Pay-as-bid Auctions with Private Information

Daniel Linnenbrink[*]

February 2024

Abstract

The pay-as-bid auction is one of the most prominent mechanisms allocating divisible goods like energy. This paper presents an equilibrium analysis of pay-as-bid auctions in a general environment where bidders are privately informed about their valuations. With two bidders the pay-as-bid auction can be represented as a continuum of asymmetric first-price auctions. I define a locally optimal Bayes-Nash equilibrium, characterize equilibrium strategies and show uniqueness. I also provide a characterization of the locally optimal equilibrium for more than two players. The equilibrium structure infers that bid flattening is less relevant than concluded by existing literature on strategic ironing. However, I show that every Bayes-Nash equilibrium has to be either locally optimal or strategic ironing. In the cases where the locally optimal equilibrium does not exist there has to be a strategic ironing equilibrium.

Keywords: auctions, discriminatory auction, equilibrium uniqueness

JEL codes: C72, D44, D47, D82

---

[*] University of Mannheim, ZEW Mannheim; `linnenbrink@uni-mannheim.de`

I would like to thank Nicolas Schutz as well as Stefan Fraunholz, Jakob Rohrbeck, Leticia Pieraerts, Hans Peter Grüner, Marion Ott, Thomas Tröger, Kornelia Aigner and audiences at University of Mannheim, ZEW Mannheim, University of the Balearic Islands, Goethe University Frankfurt, Toulouse School of Economics, UCLouvain and Université libre de Bruxelles.

1 Introduction

Auction theory has extensively studied first-price auctions but is still lacking a deeper understanding of their natural extension to divisible goods. Pay-as-bid auctions, also known as discriminatory-price auctions, are a standard auction format for allocating divisible goods or multiple homogenous goods. They are one of the most common market mechanisms allocating trillions of dollars in electricity markets and treasury auctions (Song and Zhu (2018)). The ongoing climate crisis connected to the crisis in energy markets calls for a better understanding of procurement mechanisms. The importance of pay-as-bid auctions stands in stark contrast to the lack of its theoretical understanding. How do the rules of market mechanisms translate to profits of auctioneers and bidders as well as social welfare? The key ingredient in the theory of mechanism design is private information. However, the theory of pay-as-bid auctions with private information is still mostly uncharted territory. What are equilibrium properties of pay-as-bid auctions with private information? A new foundation for the theoretical analysis of pay-as-bid auctions would be a characterization of equilibrium strategies and a proof of equilibrium uniqueness. I provide this foundation with an in-depth analysis of the setup with two bidders and an equilibrium characterization that extends to the case with more than two bidders.

An auctioneer wishes to procure one unit of a perfectly divisible good and faces competing bidders with private information about their cost functions. This could be understood as a procurement auction in energy markets. The private information parameters are independent draws from a commonly known distribution. The auctioneer runs a pay-as-bid auction. Each bidder submits a weakly increasing bid function indicating a price for every infinitesimal unit. The mechanism sets the clearing price such that the totality of units offered at the clearing price or below add up to one. All infinitesimal units offered below the clearing price are accepted. The respective payment for each accepted unit is "as-bid" in the bid function.

I prove that Bayes-Nash equilibria can only be of two types. First, I introduce *locally optimal* equilibria as a class of Bayes-Nash equilibria where first-order conditions hold at every infinitesimal unit. To solve for this equilibrium in a setup with two bidders I view the pay-as-bid auction as a continuum of asymmetric first-price auctions where a bidder competes for infinitesimal unit y with the bid of its rival for unit $1 - y$. Then, existing results on single-unit auctions can be exploited. I characterize equilibrium strategies with a system of ordinary differential equations. Using the Picard-Lindelöf theorem and an appropriate initial condition I show that there is a unique *locally optimal* equilibrium.

Second, in *strategic ironing* equilibria players flatten their bid functions such that first-order conditions do not hold at every infinitesimal unit but they hold on average over the flat part of the bid function. The name and the idea for this equilibrium type rely on the work by Woodward (2016).

I prove that any Bayes-Nash equilibrium has to have either the *locally optimal* property or the *strategic ironing* property. When the characterized offer curves for the *locally optimal* equilibrium do not satisfy monotonicity the *locally optimal* equilibrium does not exist. Combining with the proof of equilibrium existence by Woodward (2016) this implies that in such a case there has to be a *strategic ironing* equilibrium. It remains unclear whether a given cost function can give rise to both equilibrium types. Finally, I demonstrate closed form solutions of the locally optimal equilibrium for a class of linear marginal cost functions. To the best of my knowledge, this is the first closed form representation of a Bayes-Nash equilibrium in a pay-as-bid auction of a perfectly divisible good with private information. In previous studies solutions have been available only in very restrictive setups that I discuss below. I provide a backwards engineering method creating solution examples. I demonstrate the derivation using one example and evaluate equilibrium properties.

In my example the pay-as-bid auction creates lower social welfare than the ex-post efficient Vickrey auction (as expected) but is more favorable for the auctioneer in terms of revenue (procurement costs). This could motivate auctioneers in practice to use the pay-as-bid auction instead of the Vickrey auction.

The rest of this work is organized as follows. The remainder of this section situates my findings in the related literature. The following section sets up the pay-as-bid auction as a Bayesian game. I describe the relevant action spaces and private information as well as the payoff function defined by the rules of the auction format. Section 3 defines Bayes-Nash equilibria of the pay-as-bid game, describes the two possible forms and proves uniqueness for the *locally optimal* form. I show the derivation of closed form solutions and give one example. Section 5 compares welfare and revenue of the pay-as-bid auction in the closed form solution example to the Vickrey auction. Section 6 concludes.

The practical relevance of auctions for goods like energy motivates a large body of theoretical literature. Which auction format should be used to allocate divisible goods? Existing literature has shown difficulties in answering this question with an equilibrium analysis.

The literature finds equilibria in the pay-as-bid auction only in very restrictive settings. These restrictions include monopoly and perfect competition (Federico and Rahman (2003)) and setups with constant marginal costs (Fabra, von der Fehr, and Harbord (2006), Holmberg and Wolak (2018), Ausubel et al. (2014)). Another body of literature identifies equilibria in setups with linear marginal values and uncertainty of the auctioned quantity that follows some form of the generalized Pareto distribution (Wang and Zender (2002), Hästö and Holmberg (2006), Ewerhart, Cassola, and Valla (2010), Ausubel et al. (2014)). In a setup without restrictions on the uncertainty of demand but with no private information Pycia and Woodward (2023) provide a closed form representation of equilibrium strategies and show uniqueness. Including private information, Engelbrecht-Wiggans and Kahn (1998) and Chakraborty (2006) develop an equilibrium characterization and numerical solutions for a pay-as-bid auction for two homogenous goods. My work extends the above findings by providing an equilibrium characterization, a uniqueness result and examples of closed form solutions in a setup with a perfectly divisible good and without strong parametric assumptions about the distribution of private information parameters.

In my procurement setting for a divisible good the Vickrey auction, an application of the Vickrey-Clarke-Groves mechanism in the spirit of Vickrey (1961), allows ex-post efficient allocations. However, the Vickrey auction is hardly used in practice. A possible reason is that the pay-as-bid auction dominates the Vickrey auction in terms of revenue (Ausubel et al. (2014)). My analysis endorses this proposition.

Besides pay-as-bid auctions, the uniform price auction is the most common format in divisible good auctions. In the uniform price auction, also known as pay-as-clear auction, the payment for all assigned infinitesimal quantities equals the clearing price. In some setups of the uniform price auction equilibria have been identified. Klemperer and Meyer (1989) describe an equilibrium when the auctioned quantity is uncertain and there is no private information. Vives (2011) and Ausubel et al.

(2014) analyze equilibria with linear marginal valuations where the intercept is private information.

Similarly to the above studies of uniform price auctions, my analysis of pay-as-bid auctions relies on a pointwise maximization argument. However, the two approaches are inherently different. In uniform price auctions Klemperer and Meyer (1989) argue that bid functions are ex post optimal. That is, a bidder would not wish to change their bid function after learning the realization of uncertainty. Each point on the bid function is optimal with regard to one realization of residual demand (or supply). In pay-as-bid auctions this approach does not work because the total payment not only depends on one point of the bid function but on the entire part corresponding to the share of the good assigned to the respective bidder. Instead, I view the maximization problem for each infinitesimal quantity separately. In contrast to the total payment, the payment for a particular infinitesimal unit and the probability of being assigned the infinitesimal unit depend only on the bid for this particular unit.

Due to the difficulties in analyzing equilibria of pay-as-bid and uniform price auctions, an emerging body of literature uses structural approaches to calculate counterfactuals when comparing revenue of competing auction formats in divisible good auctions in practice.

Hortaçsu and McAdams (2010) use a first-order condition argument to derive valuations from observed bid data. They conclude that revenue differences between the pay-as-bid auction and the uniform price auction are not significant. Armantier and Sbaï (2006) find that the uniform price auction is more favorable for both the bidders and the auctioneer. In renewable energy auctions Lamp, Samano, and Tiedemann (2024) find that the uniform price auction would decrease market power and reduce subsidy expenses. Most papers in this section of the literature can only approximate counterfactual outcomes based on derived valuations. With data on both auction formats Kang and Puller (2008) state that the pay-as-bid auction creates higher revenue and a more efficient allocation. My equilibrium characterization allows future research to calculate counterfactual equilibrium outcomes from the pay-as-bid auction when only the uniform price auction is observed.

My setup of the pay-as-bid auction relates most closely to the work by Woodward (2016) and Woodward (2019), which establishes existence of pure-strategy Bayes-Nash equilibria. The proof of equilibrium existence relies on a discretization of the action space as analyzed by Athey (2001), McAdams (2003) and Reny (2011). In pay-as-bid auctions bidders are required to submit weakly monotonic bid functions. My analysis of *locally optimal* and *strategic ironing* equilibria - albeit questioning the relevance of bid flattening - reconciles the work of Woodward (2016) with standard results from auction theory. I extend the work of Pycia and Woodward (2023) with my analysis of private information.

2 Model

I consider the procurement problem of a divisible good in a setup with private information. All of my results extend to the sale of a divisible good. The auction is defined by the environment of the problem and the rules of the economic game set by the mechanism designer. Motivation for this model is the central energy procurement by a government facing competing energy producers. In such cases the designer is usually also the institution that collects potential revenue or pays procurement costs.

