: Peterson Institute for International Economics Working Paper No. 16-1, doi:10.2139/ssrn.2731353.

García, Miguel Angel and Richard J. Griego (1994). “An Elementary Theory of Stochastic Differential Equations Driven by A Poisson Process”. In: *Communications in Statistics - Stochastic Models* 10.2, pp. 335–363.

German Statistical Office (2023a). *Erbschaft- und Schenkungsteuerstatistik*. URL: `https://www-genesis.destatis.de/genesis//online?operation=table&code=73611-0001&bypass=true&levelindex=0&levelid=1629807760175#abreadcrumb`.

— (2023b). *Lohn- und Einkommensteuerstatistik*. URL: `https://www-genesis.destatis.de/genesis//online?operation=table&code=73111-0001&bypass=true&levelindex=0&levelid=1710771445549`.

Goebel, Jan et al. (2019). “The German Socio-Economic Panel (SOEP)”. In: *Jahrbücher für Nationalökonomie und Statistik* 239.2, pp. 345–360.

Hurd, Michael D. (1987). “Savings of the Elderly and Desired Bequests”. In: *American Economic Review* 77.3, pp. 298–312.

Kasa, Kenneth and Xiaowen Lei (2018). “Risk, uncertainty, and the dynamics of inequality”. In: *Journal of Monetary Economics* 94, pp. 60–78.

Kaymak, Boriş and Markus Poschke (2016). “The evolution of wealth inequality over half a century: The role of taxes, transfers and technology”. In: *Journal of Monetary Economics* 77, pp. 1–25.

Klenke, Achim (2014). *Probability Theory: A Comprehensive Course*. 2nd. Springer-Verlag.

Nekoei, Arash and David Seim (2023). “How do Inheritances Shape Wealth Inequality? Theory and Evidence from Sweden”. In: *The Review of Economic Studies* 90.1, pp. 463–498.

Ohlsson, Henry, Jesper Roine, and Daniel Waldenström (2020). “Inherited Wealth over the Path of Development: Sweden, 1810–2016”. In: *Journal of the European Economic Association* 18.3, pp. 1123–1157.

Piketty, Thomas and Gabriel Zucman (2015). “Chapter 15 - Wealth and Inheritance in the Long Run”. In: *Handbook of Income Distribution*. Ed. by Anthony B. Atkinson and François Bourguignon. Vol. 2. Handbook of Income Distribution. Elsevier, pp. 1303–1368.

Platen, Eckhard and Nicola Bruti-Liberati (2010). *Numerical Solution of Stochastic Differential Equations with Jumps in Finance*. Vol. 64. Stochastic Modelling and Applied Probability. Springer-Verlag.
Privault, Nicolas (2022). *Introduction to Stochastic Finance with Market Examples*. CRC Press.
Rényi, A. (2007). *Probability Theory*. Dover Publications, Inc.
The World Bank (2024). *World Development Indicators*. URL: `https://databank.worldbank.org/reports.aspx?dsid=2&series=SP.DYN.LE00.IN`.
Wan, Jing and Shenghao Zhu (2019). "Bequests, estate taxes, and wealth distributions". In: *Economic Theory* 67, pp. 179–210.
Wilhelm, Mark O. (1996). "Bequest Behavior and the Effect of Heirs' Earnings: Testing the Altruistic Model of Bequests". In: *American Economic Review* 86.4, pp. 874–892.
Wolff, Edward N. (2002). "Inheritances and Wealth Inequality, 1989–1998". In: *American Economic Review* 92.2, pp. 260–264.
Wolff, Edward N. and Maury Gittleman (2014). "Inheritances and the distribution of wealth or whatever happened to the great inheritance boom?" In: *The Journal of Economic Inequality* 12, pp. 439–468.
Yang, Fang (2013). "Social security reform with impure intergenerational altruism". In: *Journal of Economic Dynamics and Control* 37.1, pp. 52–67.
Yitzhaki, Shlomo and Edna Schechtman (2013). *The Gini Methodology: A Primer on a Statistical Methodology*. Springer.
Yntema, Dwight B. (1933). "Measures of the Inequality in the Personal Distribution of Wealth or Income". In: *Journal of the American Statistical Association* 28.184, pp. 423–433.
Zhu, Shenghao (2019). "A Becker–Tomes model with investment risk". In: *Economic Theory* 67.4, pp. 951–981.