2.1 Environment

A designer needs to procure one unit of a perfectly divisible good. The designer sets the rules of the game but is not modeled as a player. N players are potential producers of the good. Each player i has private information about her efficiency parameter θ^i which is drawn independently from a uniform distribution on the interval $[0,1]$, which is the type space.[1] The efficiency parameter θ enters the commonly known marginal cost function $c(\theta, y)$. Marginal units are denoted by $y \in [0,1]$. The marginal cost function is a Lipschitz continuous and Lebesgue measurable function $c: [0,1] \times [0,1] \to \mathbb{R}$. I assume that $c(\theta, y)$ is strictly decreasing in θ for any $y \in [0,1]$. That is, for any infinitesimal unit, a producer has lower production costs when efficiency parameter θ is higher. This assumption implies that marginal cost functions of different types never intersect. I also assume that $c(\theta, y)$ is strictly increasing in y for any $\theta \in [0,1]$. In energy production this assumption is backed by the internal merit order of producers. Cheaper power plants are activated first.

2.2 The pay-as-bid game

The environment in combination with the rules of the auction constitutes the pay-as-bid game G as a Bayesian game. This game is defined by the number of players N, their type space, the distribution of types, their action set and their payoff functions. All players have the same action set which is given by the set of weakly increasing functions $\tilde{p}: [0,1] \to \mathbb{R}$.[2] Throughout the paper I use superscripts to indicate players and subscripts to indicate partial derivatives. Then, $\tilde{p}^i(y)$ is the

---

[1] The restriction to the uniform distribution is without loss of generality. Through a change of variable, a setup with any distribution of θ that has no mass points can be transformed into the uniform case. Suppose θ is distributed according to cumulative density function F_θ. Then, $\theta' = F_\theta(\theta)$ is uniformly distributed on $[0,1]$. This transformation does not alter continuity and monotonicity behavior of marginal costs in terms of θ'.

[2] Offer curves have to be increasing in electricity markets due to the merit order adhered to by the designer. An offer curve in existing markets is usually a collection of small chunks offered by the producer at different prices. Cheap chunks are accepted first by the mechanism.

price demanded for infinitesimal unit y by player i. I call such a function offer curve. A pure strategy is a function mapping from the type space $[0,1]$ into the action set. For any type $\theta \in [0,1]$ the strategy gives an offer curve. I write a strategy as a multivariate function out of the strategy set $\mathcal{S}$ which is the set of functions $\tilde{p} : [0,1] \times [0,1] \to \mathbb{R}$ that are weakly increasing in their second argument. In this paper I only consider pure strategies. $\tilde{p}^i(\theta, y)$ is the price demanded by player i for marginal unit y when having private information parameter θ. It makes sense to also define the two inverses of $\tilde{p}^i$ when they exist. Throughout this study I denote the functions by $(\tilde{p}, \tilde{\theta}, \tilde{y})$ and variables entering the functions by (p, θ, y).

The payoff functions $g^i : \mathcal{S}^N \times [0,1]^N \to \mathbb{R}$ are given by the rules of the pay-as-bid auction. Player i receives payoff $g\big((\tilde{p}^i, \tilde{p}^{-i}), (\theta^i, \theta^{-i})\big)$ when the type profile is (θ^i, θ^{-i}) and the strategy profile is $(\tilde{p}^i, \tilde{p}^{-i})$. The definition of payoffs in the pay-as-bid game follows the work by Woodward (2016). As offer curves can be flat in quantities consider the following two functions instead of an inverse.

$$\underline{y}^i(\theta, p) \coloneqq \inf\{y \in [0,1] : \tilde{p}^i(\theta, y) \geq p\}$$

$$\overline{y}^i(\theta, p) \coloneqq \sup\{y \in [0,1] : \tilde{p}^i(\theta, y) \leq p\}$$

If there is no $y \in [0,1]$ such that $\tilde{p}^i(\theta, y) \geq p$, then $\underline{y}^i(\theta, p) \coloneqq 1$. If there is no $y \in [0,1]$ such that $\tilde{p}^i(\theta, y) \leq p$, then $\overline{y}^i(\theta, p) \coloneqq 0$. When $\tilde{p}^i(\theta, y)$ is strictly increasing in y, the inverse with respect to y exists and is given by

$$\tilde{y}^i(\theta, y) = \underline{y}^i(\theta, p) = \overline{y}^i(\theta, p).$$

The clearing price $\hat{p}$ is a function of the strategy profile and the private information profile and is given by

$$\hat{p}\Big((\tilde{p}^k)_{k=1}^N, (\theta^k)_{k=1}^N\Big) = \inf\Big\{p : \sum_{k=1}^N \underline{y}^k(\theta^k, p) \leq 1 \leq \sum_{k=1}^N \overline{y}^k(\theta^k, p)\Big\}.$$

The infimum is required because there can be two prices satisfying the market clearing condition when there is a bidder with a discontinuous bid function. When the inverses $\{\tilde{y}^k\}_{k=1}^N$ exist the above is equivalent to the following implicit definition of the clearing price.

$$\sum_{k=1}^N \tilde{y}^k(\theta^k, \hat{p}) = 1.$$

The quantity assigned to each player is also a function of the strategy profile and the private information profile. When the inverse $\tilde{y}^i$ exists, the quantity assigned to player i is given by

$$\hat{y}^i\Big((\tilde{p}^k)_{k=1}^N, (\theta^k)_{k=1}^N\Big) \coloneqq \tilde{y}^i(\theta^i, \hat{p}).$$

Flat offer curves and ties are handled by a rationing rule. Note that in equilibrium ties are a probability zero event. When offer curves are flat at the clearing price, quantities are rationed such that the smallest quantity assigned to some player is maximized. As it is not essential for equilibrium properties the formal statement of the rationing rule is provided in the appendix. Player i has to produce $\hat{y}^i$ and in return receives a payment equal to the area below i's offer curve. The profit of producer i is then formally given by

$$g^i\Big((\tilde{p}^k)_{k=1}^N, (\theta^k)_{k=1}^N\Big) = \int_0^{\hat{y}^i} \big(\tilde{p}^i(\theta^i, y) - c(\theta^i, y)\big) dy.$$

3 Bayes-Nash equilibrium and local optimality

Consider the maximization problem of player i. Denote by $Pr^i(y,p)$ the probability that player i successfully sells the y'th infinitesimal unit when offered at price p.[3] This is a function of quantity y, price p and the strategy profile of the competitors $(\tilde{p}^j)_{j\neq i}$ together with their types $(\tilde{\theta}^j)_{j\neq i}$. The first important finding here is that $Pr^i(y,p)$ does not depend on i's bids for units other than y. The formal statement for $Pr^i(y,p)$ follows from the definition of the market clearing price and the rationing rule and is given below.

$$\begin{aligned} Pr^i(y,p) = Pr\big[y + \sum_{j\neq i} \overline{y}^j(\theta^j,p) < 1\big] \\ + Pr\big[y + \sum_{j\neq i} \underline{y}^j(\theta^j,p) \leq 1 \leq y + \sum_{j\neq i} \overline{y}^j(\theta^j,p)\big] \; \mathbb{1}^i\big(y,p\big) \end{aligned}$$

The first part is the probability of strictly outbidding the competition. The second part is the part of the winning probability that stems from a potential tie at the clearing price. The indicator function $\mathbb{1}^i\big(y,p\big)$ takes value one when player i sells unit y at price p in the case that it is affected by the tie and not won by strictly outbidding the competition. Else it takes value zero. Note that $\mathbb{1}^i\big(y,p\big)$ is a function of quantity y, price p and the strategy profile of the competitors $(\tilde{p}^j)_{j\neq i}$ together with their types $(\tilde{\theta}^j)_{j\neq i}$ but it does not depend on the rest of i's offer curve. Player i's bids for units other than y do not affect the probability of winning infinitesimal unit y. The definition of $\mathbb{1}^i\big(y,p\big)$ requires special caution regarding the possibility of flat offer curves at the market clearing price. The formal definition follows from the rationing rule and is moved to the appendix as it is not essential for the equilibrium analysis. In equilibrium the probability of a tie at given price p is zero and the second part takes value zero.

---

[3] When the y'th infinitesimal unit is sold that means all "lower" units have also been sold and "higher" units are potentially also sold. That is, $Pr^i(y,p)$ is actually a probability and not a density.

Expected payoff of player i with marginal cost function $c(\theta^i, \cdot)$ when offering $\tilde{p}^i(\theta^i, \cdot)$ is

$$\tilde{g}^i\Big((\tilde{p}^k)_{k=1}^N, \theta^i\Big) = \int_0^1 Pr^i\big(y, \tilde{p}^i(\theta^i, y)\big)\big(\tilde{p}^i(\theta^i, y) - c(\theta^i, y)\big)dy.$$

This suffices to define Bayes-Nash equilibria of the auction.

Definition 1 (Bayes-Nash equilibrium)**.** *A pure-strategy Bayes-Nash equilibrium of the pay-as-bid game G is a profile of strategies $(\tilde{p}^i)_{i=1}^N$ with*

$$\tilde{p}^i \in arg \max_{\tilde{p}^i \in \mathcal{S}} \tilde{g}^i\Big((\tilde{p}^k)_{k=1}^N, \theta^i\Big) \quad \forall \theta^i \in [0,1] \;\; \forall i \in \{1, ..., N\}.$$

I distinguish two forms of Bayes-Nash equilibria. In the following subsection I introduce the *locally optimal equilibrium* which is a solution of pointwise profit maximization. I show that the unique *locally optimal equilibrium* is characterized by a system of ordinary differential equations. In section 4 I discuss *strategic ironing* as introduced by Woodward (2016) where players flatten their bid functions over certain regions. I show that every Bayes-Nash equilibrium is either of the *locally optimal* or of the *strategic ironing* form.

3.1 Uniqueness and other properties

In this subsection I develop my main findings regarding equilibria of pay-as-bid auctions that satisfy local optimality at every relevant infinitesimal unit. To formalize a notion of equilibrium uniqueness some preliminaries are required.

A common phenomenon in auction theory is that some types may win an auction with probability zero in equilibrium. Their bid can be changed without changing the equilibrium allocation. In multi-unit auctions Pycia and Woodward (2023) call the set of units that are won with positive probability *relevant* units.

Definition 2 (Relevant units)**.** *The set of relevant units for player i with type $\theta^i \in [0,1]$ and strategy $\tilde{p}^i$ is*

$$\mathcal{Y}^i = \{y \in (0,1):\ Pr^i(y, \tilde{p}^i(\theta^i, y)) > 0\}.$$

Note that any modification of a Bayes-Nash equilibrium such that offer curves differ at a set of infinitesimal quantities with measure zero is also a Bayes-Nash equilibrium as long as the resulting offer curves do not violate the monotonicity restriction of the strategy space. However, when the equilibrium offer curves are continuous such a modification would only be possible for $y \in \{0,1\}$. To account for this the above definition establishes that units $y \in \{0,1\}$ are never *relevant*.

Definition 3 (Locally optimal equilibrium)**.** *A locally optimal equilibrium of the pay-as-bid game G is a profile of strategies $(\tilde{p}^i)_{i=1}^2$ that form a pure-strategy Bayes-Nash equilibrium of G and locally maximize expected profit for every relevant infinitesimal unit and every player in the sense that*

$$\tilde{p}^i(\theta, y) \in arg\,\underset{p}{max}\ Pr^i(y,p)(p - c^i(\theta, y))$$

for every $\theta \in [0,1]$, for every i and for every relevant y.

To account for the fact that two equilibria could differ in strategies without ever resulting in different allocations Pycia and Woodward (2023) use the notion of an *essentially unique* equilibrium. I describe a locally optimal equilibrium as *essentially unique* when the set of *relevant* units and offer curves on all *relevant* units are identical in all locally optimal equilibria for all types $\theta \in [0,1]$.

I assume $N = 2$ for the remainder of this subsection. The pay-as-bid auction with two players exhibits some convenient properties that make equilibria especially tractable. Player i competes locally for the infinitesimal unit y with the bid of player j for the infinitesimal unit $1-y$. This realization is essential for the equilibrium analysis. For this form of local competition I use the term infinitesimal auction. When $\tilde{p}^i(\theta^i, y) < \tilde{p}^j(\theta^j, 1-y)$ player i wins this auction and vice versa. The pay-as-bid auction with two players can thus be described as a continuum of infinitesimal auctions that are essentially asymmetric first-price auctions. I now state the main result regarding uniqueness of a characterization for the locally optimal equilibrium and examine some of its properties.

Proposition 1 (Uniqueness of the locally optimal equilibrium)**.** *The pay-as-bid game G with two players has at most one locally optimal equilibrium in the sense of essential uniqueness. Define* $\underline{p} := \int_0^1 c(\theta, \frac{1}{2})d\theta$*. When the equilibrium exists, it is symmetric and strategies* $(\tilde{p}^i)_{i=1}^2$ *for relevant units are given by the inverse of the function* $\tilde{\theta}$ *solving the differential equation* (1) *- considered over the domain* $D = \{(\tilde{\theta}, y, p) \,|\, \tilde{\theta} \in (0,1], y \in (0,1), p \in (c(\tilde{\theta}(y,p), y), c(0, \frac{1}{2}))\}$ *- with initial condition* (2).

$$\tilde{\theta}_p(y,p) = \frac{\tilde{\theta}(y,p)}{c\Big(\tilde{\theta}(1-y,p), 1-y\Big) - p}, \tag{1}$$

$$\tilde{\theta}\Big(y, \underline{p}\Big) = 1 \quad \textit{for} \quad y \in (0,1). \tag{2}$$

Proof. See section 3.2. □

Proposition 1 characterizes bids in the unique locally optimal equilibrium with a differential equation and an initial condition. Existing literature on first-price auctions by Lebrun (2006) entails two boundary conditions, because it is restricted to equilibria where players never bid below costs. In my analysis this is possible when the respective bid is accepted with probability zero. In this case I call the respective unit not *relevant*. For units that are not relevant there are potentially infinite ways possibilities of equilibrium bid functions. As those bids are accepted with probability zero and never affect the allocation I still call the equilibrium *essentially unique.*

The proof of Proposition 1 is based on existing results regarding first-price auctions for single units. I first show that $\tilde{p}(\cdot, y)$ has to be continuous and strictly decreasing and Pr^i has to be continuously differentiable. Then, to satisfy the first-order condition of profit maximization at every unit y, the differential equation (1) has to hold. Now, view $\tilde{\theta}(y, \cdot)$ and $\tilde{\theta}(1-y, \cdot)$ as two separate functions. Equation (1) and

$$\tilde{\theta}_p(1-y, p) = \frac{\tilde{\theta}(1-y, p)}{c\Big(\tilde{\theta}(y, p), y\Big) - p}$$

form a system of ordinary differential equations. Then, the local Picard-Lindelöf theorem can be applied to show that there is a unique solution to (1) and (2). The following subsection provides a detailed proof.

Let $\tilde{p}$ and $\tilde{\theta}$ now be the functions characterized by Proposition 1. The following properties can all be derived directly from the characterization. The properties are visualized in Figure 1.

Property 1. *The offer curve for the most efficient type* $\tilde{p}(1, \cdot)$ *is flat at* $\tilde{p}(1, y) = \underline{p}$ *with* $y \in (0, 1)$.

Proof. This follows directly from inverting initial condition (2) in Proposition 1. □

Property 2. *Offer curves* $\tilde{p}(\theta, \cdot)$ *are Lipschitz continuous at all relevant units.*

Proof. This proof can be found in the appendix. □

Property 3. *For* $y < \frac{1}{2}$ *it holds that* $\tilde{p}(\theta, y) < \tilde{p}(\theta, 1-y)$ *for any* θ *in the image of* $\tilde{\theta}(y, \cdot)$.

Proof. This follows from the proof of Lemma 5. □

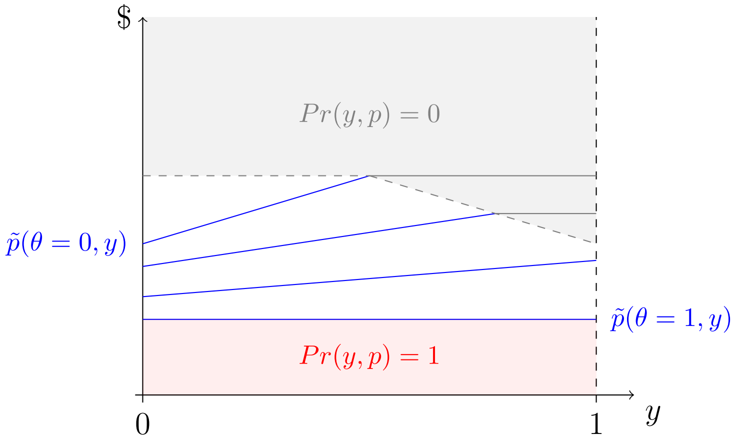


Figure 1: Solution properties

Note that Property 3 is weaker than monotonicity in y. Monotonicity of $\tilde{p}$ does not follow directly from the monotonicity of c. This is discussed in section 3.3. Figure 1 shows an example of offer curves $\tilde{p}$ solving the characterization in Proposition 1. Offer curves are continuous and $\tilde{p}(1,y)=\underline{p}$. In this example offer curves are weakly increasing everywhere. Bids in the shaded gray area are accepted with probability zero. The respective units are not *relevant* for the types bidding in this area.

3.2 Proof of uniqueness

In this part I outline the proof of Proposition 1. Technical details can be found in the appendix. To show that any *locally optimal* equilibrium has to satisfy some regularity conditions I exploit three existing results on first-price auctions. I consider the infinitesimal auction where bidder i competes for infinitesimal unit y with bidder j's bid for unit $1-y$.

Lemma 1 (Maskin and Riley (2000), Proposition 3)**.** *In any locally optimal equilibrium of the pay-as-bid game G, for any given $y\in(0,1)$, the cumulative distribution function of winning bids* $\min\{\tilde{p}_i(\cdot,y),\tilde{p}_j(\cdot,1-y)\}$ *is continuous and the support of the distribution is an interval* $[\underline{p},\overline{p}]$.

Lemma 1 implies that the distribution of bids of any bidder has no mass points at any *relevant* y. This is helpful to prove the following result.

Lemma 2 (Maskin and Riley (2000), Proposition 1)**.** *In any locally optimal equilibrium of the pay-as-bid game* G*,* $\tilde{p}_i(\cdot, y)$ *is a strictly decreasing function for* $i \in \{1, 2\}$ *in the sense that* $\theta'' > \theta'$ *implies that* $\tilde{p}_i(\theta'', y) < \tilde{p}_i(\theta', y)$ *for any* θ' *and* θ'' *such that the infinitesimal unit* y *is relevant.*

Lemma 2 implies that the inverse bid functions $(\tilde{\theta})_{i=1}^2$ exist.

Lemma 3 (Maskin and Riley (2003), Lemma 7)**.** *In any locally optimal equilibrium of the pay-as-bid game* G*, at any* $y \in (0, 1)$*,* $\tilde{\theta}(y, \cdot)$ *is continuously differentiable.*

It follows from the proof of Lemma 3 that the following system has to be satisfied and the derivatives exist. The same system follows from first-order conditions of profit maximization by both bidders.

$$\begin{aligned} \tilde{\theta}^j_p(1-y, p) &= \frac{\tilde{\theta}^j(1-y, p)}{c\Big(\tilde{\theta}^i(y, p), y\Big) - p} \\ \tilde{\theta}^i_p(y, p) &= \frac{\tilde{\theta}^i(y, p)}{c\Big(\tilde{\theta}^j(1-y, p), 1-y\Big) - p} \end{aligned} \tag{3}$$

For given $y \in [0, 1]$ this is an ordinary differential equation. To apply the usual theorems an initial condition is needed.

Lemma 4. *In any locally optimal equilibrium of the pay-as-bid game* G*, at any* $y \in (0, 1)$*, it holds that* $\tilde{\theta}\Big(y, \underline{p}\Big) = 1$ *with* $\underline{p} = \int_0^1 c(\theta, \frac{1}{2}) d\theta$*.*

Lemma 4 establishes that the initial condition for the differential equation in Proposition 1 is actually a necessary condition for a locally optimal equilibrium. It can be easily seen that the offer curve of type $\theta = 1$ has to be flat. Suppose for a contradiction that $\tilde{p}(1, \cdot)$ is strictly increasing for some y. Recall that by Lemma

2 bids of all other types are strictly higher for given y in equilibrium. Then type $\theta = 1$ has a profitable deviation. The bid for low units y close to zero can be slightly increased while maintaining winning probability one as long the bid is below the respective bid for the competing unit $1 - y$. Setting $\tilde{p}(1, y) = \int_0^1 c(\theta, \frac{1}{2})d\theta$ ensures that $\tilde{p}(0, 1/2) = c(0, 1/2)$. Otherwise the bid of type $\theta = 0$ at $y = \frac{1}{2}$ could not be optimal as profits for this infinitesimal unit could be increased by slightly lowering the bid. This is also a consequence of Theorem 1 of Pycia and Woodward (2023) and earlier results on symmetric first-price auctions. A complete proof of Lemma 4 is included in the appendix.

Taking the ordinary differential equation (3) with initial condition (2) as derived in Lemma 4 I can apply the local Picard-Lindelöf theorem to see that there is a unique solution. Since the same system has to hold also when indices j and i are exchanged, the solution has to be symmetric and can therefore be written without indices as in equation (1). This completes the proof of Proposition 1.