A. Appendix

A.1. Proof of Lemma 3.1

The stochastic differential equation (6) can be rewritten as

$$\mathrm{d}A_i(t) = f(t, A_i(t))\mathrm{d}t + g(t, A_i(t-))\mathrm{d}N_i(t)$$

where

$$f(t,a) = z(a+W) \quad \text{and} \quad g(t,a) = -\mathbb{I}_{\{a>\Phi\}}\tau_B(a-\Phi).$$

We note that $\mathbb{I}_{\{a>\Phi\}}\tau_B(a-\Phi) = \tau_B(\max\{a,\Phi\}-\Phi)$. Hence,

$$\begin{aligned}
&|f(t,a) - f(t,a')| + |g(t,a) - g(t,a')| \\
&= |z(a+W) - z(a'+W)| + |\tau_B(\max\{a,\Phi\}-\Phi) - \tau_B(\max\{a',\Phi\}-\Phi)| \\
&= z|a-a'| + \tau_B|\max\{a,\Phi\} - \max\{a',\Phi\}| \\
&\leq (z+\tau_B)|a-a'|
\end{aligned}$$

and

$$|f(t,a)|^2 + |g(t,a)|^2 = |z(a+W)|^2 + \left|\mathbb{I}_{\{a>\Phi\}}\tau_B(a-\Phi)\right|^2$$

$$
\begin{aligned}
&\leq za^2 + 2zW|a| + zW^2 + \tau_B a^2 + \tau_B \Phi^2 \\
&\leq za^2 + 2zWa^2 + 2zW + zW^2 + \tau_B a^2 + \tau_B \Phi^2, \\
&\leq (z + 2zW + \tau_B)a^2 + (2zW + zW^2 + \tau_B \Phi^2)
\end{aligned}
$$

which implies that there is a constant $K > 0$ such that

$$
\begin{aligned}
|f(t,a) - f(t,a')| + |g(t,a) - g(t,a')| &\leq K|a - a'| \\
|f(t,a)|^2 + |g(t,a)|^2 &\leq K^2(1 + a^2)
\end{aligned}
$$

hold for all $a, a' \in \mathbb{R}$ and all $t \geq 0$. Hence, we are in the situation of García and Griego (1994, Theorem 6.2) and Platen and Bruti-Liberati (2010, Theorem 1.9.3) and obtain that a unique solution of (6) exists and satisfies

$$
\mathbb{E}\left[\sup_{t \in [0,T]} A_i(t)^2\right] < \infty \quad \text{for all } \; T > 0.
$$

Moreover, the process $(A_i(t))_{t\geq 0}$ is a piecewise-deterministic Markov process in the sense of Davis (1993, Section 24) with the following local characteristics: The vector field is given as the vector field of the ordinary differential equation (2), the jump rate is $\lambda(A_i(t)) = \delta \mathbb{I}_{\{A_i(t) > \Phi\}}$ and the transition measure is given by

$$
Q(x, A) = \delta_{\Phi + (1-\tau_B)(x - \Phi)}(A),
$$

where $\delta_{\Phi+(1-\tau_B)(x-\Phi)}$ denotes the Dirac measure at $\Phi + (1-\tau_B)(x-\Phi)$. It is immediately verified that these local characteristics satisfy the standard conditions. Hence, by Theorem 25.5 in Davis (1993) we obtain that $(A_i(t))_{t\geq 0}$ is indeed a homogeneous strong Markov process.

A.2. Proof of Lemma 3.2

As the drift of the process $z(A_i(t) + W)$ is strictly positive for all $t > 0$ the process increases between jumps. Hence, it cannot fall below the level Φ between jumps. At jump times T_k the dynasty wealth remains unchanged if $A_i(T_k-) \leq \Phi$. Otherwise, we have

$$
A_i(T_k) = A_i(T_k-) - \tau_B(A_i(T_k-) - \Phi) = \Phi + (1 - \tau_B)(A_i(T_k-) - \Phi).
$$

As the second summand is non-negative, we immediately obtain that also upon jump times the wealth cannot fall below the level Φ.

A.3. Proof of Theorem 3.3

As a first step, we prove the following lemma, which contains a recursive solution of (6):

Lemma A.1. *The wealth process of the dynasty satisfies*

$$
A_i(t) = (a_i + W)e^{zt} - \tau_B \sum_{k=1}^{N_i(t)} \mathbb{I}_{\{A_i(T_k-) > \Phi\}}(A_i(T_k-) - \Phi)e^{z(t-T_k)} - W. \tag{13}
$$

Proof. It suffices to verify that (13) satisfies (6), i.e., to verify that between jump times the candidate solution (13) is differentiable with derivative $z(A_i(t)+W)$ and at jump times that process jumps down by $\mathbb{I}_{\{A_i(t-)>\Phi\}}\tau_B(A_i(t-)-\Phi)$.

If $\Delta N_i(t) = 0$[10], the candidate solution is differentiable in t and satisfies

$$\dot{A}_i(t) = z\cdot(a_i+W)\,e^{zt} - z\cdot\tau_B\sum_{k=1}^{N_i(t)}\mathbb{I}_{\{A_i(T_k-)>\Phi\}}\,(A_i(T_k-)-\Phi)\,e^{z(t-T_k)}.$$

Factoring out z and rewriting we obtain (recall the $-W$ in (13))

$$\dot{A}(t) = z(A_i(t)+W),$$

as desired. If $\Delta N_i(t)\neq 0$, then $t = T_{N_i(t)}$ and we obtain

$$\Delta A_i(t) = -\tau_B\mathbb{I}_{\{A_i(T_{N_i(t)}-)>\Phi\}}(A_i(T_{N_i(t)}-)-\Phi),$$

again as desired. $\square$

With this lemma at hand, we turn to the proof of Lemma 3.3: As $a_i\geq\Phi$, we immediately obtain by Lemma 3.2 that $A_i(t)\geq\Phi$ for all $t\geq 0$. Hence, (13) simplifies to

$$A_i(t) = (a_i+W)e^{zt} - \tau_B\sum_{k=1}^{N_i(t)}(A_i(T_k-)-\Phi)e^{z(t-T_k)} - W.$$

Let $(\Omega,\mathcal{A},\mathbb{P})$ be the abstract probability space that supports the Poisson process $N_i(t)$. We now prove by induction on n that for all $\omega\in\Omega$ and all $t\geq 0$ such that $N_i(t)\leq n$ the dynasty wealth process satisfies (8). The claim is clear for $n=0$. Let $n\geq 0$ be arbitrary and assume that for all $\omega\in\Omega$ and all $t\geq 0$ such that $N_i(t,\omega)\leq n$ the wealth process satisfies (8). Fix $\omega\in\Omega$ and $t\geq 0$ such that $N_i(t,\omega)=n+1$. For any $s<T_{n+1}(\omega)$ we have that $N_i(s,\omega)\leq n$. Hence, we obtain by the induction hypothesis that

$$A_i(T_k-) = (a_i+W)e^{zT_k}(1-\tau_B)^{k-1} + \tau_B(\Phi+W)\sum_{j=1}^{k-1}e^{z(T_k-T_j)}(1-\tau_B)^{k-1-j} - W$$

for all $1\leq k\leq n+1$. Plugging these expressions into our recursive representation (13), we obtain

$$\begin{aligned}
A_i(t) &= (a_i+W)e^{zt} - W\\
&\quad - \tau_B\sum_{k=1}^{n+1}e^{z(t-T_k)}(a_i+W)e^{zT_k}(1-\tau_B)^{k-1}\\
&\quad - \tau_B\sum_{k=1}^{n+1}e^{z(t-T_k)}\left(\tau_B(\Phi+W)\sum_{j=1}^{k-1}e^{z(T_k-T_j)}(1-\tau_B)^{k-1-j} - \Phi - W\right).
\end{aligned}$$

[10]The quantity $\Delta N_i(t)$ describes the instantaneous change of the Poisson process and is formally defined as $\Delta N_i(t) = N_i(t) - N_i(t-)$. As a Poisson process exhibits jumps of height 1, we either have $\Delta N_i(t) = 1$, indicating that a jump has taken place at time t, or $\Delta N_i(t) = 0$, indicating that no jump has happened at time t.

Combining the exponential terms and grouping together yields

$$\begin{aligned}A_i(t) &= (a_i + W)e^{zt}\left(1 - \tau_B \sum_{k=1}^{n+1}(1-\tau_B)^{k-1}\right) - W \\ &\quad - \tau_B^2(\Phi + W)\sum_{k=1}^{n+1}\sum_{j=1}^{k-1} e^{z(t-T_j)}(1-\tau_B)^{k-j-1} + \tau_B(\Phi+W)\sum_{k=1}^{n+1} e^{z(t-T_k)}.\end{aligned}$$

Changing the summation sequence in the third term we obtain

$$\begin{aligned}\tau_B^2(\Phi + W)\sum_{k=1}^{n+1}\sum_{j=1}^{k-1} e^{z(t-T_j)}(1-\tau_B)^{k-j-1} &= \tau_B^2(\Phi+W)\sum_{j=1}^{n+1}\sum_{k=j+1}^{n+1} e^{z(t-T_j)}(1-\tau_B)^{k-j-1} \\ &= \tau_B^2(\Phi+W)\sum_{j=1}^{n+1}\sum_{k=0}^{n+1-1-j} e^{z(t-T_j)}(1-\tau_B)^{k}.\end{aligned}$$