3.3 Existence

Existence of the *locally optimal* equilibrium in pay-as-bid auctions is more complicated than equilibrium existence in first-price auctions. However, using Proposition 2 it is straightforward to check for existence with a given marginal cost function c. At the end of this subsection I give an example of a *locally optimal* equilibrium in closed form.

Proposition 2. *The pay-as-bid game G with 2 players has a locally optimal equilibrium if and only if the inverse bid function $\tilde{\theta}(y, p)$ characterized by ODE* (1) *and initial condition* (2) *is weakly increasing in y and $\tilde{\theta}(y, \cdot)$ has image* $(0, 1]$ *for every $y \in [0, 1/2]$.*

Proof. To constitute an equilibrium, the characterized strategy $\tilde{p}$ needs to actually be an admissible strategy in the sense that $\tilde{p}$ in $\mathcal{S}$. Therefore, $\tilde{p}(\theta, y)$ has to be defined on the domain $[0,1] \times [0,1]$ and needs to be weakly increasing in y. This is the case if and only if the inverse bid function $\tilde{\theta}(y,p)$ is weakly increasing in y and $\tilde{\theta}(y,\cdot)$ has image $(0,1]$ for every $y \in [0,1/2]$. To complete the proof of Proposition 2 I show that there is no player i and no type θ^i that has a profitable deviation from the characterized bid function. I split the proof into the three following Lemmas and include their complete proofs in the appendix.

The common support of bids in the infinitesimal auction where i bids for unit y and j bids for unit $1-y$ is given by $[\underline{p}, \overline{p}]$ with $\overline{p} := \min\{\tilde{p}^i(0,y), \tilde{p}^j(0,1-y)\}$.

Lemma 5. *Let* $\overline{p} := \min\{\tilde{p}^i(0,y), \tilde{p}^j(0,1-y)\}$. *There is no profitable deviation above the common support of bids in the sense that*

$$Pr^i\big(y, \tilde{p}^i(\theta,y)\big)\big(\tilde{p}^i(\theta,y) - c^i(\theta,y)\big) \geq Pr^i(y,p)(p - c^i(\theta,y)) \quad \forall p > \overline{p}.$$

Lemma 5 is proven using a revealed preference argument. For the proof it needs to be shown that it is possible to define bids for units that are not *relevant* such that there is no profitable deviation on the *relevant* units. A simple way to do this is to make bids for units that are not *relevant* a continuous and flat extension of the bids for *relevant* units. However, this is not the only possibility as can be seen with the example below.

Lemma 6. *Let* $\overline{p} := \min\{\tilde{p}^i(0,y), \tilde{p}^j(0,1-y)\}$. *There is no profitable deviation within the common support of bids in the sense that*

$$\tilde{p}^i(\theta,y) \in arg \max_{p \in [\underline{p}, \overline{p}]} Pr^i(y,p)(p - c^i(\theta,y))$$

for every $\theta \in [0,1]$, *for every relevant* y *and for every* i.

Lemma 6 follows from a second order condition of profit maximization that is given by Wolfstetter (1996).

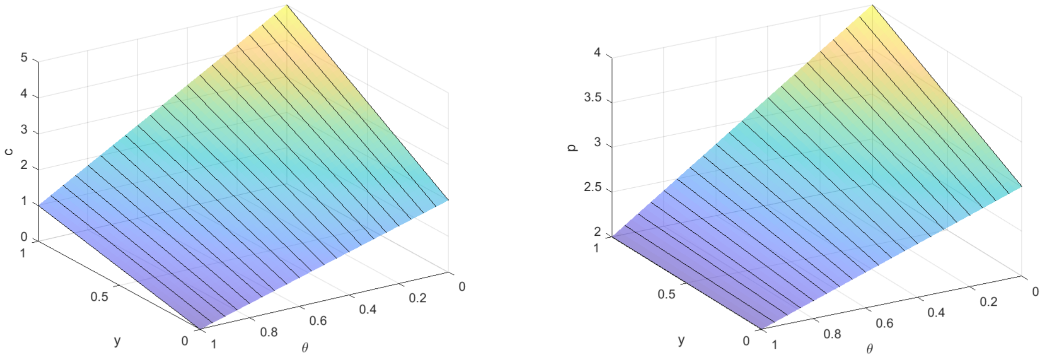
Lemma 7. *There is no profitable deviation below the common support of bids in the sense that*

$$Pr^i\big(y, \tilde{p}^i(\theta, y)\big)\big(\tilde{p}^i(\theta, y) - c^i(\theta, y)\big) \geq Pr^i(y, p)(p - c^i(\theta, y)) \quad \forall p < \underline{p}.$$

Bidding $\underline{p}$ gives higher profit than bidding below $\underline{p}$. Since $\underline{p}$ belongs to the common support of bids Lemma 7 then follows from Lemma 6. In conclusion, there is no type $\theta \in [0, 1]$ that has a profitable deviation at any *relevant* y from $\tilde{p}(\theta, y)$ to some $p \in \mathbb{R}$. $\square$

One might presume that when marginal costs c are monotonous in y the resulting bid function $\tilde{p}$ also satisfies the monotonicity condition. However, monotonicity of c is not sufficient. An example where the equilibrium candidate $\tilde{p}$ violates monotonicity is included in the appendix. The work on first-price auctions by Lebrun (1997) allows sufficient conditions on the cost function c such that the condition on the image is satisfied.

Since both of the properties required in Proposition 2 cannot be easily written in terms of the marginal cost function c a simple and useful approach is to check whether the equilibrium candidate bid function satisfies the properties. As an example where both properties hold as required consider the game with two players and marginal cost function $c(\theta, y) = \alpha + \beta\theta + 3\gamma y - 2\gamma y\theta$ with $\alpha > 0$, $\beta < 0$ and $\gamma > 0$. This is the class of linear marginal cost functions that results in linear equilibrium bids. The equilibrium offer curve is $\tilde{p}(\theta, y) = \alpha + \gamma + \frac{1}{2}\beta\theta + \gamma y - \gamma y\theta$. One example from this class is shown in Figure 2. A simple backwards-engineering procedure to create examples of pay-as-bid auctions with equilibria in closed form is to first think of an offer curve $\tilde{p}$ such that $\tilde{p}(1, y)$ is flat and then find c by



(a) Marginal costs (b) Bid functions

Figure 2: Equilibrium example

plugging into the following equation.

$$\tilde{\theta}_p\big(1-y,\tilde{p}(\theta,y)\big)=\frac{\tilde{\theta}\big(1-y,\tilde{p}(\theta,y)\big)}{c(\theta,y)-\tilde{p}(\theta,y)}$$

This can be easily solved for c. The Picard-Lindelöf theorem implies that for any cost function satisfying Lipschitz continuity in the ODE (1) a solution $\tilde{\theta}$ exists. However, as usual with differential equations, a closed form expression often does not exist and the researcher would have to use numerical methods to derive equilibrium offer curves from a given cost function.

3.4 $N > 2$ players

In the case with more than two players the pay-as-bid auction cannot be easily represented as a continuum of first-price-auctions and my results are considerably weaker. In the case with two players the bid for unit y made by player 1 competes with the bid of player 2 for unit $1-y$. Now the bid for unit y of player 1 competes with the aggregate supply from all other players which is a more complex function of offer curves. A natural approach to the locally optimal equilibrium is to solve first-order conditions as in the case with two players. I state the resulting characterization in Proposition 3.

Proposition 3. *In any symmetric locally optimal equilibrium of G with differentiable strategies, the inverse bid function* $\tilde{\theta}$ *has to satisfy*

$$0 = Pr(y,p) + \Big(p - c\big(\tilde{\theta}(y,p), y\big)\Big)\frac{d}{dp}Pr(y,p) \tag{4}$$

with

$$\begin{aligned} Pr(y,p) = &\int_{\underline{y}(0,p)}^{1-y-(N-2)\underline{y}(0,p)} \int_{\underline{y}(0,p)}^{1-y-(N-3)\underline{y}(0,p)-s_1} \cdots \int_{\underline{y}(0,p)}^{1-y-\underline{y}(0,p)-\sum_{j=1}^{N-3} s_{N-3}} \\ &\quad \tilde{\theta}\big(1 - y - \sum_{j=1}^{N-2} s_j, p\big)\ \Pi_{j=1}^{N-2}\tilde{\theta}_y(s_j,p)ds_{N-2}...ds_2ds_1 \\ &+ \sum_{k=1}^{N-2}\binom{N-2}{k}\tilde{\theta}(\underline{y}(0,p),p)^k \int_0^{1-y}\int_0^{1-y-s_1}\cdots\int_0^{1-y-\sum_{j=1}^{N-3-k} s_j} \\ &\quad \tilde{\theta}\big(1 - y - \sum_{j=1}^{N-2} s_j, p\big)\ \Pi_{j=1}^{N-2-k}\tilde{\theta}_y(s_j,p)ds_{N-2-k}...ds_2ds_1. \end{aligned} \tag{5}$$

Proof. The proof of Proposition 3 is included in the appendix. □

Note that Proposition 3 does not state equilibrium uniqueness. The order of the cross-derivatives forbids potential applications of the Cauchy-Kovalevskaya theorem. Additionally, the setting with more than two players allows different initial conditions compared to the unique flat offer curve of type $\theta^i = 1$ in the setting with two players.

Most steps for equilibrium sufficiency of the above characterization work as in the proof of Proposition 2. Lemmas 6 and 7 can also be applied in this setting. However, the proof of Lemma 5 does not go through here and deviations to prices above the support of bids for quantities $y < 1/n$ need to be ruled out differently.

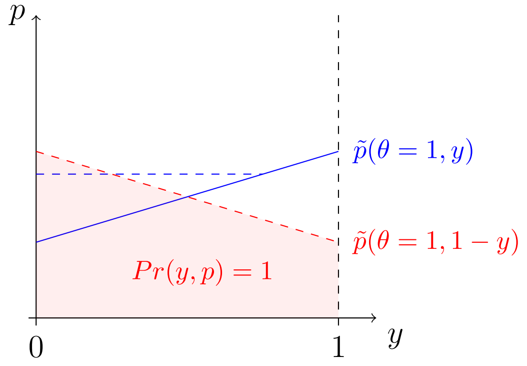


Figure 3: Strategic ironing

Again, equilibrium examples can be backwards-engineered by thinking of some offer curves $\tilde{p}$ and solving the above characterization for c. I provide such an equilibrium example in closed form with three players in the appendix.