This and the rule for geometric sums yields

$$\begin{aligned}A_i(t) &= (a_i + W)e^{zt}\left(1 - \tau_B \frac{1-(1-\tau_B)^{n+1}}{\tau_B}\right) - W \\ &\quad - \sum_{j=1}^{n+1} \tau_B^2(\Phi+W)\frac{1-(1-\tau_B)^{n-j+1}}{\tau_B} e^{z(t-T_j)} + \tau_B(\Phi+W)\sum_{k=1}^{n+1} e^{z(t-T_k)} \\ &= (a_i+W)e^{zt}(1-\tau_B)^{n+1} - W \\ &\quad - \tau_B(\Phi+W)\sum_{j=1}^{n+1}\left(1-(1-\tau_B)^{n+1-j}\right)e^{z(t-T_j)} + \tau_B(\Phi+W)\sum_{k=1}^{n+1} e^{z(t-T_k)}.\end{aligned}$$

Combining the last two sums yields the desired result.

A.4. Proof of Lemma 4.1

For any random variable X we introduce the shorthand notation

$$\mathbb{E}_{a_i}[X] := \mathbb{E}[X|A_i(0) = a_i].$$

Since $a_i \geq \Phi$, we obtain by Lemma 3.2 that $A_i(t) \geq \Phi$ for all $t \geq 0$. Hence, the stochastic differential equation (6) simplifies to

$$\mathrm{d}A_i(t) = z(A_i(t) + W)\mathrm{d}t - \tau_B(A_i(t-) - \Phi)\mathrm{d}N_i(t).$$

Rewriting this stochastic differential equation in integral notation we get

$$A_i(t) = a_i + \int_0^t z(A_i(s) + W)\mathrm{d}s - \int_0^t \tau_B(A_i(s-) - \Phi)\mathrm{d}N_i(s).$$

Taking expectations and rewriting the expectation of the integral with respect to the Poisson process according to Proposition 20.10 in Privault (2022) (note that $\sup_{s\in[0,t]}|A_i(s)|$ is integrable by Lemma 3.1), we obtain

$$\begin{aligned}\mathbb{E}_{a_i}\left[A_i(t)\right] &= a_i + \mathbb{E}_{a_i}\left[\int_0^t z\left(A_i(s)+W\right)\mathrm{d}s\right] - \mathbb{E}_{a_i}\left[\int_0^t \tau_B(A_i(s-)-\Phi)\mathrm{d}N_i(s)\right] \\ &= a_i + \mathbb{E}_{a_i}\left[\int_0^t z\left(A_i(s)+W\right)\mathrm{d}s\right] - \delta\mathbb{E}_{a_i}\left[\int_0^t \tau_B(A_i(s-)-\Phi)\mathrm{d}s\right].\end{aligned}$$

Applying Fubini's Theorem (Klenke, 2014, Theorem 14.16) (again note that $\sup_{s\in[0,t]}|A_i(s)|$ is integrable by Lemma 3.1) yields

$$\mathbb{E}_{a_i}\left[A_i(t)\right] = a_i + \int_0^t \Big(z\left(\mathbb{E}_{a_i}[A_i(s)]+W\right) - \tau_B\delta\left(\mathbb{E}_{a_i}[A_i(s)]-\Phi\right)\Big)\mathrm{d}s.$$

Using our shorthand notation $\eta(a_i,t)$ and regrouping we obtain

$$\eta(a_i,t) = a_i + \int_0^t \Big((z-\tau_B\delta)\eta(a_i,s) + zW + \tau_B\delta\Phi\Big)\mathrm{d}s,$$

which is the integral notation of the ODE (9). Hence, we verified that $\eta(a_i,t)$ satisfies the ODE (9) subject to the initial condition $\eta(a_i,0)=a_i$.

Let us now turn to the second moment: Using Itô's lemma for Poisson processes (Privault, 2022, Proposition 20.14) for $f(x)=x^2$, we obtain using $f'(x)=2x$ that

$$\begin{aligned}\mathrm{d}(A_i(t))^2 &= \mathrm{d}f(A_i(t)) \\ &= f'(A_i(t))z\Big(A_i(t)+W\Big)\mathrm{d}t + \Bigg(f\Big(A_i(t-)-\tau_B(A_i(t-)-\Phi)\Big) - f(A_i(t-))\Bigg)\mathrm{d}N_i(t) \\ &= 2A_i(t)z\Big(A_i(t)+W\Big)\mathrm{d}t + \Bigg(\Big((1-\tau_B)A_i(t-)+\tau_B\Phi\Big)^2 - \Big(A_i(t-)\Big)^2\Bigg)\mathrm{d}N_i(t).\end{aligned}$$