4 Strategic ironing

Woodward (2016) postulates that in a class of equilibria players flatten their bids for a range of units compared to the solution of first-order conditions. This is termed *strategic ironing.* According to the definition of Woodward (2016) a profile of equilibrium strategies exhibits *strategic ironing* when there is at least one type that submits a bid function with a flat part at low quantities. Consider a symmetric equilibrium candidate, where the most efficient type $\theta = 1$ submits a strictly increasing offer curve. This is depicted by the solid blue line in Figure 3. Assume that types $\theta < 1$ demand higher prices for each unit. Then the dashed red line shows the most restrictive residual demand curve a player can face. This is given by the mirrored offer curve of type $\theta = 1$. Every unit offered at a price in the shaded red area is sold with probability $Pr(y, p) = 1$. Then, the blue offer curve cannot be part of an equilibrium as the shaded red area leaves room to deviate by slightly increasing bids for low units while still winning them with probability one. This pattern motivates the analysis of flattened bids such as the dashed blue line where first-order conditions hold *on average* over the flattened part. Local deviations at low units are eliminated by the monotonicity constraint on offer curves. The new offer curve is flattened. This averaging of first-order conditions dates back to Kastl (2012). Note however that this also changes incentives for the remaining part of the offer curve. The derivation of such equilibria is more complicated and left for further research. To reflect the intuition that *strategic ironing* is a flattening resulting in an equilibrium strategy that differs from the solution to local first-order conditions I use a slightly different definition than Woodward (2016).

Definition 4 (Strategic ironing equilibrium)**.** *A Bayes-Nash equilibrium of the pay-as-bid game G exhibits strategic ironing if there is a player i with a type $\theta^i \in [0,1]$ and an interval of relevant quantities $[\underline{y}, \overline{y}] \subseteq [0,1]$ such that for all units $y \in [\underline{y}, \overline{y}]$ it holds that $\tilde{p}^i(\theta^i, y)$ is constant and does not locally maximize expected profit in the sense that*

$$\tilde{p}^i(\theta^i, y) \notin \arg\max_p Pr^i(y,p)(p - c(\theta^i, y)).$$

I have shown above that there can be a solution to local first-order conditions such that the most efficient type submits a flat bid function. According to the definition of Woodward (2016) the resulting equilibrium would exhibit *strategic ironing* but it does not according to my definition when bid functions locally maximize profits at every infinitesimal unit. Bids are not flattened in the *locally optimal* equilibrium. This supports the claim that bid flattening is less important than postulated by Woodward (2016).

It follows immediately from the definitions that a given profile of strategies forming a Bayes-Nash equilibrium cannot be at the same time *locally optimal* and *strategic ironing.* A related finding that is not as trivial is given by Proposition 4.

Proposition 4. *Every Bayes-Nash equilibrium of the pay-as-bid game G with two players is either of the locally optimal form or of the strategic ironing form. Every Bayes-Nash equilibrium of the pay-as-bid game G in differentiable strategies is either of the locally optimal form or of the strategic ironing form.*

Proof. The proof of Proposition 4 is included in the appendix. □

As discussed in section 3.3 the *locally optimal equilibrium* does not always exist. Woodward (2016) shows that a Bayes-Nash equilibrium always exists. Together with Proposition 4 this implies that a *strategic ironing* equilibrium has to exist in those cases. To the best of my knowledge this is the first proof of existence of the *strategic ironing* equilibrium. It remains unclear whether the two equilibrium types could coexist for certain cost functions.

5 Welfare and Revenue

My characterization of equilibrium strategies in the pay-as-bid auction allows a more profound analysis of mechanism choices in divisible good settings such as electricity markets. To provide an example I compare procurement costs of the pay-as-bid auction to the Vickrey auction in the linear setup from section 3.3. Note that the finding below only applies to this particular set of linear marginal cost functions with linear equilibrium offer curves in a two-player setting.

Given that all mechanisms considered procure the same quantity of the good, welfare differences arise only from differences in production costs that result from the allocations. The VCG-mechanism can be applied to this setting in the form of the Vickrey auction to minimize the production costs and thereby maximize social welfare.

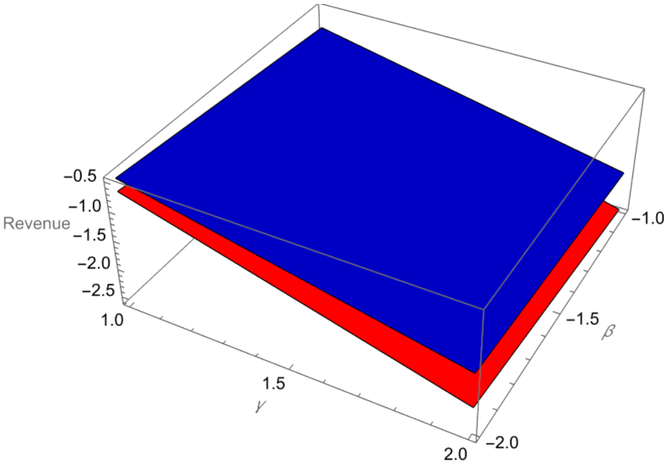


Figure 4: Revenue of pay-as-bid and VCG

When interpreting the problem as a procurement auction the revenue of the auctioneer is given by the negative of the payments to the players. For realized type profile $(\theta^i)_{i=1}^n$ and equilibrium offer curves $\tilde{p}$ the revenue of a pay-as-bid auction assigning quantities $\hat{y}^i$ is given by

$$R\big((\theta^i)_{i=1}^n\big) = -\sum_{i=1}^n \int_0^{\hat{y}^i((\theta^k)_{k=1}^n)} \tilde{p}(\theta^i, y)dy.$$

Exchanging $\tilde{p}$ by c in the above equation one receives social welfare. Consider now again the setup with two players and the class of all linear marginal cost functions that also result in linear offer curves. The marginal cost function is $c(\theta, y) = \alpha + \beta\theta + 3\gamma y - 2\gamma y\theta$ with $\alpha > 0$, $\beta < 0$ and $\gamma > 0$. The unique locally optimal equilibrium is given by $\tilde{p}(\theta, y) = \alpha + \gamma + \frac{1}{2}\beta\theta + \gamma y - \gamma y\theta$. Note that parameter α is irrelevant here as it identically shifts welfare and revenue in the pay-as-bid auction and the Vickrey auction. Figure 4 compares expected revenue of the pay-as-bid auction and the Vickrey auction for $\beta \in [-2, -1]$ and $\gamma \in [1, 2]$ while normalizing $\alpha = 0$. In this setting expected welfare of the Vickrey auction is strictly higher than expected welfare of the pay-as-bid auction. However, using the pay-as-bid auction, the designer achieves strictly higher expected revenue.

This possibly explains that in practice designers favor the pay-as-bid auction. Consequently, policy makers might consider an intervention in favor of the Vickrey auction.

6 Conclusion

In this study I improve the understanding of pay-as-bid auctions in a setup including private information about individual costs or valuations. In contrast to existing literature I show that the equilibrium does not have to entail flattening of bids. Players can choose optimal offer curves by pointwise maximization in the *locally optimal equilibrium.* I prove that there is a unique *locally optimal* equilibrium in the setup with two players and characterize equilibrium strategies. Existence of this equilibrium form can be easily checked, however, my analysis does not give an existence condition in terms of the cost function. I discuss the idea of *strategic ironing* equilibria introduced in the existing literature. I provide a definition and prove that any Bayes-Nash equilibrium of the pay-as-bid auction has to take one of the two forms. Taking into account existing results on the existence of Bayes-Nash equilibria, when the *locally optimal* equilibrium does not exist, there has to be a *strategic ironing* equilibrium. However, it remains unclear whether there are cases where both types of equilibria exist at the same time. I give a first example of a closed form solution of a *locally optimal* Bayes-Nash equilibrium in a pay-as-bid auction with private information. This solution is directly given by the solution to the differential equation. In this example I compare expected revenue and welfare of the pay-as-bid auction to the ex-post efficient Vickrey auction. The results indicate that in practice auction designers do not use the Vickrey auction because the pay-as-bid auction - and possibly other auction formats - entails lower procurement costs. Equilibrium uniqueness and low procurement costs incentivize designers to choose pay-as-bid in practice.

However, the pay-as-bid auction is not optimal in terms of social welfare. This could motivate social planners to intervene with the mechanism choice.

Appendix

A Rationing rule

Consider a fixed realization of the type profile $(\theta^j)_{j=1}^N$. When the inverse $\tilde{y}^i$ does not exist the assigned quantity $\hat{y}^i$ cannot be determined as in section 2.2. Suppose player i offers quantity y at price p. That is, $\tilde{p}(\theta^i, y) = p$. If $y > 1 - \sum_{j \neq i} \underline{y}^j(\theta^j, p)$ player i is not assigned the infinitesimal unit y and $\mathbb{1}^i\big(y, p\big) = 0$. Vice versa, if $y < 1 - \sum_{j \neq i} \overline{y}^j(\theta^j, p)$ player i is assigned the infinitesimal unit y and $\mathbb{1}^i\big(y, p\big) = 1$.

Now suppose $1 - \sum_{j \neq i} \overline{y}^j(\theta^j, p) \leq y \leq 1 - \sum_{j \neq i} \underline{y}^j(\theta^j, p)$. For given quantity y and price p let $\mathcal{L}(y, p)$ be the set of players j with $\overline{y}^j(p) < y$. Let $\mathcal{M}(y, p)$ be the set of players j with $\underline{y}^j(p) \leq y$ and $\overline{y}^j(p) \geq y$. Let $\mathcal{H}(y, p)$ be the set of players j with $\underline{y}^j(p) > y$. Player i is assigned unit y at price p if and only if $\mathbb{1}^i\big(y, p\big) = 1$ with

$$\mathbb{1}^i\big(y, p\big) := \mathbb{1}\Bigg(\sum_{j \in \mathcal{L}(y,p) \setminus i} \overline{y}^j(p) + |\mathcal{M}(y, p)| y + \sum_{j \in \mathcal{H}(y,p) \setminus i} \underline{y}^j(p) \leq 1\Bigg).$$

Now I define the total quantity assigned to player i which is $\hat{y}^i\Big((\tilde{p}^k)_{k=1}^N, (\theta^k)_{k=1}^N\Big)$. Define the auxiliary quantity y^* such that

$$\sum_{j \in \mathcal{L}(y^*, \hat{p})} \overline{y}^j(\hat{p}) + |\mathcal{M}(y^*, \hat{p})| y^* + \sum_{j \in \mathcal{H}(y^*, \hat{p})} \underline{y}^j(\hat{p}) = 1.$$

By continuity and by the definition of the clearing price there is always exactly one y^* that solves the above equation. Then $\hat{y}^j = \overline{y}^j(\hat{p})$ for all j with $\overline{y}^j(\hat{p}) < y^*$ and $\hat{y}^j = \underline{y}^j(\hat{p})$ for all j with $\underline{y}^j(\hat{p}) > y^*$. The total assigned quantity is $\hat{y}^j = y^*$ for all j with $\underline{y}^j(\hat{p}) \leq y^* \leq \overline{y}^j(\hat{p})$.

The given rationing rule ensures that quantities at the market clearing price are distributed as equally as possible. At the same time it has the convenient property that the probability player i is assigned some unit y at price p does not depend on what i bids for units other than y. Woodward (2016) uses a different rationing rule that is pro-rata at the margin.[4]

B Proof of Property 2

Note that within the domain D it holds that $\tilde{\theta}_p(y,p) < 0$. Hence, the inverse $\tilde{p}$ exists. I begin by showing that $\tilde{\theta}(\cdot,p)$ is Lipschitz continuous in the sense that it possesses a Lipschitz constant L_θ such that $|\tilde{\theta}(y,p) - \tilde{\theta}(y',p)| \leq L_\theta |y - y'|$ within the domain D.

Let $x(y,p) := \begin{pmatrix} \tilde{\theta}(y,p) \\ \tilde{\theta}(1-y,p) \end{pmatrix}$ and let $f(x,y',p) := \begin{pmatrix} \frac{\tilde{\theta}(y,p)}{c(\tilde{\theta}(1-y,p),1-y')-p} \\ \frac{\tilde{\theta}(1-y,p)}{c(\tilde{\theta}(y,p),y')-p} \end{pmatrix}$. Lipschitz continuity of $\tilde{\theta}(\cdot,p)$ is then proven by showing that within domain D it holds that $\|x(y,p) - x(y',p)\| \leq L_x \|y - y'\|$ for some $L_x \in \mathbb{R}$. Using ODE (1) in integral form gives

$$\|x(y,p) - x(y',p)\| = \|x(y,\underline{p}) - x(y',\underline{p}) + \int_{\underline{p}}^{p} f(x(y,s),y,s) - f(x(y',s),y',s)ds\|.$$

By the initial condition (2), $\|x(y,\underline{p}) - x(y',\underline{p})\| = 0$. Then, the triangle inequality can be repeatedly applied.

$$\|x(y,p) - x(y',p)\| \leq \int_{\underline{p}}^{p} \|f(x(y,s),y,s) - f(x(y',s),y',s)\|ds$$

---

[4] The statement of the rationing rule by Woodward (2016) contains a slight flaw implying that the assigned quantities do generally not sum up to the total quantity. However, this could be fixed easily to receive rationing pro-rata at the margin if desired.

$$
\begin{aligned}
&\leq \int_{\underline{p}}^{p} \|f(x(y,s),y,s) - f(x(y,s),y',s)\| ds \\
&+ \int_{\underline{p}}^{p} \|f(x(y,s),y',s) - f(x(y',s),y',s)\| ds
\end{aligned}
$$

Lipschitz continuity of c implies that f has Lipschitz constants L_1, L_2 such that

$$
\begin{aligned}
\|x(y,p) - x(y',p)\| &\leq \int_{\underline{p}}^{p} L_1\|y-y'\|ds + \int_{\underline{p}}^{p} L_2\|x(y,s) - x(y',s)\|ds \\
&= L_1\|y-y'\|(p-\underline{p}) + \int_{\underline{p}}^{p} L_2\|x(y,s) - x(y',s)\|ds
\end{aligned}
$$

Applying the integral form of Grönwall's inequality gives

$$
\begin{aligned}
\|x(y,p) - x(y',p)\| &\leq L_1\|y-y'\|(p-\underline{p})exp\big(\int_{\underline{p}}^{p} L_2 ds\big) \\
&= L_1\|y-y'\|(p-\underline{p})e^{L_2(p-\underline{p})}.
\end{aligned}
$$

Taking $L_x = L_1(c(0,\frac{1}{2}) - \underline{p})e^{L_2(c(0,\frac{1}{2})-\underline{p})}$ completes the proof of Lipschitz continuity of $\tilde{\theta}(\cdot, p)$.

I continue to show that $\tilde{p}(\theta,\cdot)$ is Lipschitz continuous in the sense that it possesses a Lipschitz constant L_p such that $|\tilde{p}(\theta,y) - \tilde{p}(\theta,y')| \leq L_p|y-y'|$ within the domain D. Note that $\tilde{p}(\cdot,y)$ is Lipschitz continuous with some Lipschitz constant L_3 because $\tilde{\theta}(y,\cdot)$ is continuously differentiable with $\tilde{\theta}_p < 0$ and domain D is bounded.

$$
\begin{aligned}
|\tilde{p}(\theta,y) - \tilde{p}(\theta,y')| &= \tilde{p}(\theta,y) - \tilde{p}(\tilde{\theta}(y,\tilde{p}(\theta,y')),y) \\
&\leq L_3|\theta - \tilde{\theta}(y,\tilde{p}(\theta,y'))| \\
&= L_3|\tilde{\theta}(y',\tilde{p}(\theta,y')) - \tilde{\theta}(y,\tilde{p}(\theta,y'))| \\
&\leq L_3 L_\theta |y-y'|
\end{aligned}
$$

All the function values used above are well defined by the characterization in Proposition 1 when y' is sufficiently close to y.

C Proof of Lemma 1

This is a version of Proposition 3 by Maskin and Riley (2000). Consider the infinitesimal auction where i's bid for unit y competes with j's bid for unit $1-y$. Let $\overline{p}$ be the supremum of the support of the distribution of winning bids:

$$\overline{p} \coloneqq \sup_{\theta^i,\theta^j} \min\{\tilde{p}^i(\theta^i, y), \tilde{p}^j(\theta^j, 1-y)\}$$

A bidder bidding below $\overline{p}$ must have strictly positive expected payoff from the infinitesimal unit. Her probability of winning the unit is strictly positive and if her payoff was negative or zero a profitable deviation would be to slightly increase the bid. In line with Maskin and Riley (2000), the next step is to show that the c.d.f. of winning bids is continuous at all $p < \overline{p}$. A discontinuity at some p would imply that at least one player bids p with positive probability. Suppose both players bid p with positive probability. A profitable deviation for at least one player (depending on the tie-breaking rule) would be to bid slightly less than p instead. Suppose only player i bids p with positive probability. Then, the support of bids by player j has to have a gap on some interval $[p, p^*]$. Suppose not. With a bid slightly above p player j makes positive profit as shown in the beginning. Due to the discontinuity in the probability of winning the unit, it is a profitable deviation for j to bid slightly below p. However, bidding p with positive probability is then not optimal for i. It is a profitable deviation to bid p^* instead. Maskin and Riley (2000) then show that the support has no gaps. Suppose for a contradiction that no player bids in the interval $(p', p'') \subset [\underline{p}, \overline{p}]$ with positive probability. Because of the above, the probability of winning the infinitesimal unit with the bids p' and p'' is the same. Then, it is a profitable deviation to bid

p'' instead of bidding p' or sufficiently close to p'. Furthermore, the support of $\tilde{p}^i(\cdot)$ cannot have a gap for one player only because then it would not be optimal for the other player to bid in the interior of the gap. Thus, the probabilities of winning the unit, $Pr^i(y,\cdot)$ and $Pr^j(1-y,\cdot)$, are continuous functions.

D Proof of Lemma 2

I begin by showing that $\tilde{p}^i(\cdot,y)$ is a weakly decreasing function. This is a version of Proposition 1 by Maskin and Riley (2000). Let $p'=\tilde{p}^i(\theta',y)$. Recall that the density of the private information parameter $\theta^j\in[0,1]$ is one. As p' is a best response of player i with type θ' it holds for any $p''>p'$ that

$$\int_{\tilde{p}^j(\theta^j,1-y)\geq p'}\big(p'-c(\theta',y)\big)d\theta^j \geq \int_{\tilde{p}^j(\theta^j,1-y)\geq p''}\big(p''-c(\theta',y)\big)d\theta^j.$$

Weak monotonicity then follows, as

$$\frac{d}{d\theta'}\int_{\tilde{p}^j(\theta^j,1-y)\geq p'}\big(p'-c(\theta',y)\big)d\theta^j > \frac{d}{d\theta'}\int_{\tilde{p}^j(\theta^j,1-y)\geq p''}\big(p''-c(\theta',y)\big)d\theta^j.$$

It is never optimal for a type $\theta''>\theta'$ to bid above p'. The above inequality can be written as

$$\int_{\tilde{p}^j(\theta^j,1-y)\geq p'}-c_\theta(\theta',y)d\theta^j > \int_{\tilde{p}^j(\theta^j,1-y)\geq p''}-c_\theta(\theta',y)d\theta^j.$$

By the properties of the cost function, $-c_\theta$ is positive everywhere. The probability of winning with p' must be strictly larger than the probability of winning with p''. Otherwise it would not be optimal to bid p' for type θ'. The statement then holds by the finding of Lemma 1. Combining the result with the continuity argument from Lemma 1 gives strict monotonicity. As the distribution of bids has no mass points in relevant regions it holds that $\theta''>\theta'$ implies that $\tilde{p}^i(\theta'',y)<\tilde{p}^i(\theta',y)$.

The inverse $\tilde{\theta}^i(y,\cdot)$ exists. Note that the proof of weak monotonicity also goes through for $N>2$ by adjusting the notation of integration.