Expanding the quadratic term and regrouping yields

$$\begin{aligned}\mathrm{d}(A_i(t))^2 = &\Big(2z(A_i(t))^2 + 2zWA_i(t)\Big)\mathrm{d}t \\ &+ \Bigg(((1-\tau_B)^2-1)(A_i(t-))^2 + 2(1-\tau_B)\tau_B\Phi A_i(t-) + \tau_B^2\Phi^2\Bigg)\mathrm{d}N_i(t).\end{aligned}$$

Rewriting this stochastic differential equation in integral form using that $(1-\tau_B)^2-1 = \tau_B^2 - 2\tau_B$ gives

$$\begin{aligned}(A_i(t))^2 = a_i^2 &+ \int_0^t \Big(2z(A_i(s))^2 + 2zWA_i(s)\Big)\mathrm{d}s \\ &+ \int_0^t \Big((\tau_B^2-2\tau_B)(A_i(s-))^2 + 2(1-\tau_B)\tau_B\Phi A_i(s-) + \tau_B^2\Phi^2\Big)\mathrm{d}N_i(s).\end{aligned}$$

Taking expectations we arrive at

$$\begin{aligned}\mathbb{E}_{a_i}\left[(A_i(t))^2\right] = a_i^2 &+ \mathbb{E}_{a_i}\left[\int_0^t \Big(2z(A_i(s))^2 + 2zWA_i(s)\Big)\mathrm{d}s\right] \\ &+ \mathbb{E}_{a_i}\left[\int_0^t \Big((\tau_B^2 - 2\tau_B)(A_i(s-))^2 + 2(1-\tau_B)\tau_B\Phi A_i(s-) + \tau_B^2\Phi^2\Big)\mathrm{d}N_i(s)\right].\end{aligned}$$

Again we rewrite the integral with respect to the Poisson process according to Proposition 20.10 in Privault (2022) (note that $\sup_{s\in[0,t]}|A_i(s)|$ and $\sup_{s\in[0,t]} A_i(s)^2$ are integrable by Lemma 3.1)

$$\begin{aligned}\mathbb{E}_{a_i}\left[(A_i(t))^2\right] = a_i^2 &+ \mathbb{E}_{a_i}\left[\int_0^t \Big(2z(A_i(s))^2 + 2zWA_i(s)\Big)\mathrm{d}s\right] \\ &+ \delta\mathbb{E}_{a_i}\left[\int_0^t \Big((\tau_B^2 - 2\tau_B)(A_i(s-))^2 + 2(1-\tau_B)\tau_B\Phi A_i(s-) + \tau_B^2\Phi^2\Big)\mathrm{d}s\right].\end{aligned}$$

Since $\sup_{s\in[0,t]}|A_i(s)|$ and $\sup_{s\in[0,t]} A_i(s)^2$ are integrable by Lemma 3.1, we obtain using Fubini's Theorem (Klenke, 2014, Theorem 14.16)

$$\begin{aligned}\mathbb{E}_{a_i}\left[(A_i(t))^2\right] = a_i^2 &+ \int_0^t \Big(2z\mathbb{E}_{a_i}\left[A_i(s)^2\right] + 2zW\mathbb{E}_{a_i}[A_i(s)]\Big)\mathrm{d}s \\ &+ \int_0^t \delta\Big((\tau_B^2 - 2\tau_B)\mathbb{E}_{a_i}\left[((A_i(s))^2\right] + 2(1-\tau_B)\tau_B\Phi\mathbb{E}_{a_i}\left[A_i(s)\right] + \tau_B^2\Phi^2\Big)\mathrm{d}s.\end{aligned}$$

Regrouping we arrive at

$$\begin{aligned}\mathbb{E}_{a_i}\left[(A_i(t))^2\right] = a_i^2 + \int_0^t \Bigg(&\Big(2z + \delta(\tau_B^2 - 2\tau_B)\Big)\mathbb{E}_{a_i}\left[(A_i(s))^2\right] \\ &+ \Big(2zW + 2\delta(1-\tau_B)\tau_B\Phi\Big)\mathbb{E}_{a_i}\left[A_i(s)\right] + \delta\tau_B^2\Phi^2\Bigg)\mathrm{d}s.\end{aligned}$$

Using our shorthand notations $\eta(a_i,t)$ and $\zeta(a_i,t)$ we obtain the integral equation

$$\begin{aligned}\zeta(a_i,t) = a_i^2 + \int_0^t \Bigg(&\Big(2z + \delta(\tau_B^2 - 2\tau_B)\Big)\zeta(a_i,s) \\ &+ \Big(2zW + 2\delta(1-\tau_B)\tau_B\Phi\Big)\eta(a_i,s) + \delta\tau_B^2\Phi^2\Bigg)\mathrm{d}s.\end{aligned}$$

Putting this last equation into differential notation we obtain that $\zeta(a_i,t)$ satisfies the ODE (10) subject to the initial condition $\zeta(a_i,0) = a_i^2$.

A.5. Proof of Theorem 4.2

Introducing the matrix

$$M = \begin{pmatrix}\alpha & 0 \\ \beta & \gamma\end{pmatrix},$$

we can rewrite the ODEs from Lemma 4.1 in matrix form

$$\begin{pmatrix} \dot{\eta}(a_i,t) \\ \dot{\zeta}(a_i,t) \end{pmatrix} = M \begin{pmatrix} \eta(a_i,t) \\ \zeta(a_i,t) \end{pmatrix} + \begin{pmatrix} zW + \delta\tau_B\Phi \\ \delta\tau_B^2\Phi^2 \end{pmatrix}$$

subject to the initial condition

$$\begin{pmatrix} \eta(a_i,0) \\ \zeta(a_i,0) \end{pmatrix} = \begin{pmatrix} a_i \\ a_i^2 \end{pmatrix}.$$

It is well-known that its solution (Dym, 2013, Chapter 13) reads

$$\begin{pmatrix} \eta(a_i,t) \\ \zeta(a_i,t) \end{pmatrix} = e^{tM} \begin{pmatrix} a_i \\ a_i^2 \end{pmatrix} + \int_0^t e^{(t-s)M} \begin{pmatrix} zW + \delta\tau_B\Phi \\ \delta\tau_B^2\Phi^2 \end{pmatrix} \mathrm{d}s, \tag{14}$$

where e^{tM} and $e^{(t-s)M}$ are matrix exponentials. To compute $\eta(a_i,t)$ and $\zeta(a_i,t)$ we start by computing these matrix exponentials through diagonalization. Since $\beta > 0$ and by assumption also $\alpha \neq \gamma$, we obtain that M can be diagonalized as follows

$$M = \underbrace{\begin{pmatrix} \frac{\alpha-\gamma}{\beta} & 0 \\ 1 & 1 \end{pmatrix}}_{=V} \underbrace{\begin{pmatrix} \alpha & 0 \\ 0 & \gamma \end{pmatrix}}_{=D} \underbrace{\begin{pmatrix} \frac{\beta}{\alpha-\gamma} & 0 \\ -\frac{\beta}{\alpha-\gamma} & 1 \end{pmatrix}}_{=V^{-1}}.$$

Hence, the matrix exponential can be computed as

$$\begin{aligned} e^{tM} &= Ve^{tD}V^{-1} \\ &= \begin{pmatrix} \frac{\alpha-\gamma}{\beta} & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{\alpha t} & 0 \\ 0 & e^{\gamma t} \end{pmatrix} \begin{pmatrix} \frac{\beta}{\alpha-\gamma} & 0 \\ -\frac{\beta}{\alpha-\gamma} & 1 \end{pmatrix} \\ &= \begin{pmatrix} e^{\alpha t} & 0 \\ \frac{\beta}{\alpha-\gamma}(e^{\alpha t} - e^{\gamma t}) & e^{\gamma t} \end{pmatrix}. \end{aligned}$$

Inserting this in equation (14) we obtain

$$\begin{aligned} \begin{pmatrix} \eta(a_i,t) \\ \zeta(a_i,t) \end{pmatrix} = &\begin{pmatrix} e^{\alpha t} & 0 \\ \frac{\beta}{\alpha-\gamma}(e^{\alpha t} - e^{\gamma t}) & e^{\gamma t} \end{pmatrix} \begin{pmatrix} a_i \\ a_i^2 \end{pmatrix} \\ &+ \int_0^t \begin{pmatrix} e^{\alpha(t-s)} & 0 \\ \frac{\beta}{\alpha-\gamma}(e^{\alpha(t-s)} - e^{\gamma(t-s)}) & e^{\gamma(t-s)} \end{pmatrix} \begin{pmatrix} zW + \delta\tau_B\Phi \\ \delta\tau_B^2\Phi^2 \end{pmatrix} \mathrm{d}s. \end{aligned}$$