E Proof of Lemma 3

This is a version of Lemma 7 by Maskin and Riley (2003). By the definition of the locally optimal equilibrium and the finding of Lemma 2, an equilibrium bid function $\tilde{p}^i(\cdot)$ has to satisfy the following for all relevant units.

$$\begin{aligned}\tilde{p}^i(\theta,y) &= \arg\max_p \tilde{\theta}^j(1-y,p)(p-c(\theta,y))\\ &= \arg\max_p log\big(\tilde{\theta}^j(1-y,p)\big)+log\big(p-c(\theta,y)\big)\end{aligned}$$

Consider any $\hat{p}$ such that $\tilde{\theta}^j(1-y,\hat{p})\in(0,1]$. I show that in any locally optimal equilibrium $\frac{d}{dp}log(\tilde{\theta}^j(1-y,p))$ exists and is continuous at $p=\hat{p}$. In the following I show right-continuity. The proof of left-continuity is analogous. As shown above, $\tilde{\theta}^j(1-y,\cdot)$ and $\tilde{\theta}^i(y,\cdot)$ are strictly decreasing and continuous. For a strictly decreasing sequence $\{p_t\}_{t=0}^{\infty}$ converging to $\hat{p}$ the sequence $\{\tilde{\theta}^i(y,p_t)\}_{t=0}^{\infty}$ converges to $\tilde{\theta}^i(y,\hat{p})$ monotonically from below. Because p_t is optimal for player i with type $\tilde{\theta}^i(y,p_t)$ it holds that

$$log(\tilde{\theta}^j(1-y,\hat{p}))+log\big(\hat{p}-c(\tilde{\theta}^i(y,p_t),y)\big)\le log(\tilde{\theta}^j(1-y,p_t))+log\big(p_t-c(\tilde{\theta}^i(y,p_t))\big)$$

and

$$\frac{log(\tilde{\theta}^j(1-y,p_t))-log(\tilde{\theta}^j(1-y,\hat{p}))}{p_t-\hat{p}}\ge\frac{log(\hat{p}-c(\tilde{\theta}^i(y,p_t),y))-log(p_t-c(\tilde{\theta}^i(y,p_t),y))}{p_t-\hat{p}}. \tag{6}$$

Because $\hat{p}$ is optimal for a player with type $\tilde{\theta}^i(y,\hat{p})$ it holds that

$$log(\tilde{\theta}^j(1-y,\hat{p}))+log(\hat{p}-c(\tilde{\theta}^i(y,\hat{p}),y))\ge log(\tilde{\theta}^j(1-y,p_t))+log(p_t-c(\tilde{\theta}^i(y,\hat{p}),y))$$

and

$$\frac{log(\tilde{\theta}^j(1-y,p_t)) - log(\tilde{\theta}^j(1-y,\hat{p}))}{p_t - \hat{p}} \leq \frac{log(\hat{p} - c(\tilde{\theta}^i(y,\hat{p}),y)) - log(p_t - c(\tilde{\theta}^i(y,\hat{p}),y))}{p_t - \hat{p}}. \tag{7}$$

Now let p_t converge to $\hat{p}$. The left-hand side of equations (6) and (7) is then equal to $\left[\frac{d}{dp}log(\tilde{\theta}^j(1-y,p))\right]_{p=\hat{p}}$. The right-hand side of equations (6) and (7) then implies that this exists and is given by

$$\left[\frac{d}{dp}log(p - c(\tilde{\theta}^i(y,\hat{p}),y))\right]_{p=\hat{p}} = \frac{1}{\hat{p} - c(\tilde{\theta}^i(y,\hat{p}),y)}.$$

This is a continuous function. Hence, in any locally optimal equilibrium it has to hold that

$$\frac{\tilde{\theta}^j_p(1-y,p)}{\tilde{\theta}^j(1-y,p)} = \frac{1}{p - c(\tilde{\theta}^i(y,p),y)}$$

for any relevant unit y. This is equivalent to ODE (1). The same differential equation can be derived using first-order conditions of profit maximization at every infinitesimal unit.

F Proof of Lemma 4

I show in the first step that the offer curve for $\theta = 1$ has to be flat in the sense that there exists some $\underline{p}$ such that $\tilde{\theta}^i\left(y,\underline{p}\right) = 1$ for every i and every $y \in (0,1)$. Suppose this is not the case. Then, there is an infinitesimal unit $\varepsilon > 0$ and a player i with type $\theta^i = 1$ that can increase her bid to some $p' > \tilde{p}^i(1,\varepsilon)$ while maintaining $Pr^i(\varepsilon,p') = 1$. This deviation would strictly increase profits. For an illustration Figure 3 in section 4 is useful.

Now I show that in any locally optimal equilibrium $\underline{p} = \int_0^1 c(\theta,\frac{1}{2})d\theta$. It has to hold that $\tilde{p}^1(0,1/2) = \tilde{p}^2(0,1/2) = c(0,1/2)$. Suppose for a contradiction that there is a player i with $\tilde{p}^i(0,1/2) > c(0,1/2)$. Then player $j \neq i$ can profit from deviating by slightly undercutting $\tilde{p}^i(0,1/2)$. Now suppose that there is a player i with $\tilde{p}^i(0,1/2) < c(0,1/2)$. Then, there is at least one player that makes losses at $y = 1/2$ for types close to $\theta = 0$. A profitable deviation is to increase the bids for these types.

Consider (3) at $y = \frac{1}{2}$ and since y is fixed drop it from the notation of all functions. By the local Picard-Lindelöf theorem this system with initial condition $\tilde{\theta}^i(c(0,1/2)) = 0$ derived above has a unique solution. As the indices in (3) can be freely exchanged the unique solution has to be symmetric and I can write the system as the single equation (8).

$$\tilde{\theta}_p(p) = \frac{\tilde{\theta}(p)}{c(\tilde{\theta}(p)) - p} \tag{8}$$

This is then an exact differential equation with potential function

$$\Phi(p,\theta) := \int_0^\theta c(t)dt - p\theta.$$

The exact differential equation is handled by rearranging equation (8).

$$-\tilde{\theta}(p) + \tilde{\theta}_p(p)\Big(c(\tilde{\theta}(p)) - p\Big) = 0$$

This can be written as a derivative of the function $\Phi(p,\tilde{\theta}(p))$. Let Φ_p and Φ_θ be the partial derivatives of Φ with respect to the first and second argument.

$$\Phi_p(p,\tilde{\theta}(p)) + \tilde{\theta}_p(p)\Phi_\theta(p,\tilde{\theta}(p)) = \frac{d}{dp}\Phi(p,\tilde{\theta}(p)) = 0$$

Integrating this gives

$$\Phi(p,\tilde{\theta}(p)) - \Phi(p_0,\theta_0) = 0.$$

With an initial condition (p_0, θ_0) this implicitly defines the solution $\tilde{\theta}(\cdot)$. From the initial condition stated above take $(p_0, \theta_0) = (c(0, 1/2), 0)$ and plug in to get $\Phi(p_0, \theta_0) = 0$. The solution to (8) is then implicitly defined by

$$\Phi(p, \tilde{\theta}(p)) = 0.$$

Plugging in, this is equivalent to

$$\int_0^{\tilde{\theta}(p)} c(t)dt - p\,\tilde{\theta}(p) = 0.$$

This is then finally solved for $\underline{p}$ by plugging in $p = \underline{p}$ and $\tilde{\theta}(\underline{p}) = 1$.

G Proof of Lemma 5

To prove Lemma 5 I first prove a useful property (Property 3) of $\tilde{p}$ that results from the equilibrium characterization. Let $y < \frac{1}{2}$. For any $\theta \in (0, 1]$ it holds that $\tilde{p}(\theta, y) \leq \tilde{p}(\theta, 1-y)$. I have shown above that $\tilde{p}(\cdot, y)$ is strictly decreasing. Then I can provide the desired proof by showing that $\tilde{\theta}(y, p) \leq \tilde{\theta}(1-y, p)$ at $p = \tilde{p}(\theta, y)$. At the initial condition $\tilde{\theta}(y, \underline{p}) = \tilde{\theta}(1-y, \underline{p}) = 1$. Note that

$$\tilde{\theta}_p(y, \underline{p}) = \frac{1}{c(1, 1-y) - \underline{p}} < \frac{1}{c(1, y) - \underline{p}} = \tilde{\theta}_p(1-y, \underline{p}).$$

Now suppose for a contradiction that $\tilde{\theta}(y, p) > \tilde{\theta}(1-y, p)$ for some price larger than $\underline{p}$. Let p^* be the infimum of all such prices. By continuity it must then hold that $\tilde{\theta}(y, p^*) = \tilde{\theta}(1-y, p^*)$. By the properties of the cost function it holds again that

$$\tilde{\theta}_p(y, p^*) = \frac{\tilde{\theta}(y, p^*)}{c(\tilde{\theta}(1-y, p^*), 1-y) - p^*} < \frac{\tilde{\theta}(1-y, p^*)}{c(\tilde{\theta}(y, p^*), y) - p^*} = \tilde{\theta}_p(1-y, p^*).$$

Note that $\tilde{\theta}$ is differentiable. The above then implies that $\tilde{\theta}(y,p) > \tilde{\theta}(1-y,p)$ for some $p \in (\underline{p}, p^*)$ which contradicts the definition of p^* and completes the proof of the auxiliary property.

Now proceed with the proof of Lemma 5. For $y \geq \frac{1}{2}$ the Lemma follows from $Pr^i(y,p) = 0$ and the fact that the profits from the characterized bids are non-negative. This is proven in the proof of Lemma 7. Consider now $y < \frac{1}{2}$. Bids for units that are not *relevant* are not characterized in Proposition 1 but are important for players considering deviations from the characterized equilibrium strategies. It has to be shown that there exists an admissible extension of the characterized equilibrium bid functions to irrelevant units such that there exists no profitable deviation from the bids for *relevant* units. For a given type θ the set of irrelevant units is the set of units $y \geq y^*$ with y^* defined by $\tilde{p}(\theta, y^*) = lim_{t \to 0} \tilde{p}(t, 1-y^*)$. For irrelevant units $y \geq y^*$ I use the bid function $\tilde{p}(\theta, y) = \tilde{p}(\theta, y^*)$. The bid function is just a flat extension of the bid function for *relevant* units. Note that then expected profits are everywhere continuous in the bid.

The expected profit from the equilibrium bid for player i for a given unit $y < \frac{1}{2}$ is $\pi(p) = \tilde{\theta}^j(1-y,p)(p - c(\theta,y)) \geq 0$. Deviating to some $p' \geq \tilde{p}(0, \frac{1}{2})$ is not profitable because then $Pr^i(y,p') = 0$. It remains to be shown that deviating to $p' \in (\tilde{p}(0,y), \tilde{p}(0,\frac{1}{2}))$ is not profitable. Consider such a p'. There exists a corresponding unit $y' \in (y, \frac{1}{2})$ such that $\tilde{p}(0,y') = p'$. Note that $lim_{p \nearrow p'} \tilde{\theta}_p(1-y',p)(p - c(0,y')) + \tilde{\theta}(1-y',p) = 0$. The first-order condition of local profit maximization for type $\theta = 0$ holds at y'. Now compare this to the profit function at unit y when deviating to $p' > p$. Because bids for irrelevant units are flat it holds that $\tilde{\theta}(1-y,p') = \tilde{\theta}(1-y',p')$. By the properties of the cost function it holds that $p' - c(\theta,y) > p' - c(\theta,y')$. The markup at y is higher than the markup at y' and the player has more to loose. Finally, because bids for irrelevant units are flat it holds that $\tilde{\theta}_p(1-y,p') < \tilde{\theta}_p(1-y',p')$. At y the probability of winning the infinitesimal auction decreases more strongly in the bid compared to y' for bids around p'. In conclusion, for all $p' \in (\tilde{p}(0,y), \tilde{p}(0,\frac{1}{2}))$ expected profits are strictly decreasing in the bid. Combining with continuity of the expected profit function this yields that deviating to such a p' would strictly decrease expected profits.