If $\alpha \neq 0$ we obtain

$$\int_0^t e^{\alpha(t-s)}\mathrm{d}s = e^{\alpha t}\int_0^t e^{-\alpha s}\mathrm{d}s = e^{\alpha t}\left[-\frac{1}{\alpha}e^{-\alpha s}\right]_0^t = \frac{1}{\alpha}(e^{\alpha t} - 1)$$

and otherwise for $\alpha = 0$ we have

$$\int_0^t e^{\alpha(t-s)}\mathrm{d}s = \int_0^t 1\mathrm{d}s = t.$$

Using indicator functions again, we can write this as

$$\int_0^t e^{\alpha(t-s)}\mathrm{d}s = \mathbb{I}_{\{\alpha\neq 0\}}\frac{1}{\alpha}(e^{\alpha t}-1)+\mathbb{I}_{\{\alpha=0\}}t.$$

A completely analogous argument gives

$$\int_0^t e^{\gamma(t-s)}\mathrm{d}s = \mathbb{I}_{\{\gamma\neq 0\}}\frac{1}{\gamma}(e^{\gamma t}-1)+\mathbb{I}_{\{\gamma=0\}}t.$$

Using this we arrive at

$$\begin{aligned}\eta(a_i,t) &= a_i e^{\alpha t}+(zW+\delta\tau_B\Phi)\int_0^t e^{\alpha(t-s)}\mathrm{d}s\\ &= a_i e^{\alpha t}+(zW+\delta\tau_B\Phi)\left(\mathbb{I}_{\{\alpha\neq 0\}}\frac{1}{\alpha}(e^{\alpha t}-1)+\mathbb{I}_{\{\alpha=0\}}t\right)\end{aligned}$$

and

$$\begin{aligned}\zeta(a_i,t) =& \frac{a_i\beta}{\alpha-\gamma}(e^{\alpha t}-e^{\gamma t})+a_i^2e^{\gamma t}+(zW+\delta\tau_B\Phi)\frac{\beta}{\alpha-\gamma}\int_0^t e^{\alpha(t-s)}-e^{\gamma(t-s)}\mathrm{d}s\\ &+\delta\tau_B^2\Phi^2\int_0^t e^{\gamma(t-s)}\mathrm{d}s\\ =& \frac{a_i\beta}{\alpha-\gamma}(e^{\alpha t}-e^{\gamma t})+a_i^2e^{\gamma t}+\frac{\beta(zW+\delta\tau_B\Phi)}{\alpha-\gamma}\left(\mathbb{I}_{\{\alpha\neq 0\}}\frac{1}{\alpha}(e^{\alpha t}-1)+\mathbb{I}_{\{\alpha=0\}}t\right)\\ &+\left(\delta\tau_B^2\Phi^2-\frac{\beta(zW+\delta\tau_B\Phi)}{\alpha-\gamma}\right)\left(\mathbb{I}_{\{\gamma\neq 0\}}\frac{1}{\gamma}(e^{\gamma t}-1)+\mathbb{I}_{\{\gamma=0\}}t\right)\end{aligned}$$

which is the desired claim.

A.6. Proof of Theorem 4.3

By the law of total expectation (Rényi, 2007, Chapter 4, §4) we have

$$\mathbb{E}[A_i(t)] = \mathbb{E}\left[\mathbb{E}[A_i(t)|A_i(0)]\right].$$

As $A_i(0)$ has the probability density function f, we obtain

$$\mathbb{E}[A_i(t)] = \int_0^\infty f(a_i)\mathbb{E}[A_i(t)|A_i(0)=a_i]\mathrm{d}a_i = \int_0^\infty f(a_i)\eta(a_i,t)\mathrm{d}a_i,$$

where in the second equality we used our shorthand notation $\eta(a_i,t)$. Similarly, we proceed with the variance: First, the law of total variance (Rényi, 2007, Chapter 4, §6) yields

$$\mathrm{Var}(A_i(t)) = \mathbb{E}\left[\mathrm{Var}(A_i(t)|A_i(0))\right]+\mathrm{Var}\left(\mathbb{E}\left[A_i(t)|A_i(0)\right]\right).$$

Using the classical rules for conditional expectation we get

$$\mathrm{Var}(A_i(t)|A_i(0)) = \mathbb{E}\left[(A_i(t)-\mathbb{E}[A_i(t)|A_i(0)])^2|A_i(0)\right] = \mathbb{E}\left[(A_i(t)^2|A_i(0)\right]-\mathbb{E}\left[A_i(t)|A_i(0)\right]^2.$$

Moreover, the variance of the random variable $\mathbb{E}[A_i(t)|A_i(0)]$ is computed as

$$\mathrm{Var}\left(\mathbb{E}\left[A_i(t)|A_i(0)\right]\right) = \mathbb{E}\left[\mathbb{E}\left[A_i(t)|A_i(0)\right]^2\right] - \mathbb{E}\left[\mathbb{E}\left[A_i(t)|A_i(0)\right]\right]^2.$$

Hence,

$$\begin{aligned}
&\mathrm{Var}(A_i(t)) \\
&= \mathbb{E}\left[\mathbb{E}\left[(A_i(t)^2|A_i(0)\right] - \mathbb{E}\left[A_i(t)|A_i(0)\right]^2\right] + \mathbb{E}\left[\mathbb{E}\left[A_i(t)|A_i(0)\right]^2\right] - \mathbb{E}\left[\mathbb{E}\left[A_i(t)|A_i(0)\right]\right]^2 \\
&= \mathbb{E}\left[\mathbb{E}\left[(A_i(t)^2|A_i(0)\right]\right] - \mathbb{E}\left[\mathbb{E}\left[A_i(t)|A_i(0)\right]\right]^2.
\end{aligned}$$

Then, again using that $A_i(0)$ has the probability density function f, we obtain

$$\mathrm{Var}(A_i(t)) = \int_0^\infty f(a_i)\mathbb{E}\left[(A_i(t)^2|A_i(0) = a_i\right]\mathrm{d}a_i - \left(\int_0^\infty f(a_i)\mathbb{E}\left[A_i(t)|A_i(0) = a_i\right]\mathrm{d}a_i\right)^2.$$

Finally, using our shorthand notations $\eta(a_i, t)$ and $\zeta(a_i, t)$ we obtain

$$\mathrm{Var}(A_i(t)) = \int_0^\infty f(a_i)\zeta(a_i, t)\mathrm{d}a_i - \left(\int_0^\infty f(a_i)\eta(a_i, t)\mathrm{d}a_i\right)^2.$$

Inserting the expressions for the expected value $\mathbb{E}[A_i(t)]$ and the variance $\mathrm{Var}(A_i(t))$ into the definition of the coefficient of variation yields the desired claim.

A.7. Derivation of equations (11) and (12)

The amount of bequest taxes paid by dynasty i up to time t is denoted by $S_i(t)$ and follows the stochastic differential equation

$$\mathrm{d}S_i(t) = \mathbb{I}_{\{A_i(t-)>\Phi\}}\tau_B(A_i(t-) - \Phi)\mathrm{d}N_i(t).$$

Introducing the short-hand notation

$$s(a_i, t) = \mathbb{E}[S_i(t)|A_i(0) = a_i],$$

we immediately obtain for $a_i < \Phi$ that

$$s(a_i, t) = \begin{cases} 0 & \text{if } t \leq t_\Phi(a_i) \\ s(\Phi, t - t_\Phi(a_i)) & \text{if } t > t_\Phi(a_i). \end{cases}$$

If $a_i \geq \Phi$, we obtain using Lemma 3.2 that

$$s(a_i, t) = \mathbb{E}_{a_i}[S_i(t)] = \mathbb{E}_{a_i}\left[\int_0^t \tau_B(A_i(s-) - \Phi)\mathrm{d}N_i(s)\right].$$

As $\sup_{s\in[0,t]}|A_i(s)|$ is integrable, we can rewrite the integral with respect to the Poisson process first using Proposition 20.10 in Privault (2022) and then using Fubini's Theorem (Klenke, 2014, Theorem 14.16) to obtain

$$s(a_i, t) = \delta\mathbb{E}_{a_i}\left[\int_0^t \tau_B(A_i(s) - \Phi)\mathrm{d}s\right] = \delta\tau_B\int_0^t \left(\mathbb{E}_{a_i}[A_i(s)] - \Phi\right)\mathrm{d}s.$$

Using the explicit expression for $\eta(a_i, s) = \mathbb{E}_{a_i}[A_i(s)]$ from Theorem 4.2 we obtain

$$s(a_i, t) = \delta\tau_B \int_0^t \left(a_i e^{\alpha s} + (zW + \delta\tau_B \Phi) \left(\mathbb{I}_{\{\alpha \neq 0\}} \frac{1}{\alpha}(e^{\alpha s} - 1) + \mathbb{I}_{\{\alpha = 0\}} s \right) - \Phi \right) \mathrm{d}s.$$

Rearranging and computing the integrals yields

$$s(a_i, t) = \begin{cases} \delta\tau_B \left(\left(\frac{a_i}{\alpha} + \frac{zW + \delta\tau_B \Phi}{\alpha^2} \right) (e^{\alpha t} - 1) - \left(\frac{zW + \delta\tau_B \Phi}{\alpha} + \Phi \right) t \right) & \text{if } \alpha \neq 0 \\ \delta\tau_B \left((a_i - \Phi) t + \frac{zW + \delta\tau_B \Phi}{2} t^2 \right) & \text{if } \alpha = 0. \end{cases}$$