H Proof of Lemma 6

The necessary condition for this is given by the system of ordinary differential equations (1) with initial condition (2). A sufficient condition is given by Wolfstetter (1996) for single-unit auctions, which can be readily applied to all infinitesimal units in my setup as follows.

Let $(\tilde{p}^j)_{j=1}^N$ be the strategies characterized by (1) and (2) and let $Pr^i(y,p)$ be the winning probability function of player i following from $(\tilde{p}^j)_{j\neq i}$. Consider player i with type θ^i who optimally chooses a bid for unit y. The expected utility from that unit when bidding p is

$$EU^i(\theta^i, y, p) := Pr^i(y,p)(p - c(\theta^i, y)).$$

The characterized $\tilde{p}^i$ satisfies

$$EU^i_p\big(\theta^i, y, \tilde{p}^i(\theta^i,y)\big) = Pr^i_p\big(y, \tilde{p}^i(\theta^i,y)\big)\big(\tilde{p}^i(\theta^i,y) - c(\theta^i,y)\big) + Pr^i\big(y, \tilde{p}^i(\theta^i,y)\big) = 0.$$

The proof is completed by showing that $EU^i_p(\theta^i,y,p) > 0$ for $p < \tilde{p}^i(\theta^i,y)$ and $EU^i_p(\theta^i,y,p) < 0$ for $p > \tilde{p}^i(\theta^i,y)$. Differentiate $EU^i_p(\theta^i,y,p)$ with respect to θ^i to get

$$EU^i_{\theta^i p} = -Pr^i_p(y,p)c_\theta(\theta^i,y) < 0.$$

Let $\underline{p} \leq p < \tilde{p}^i(\theta^i, y)$. Let $\hat{\theta}$ be the type with $\tilde{p}^i(\hat{\theta}, y) = p$. Then it holds that $\hat{\theta} > \theta^i$. Combining all the above yields

$$EU_p^i(\theta^i, y, p) > EU_p^i(\hat{\theta}, y, p) = 0.$$

The proof works analogously for $p > \tilde{p}^i(\theta^i, y)$.

I Proof of Lemma 7

Bidding strictly below $\underline{p}$ gives zero expected profit. Bidding according to the characterized equilibrium strategies gives nonnegative expected profit. This is because a solution $(\tilde{\theta}^i)_{i=1}^2$ to (1) and (2) satisfies $c\big(\tilde{\theta}^i(y, p), y\big) < p$ for any $p \in [\underline{p}, \overline{p})$. To see this, note first that at the initial condition it holds that $c\big(\tilde{\theta}^i(y, \underline{p}), y\big) < \underline{p}$. Even though $y = 1$ is not relevant, by monotonicity of the cost function, expected profits are nonnegative for all types and all relevant units if and only if expected profits are non-negative for $\theta = 1$ and $y = 1$ at the initial value. This is the case if and only if $c(1, 1) \leq \underline{p}$ which has to hold because otherwise the condition on the image would be violated. Suppose now that $c\big(\tilde{\theta}^i(y, p), y\big) < p$ is violated for some $p' \in (\underline{p}, \overline{p})$. By continuity this implies the existence of a $p'' \in (\underline{p}, p']$ such that $c\big(\tilde{\theta}^i(y, p''), y\big) = p''$. This violates Lipschitz continuity of the ODE and the respective unit cannot be relevant.

J Proof of Proposition 3

First, note that the symmetry of strategies implies that Pr^i is symmetric and can simply be denoted by Pr. Lemma 3 can be applied to show that $Pr(y, \cdot)$ has to be differentiable and has to satisfy (4). I continue with the derivation of the expression for $Pr(y, p)$. Note that monotonicity of $\tilde{\theta}(y, \cdot)$ holds by the proof of Lemma 2. Differentiability implies that Pr is continuous. Therefore, ties have probability zero and I can write by using density one for θ and by consecutively applying integration by substitution with $\theta^i = \tilde{\theta}(s_i, p)$ that

$$\begin{aligned}
Pr(y,p) = &\int_0^{\tilde{\theta}(1-y-(N-2)\underline{y}(0,p),p)} \int_0^{\tilde{\theta}(1-y-(N-3)\underline{y}(0,p)-\underline{y}(\theta^1,p),p)} \ldots \int_0^{\tilde{\theta}(1-y-\sum_{j=1}^{N-2}\underline{y}(\theta^j,p),p)} \\
&1\, d\theta^{N-1} \ldots d\theta^2\, d\theta^1 \\
= &\int_0^{\tilde{\theta}(1-y-(N-2)\underline{y}(0,p),p)} \int_0^{\tilde{\theta}(1-y-(N-3)\underline{y}(0,p)-\underline{y}(\theta^1,p),p)} \ldots \int_0^{\tilde{\theta}(1-y-\underline{y}(0,p)-\sum_{j=1}^{N-3}\underline{y}(\theta^j,p),p)} \\
&\tilde{\theta}(1-y-\sum_{j=1}^{N-2}\underline{y}(\theta^j,p),p)\, d\theta^{N-2} \ldots d\theta^2\, d\theta^1 \\
= &\int_{\underline{y}(0,p)}^{1-y-(N-2)\underline{y}(0,p)} \int_{\underline{y}(0,p)}^{1-y-(N-3)\underline{y}(0,p)-s_1} \ldots \int_{\underline{y}(0,p)}^{1-y-\underline{y}(0,p)-\sum_{j=1}^{N-3}s_{N-3}} \\
&\tilde{\theta}\big(1-y-\sum_{j=1}^{N-2}s_j,p\big)\; \Pi_{j=1}^{N-2}\tilde{\theta}_y(s_j,p)ds_{N-2}\ldots ds_2 ds_1 \\
+ &\sum_{k=1}^{N-2}\binom{N-2}{k}\tilde{\theta}(\underline{y}(0,p),p)^k \int_0^{1-y}\int_0^{1-y-s_1} \ldots \int_0^{1-y-\sum_{j=1}^{N-3-k}s_j} \\
&\tilde{\theta}\big(1-y-\sum_{j=1}^{N-2}s_j,p\big)\; \Pi_{j=1}^{N-2-k}\tilde{\theta}_y(s_j,p)ds_{N-2-k}\ldots ds_2 ds_1.
\end{aligned}$$

K Proof of Proposition 4

Suppose that $(\tilde{p}^i)_{i=1}^N$ form a Bayes-Nash equilibrium of G that is not *locally optimal* and not *strategic ironing*. Begin with an auxiliary result for the main argument of this proof.

Lemma K.1. *For every i, every $\theta^i \in [0,1]$ and almost every relevant $y \in [0,1]$, $\tilde{p}^i(\theta^i, y)$ is a local maximizer of expected utility in the sense that there exists an $\varepsilon > 0$ such that*

$$\tilde{p}^i(\theta^i, y) \in \arg\max_{p\in(\tilde{p}^i(\theta^i,y)-\varepsilon,\tilde{p}^i(\theta^i,y)+\varepsilon)} Pr^i(y,p)(p - c(\theta^i, y)).$$

Proof. Suppose for a contradiction that there is an i and a θ^i with a set $\underline{\mathcal{Y}}^i \subset \mathcal{Y}^i$ of nonzero measure where $\tilde{p}^i(\theta^i, y)$ is not a local maximizer for $y \in \underline{\mathcal{Y}}^i$. Since $\underline{\mathcal{Y}}^i$ has nonzero measure, by Lebesgue's density theorem, I can find some relevant units y^* and y^{**} with $0 < y^* < y^{**} < 1$ such that the density of $\underline{\mathcal{Y}}^i$ in $[y^*, y^{**}]$ is arbitrarily close to one where the density is defined as

$$\frac{\mu\big(\underline{\mathcal{Y}}^i \cap [y^*, y^{**}]\big)}{\mu\big([y^*, y^{**}]\big)}$$

with μ being the corresponding Lebesgue measure. Now define the two sets $\mathcal{Y}_H$ and $\mathcal{Y}_L$. To shorten notation I write $EU^i(y,p) := Pr^i(y,p)(p - c(\theta^i, y))$.

$$\mathcal{Y}_H := \{y \in \underline{\mathcal{Y}}^i \,|\, \exists \varepsilon > 0 : EU^i(y,p) > EU^i(y, \tilde{p}^i(\theta^i, y)) \,\forall p \in (\tilde{p}^i(\theta^i, y), \tilde{p}^i(\theta^i, y) + \varepsilon]\}$$

$$\mathcal{Y}_L := \{y \in \underline{\mathcal{Y}}^i \,|\, \exists \varepsilon > 0 : EU^i(y,p) > EU^i(y, \tilde{p}^i(\theta^i, y)) \,\forall p \in (\tilde{p}^i(\theta^i, y) - \varepsilon, \tilde{p}^i(\theta^i, y)]\}$$

Note that $\mathcal{Y}_H \cup \mathcal{Y}_L = \underline{\mathcal{Y}}^i$. Suppose there is an element y_0 of $\underline{\mathcal{Y}}^i$ that is not an element of either $\mathcal{Y}_H$ or $\mathcal{Y}_L$. Then, to be an element of $\underline{\mathcal{Y}}^i$ the function EU^i would need to be discontinuous in p everywhere in a neighborhood of $\tilde{p}^i(\theta^i, y_0)$ such that it jumps above and weakly below $EU^i(y_0, \tilde{p}^i(\theta^i, y_0))$ as p changes. This is not possible because Pr^i can only jump downwards as p increases but not upwards. Having established $\mathcal{Y}_H \cup \mathcal{Y}_L = \underline{\mathcal{Y}}^i$ implies that at least one of the two sets needs to have nonzero measure. Suppose $\mathcal{Y}_H$ has nonzero measure. Again by Lebesgue's density theorem, I can find some relevant units y' and y'' with $y^* \leq y' < y'' \leq y^{**}$ such that the density of $\mathcal{Y}_H$ in $[y', y'']$ is arbitrarily close to 1. Now define $\underline{\varepsilon}$ as follows.

$$\underline{\varepsilon} := \inf\{\varepsilon \,|\, EU^i(y,p) > EU^i(y, \tilde{p}^i(\theta^i, y)) \,\forall p \in (\tilde{p}^i(\theta^i, y), \tilde{p}^i(\theta^i, y) + \varepsilon] \,\forall y \in [y', y''] \cap \mathcal{Y}_H\}$$

Then I get a contradiction because $(\tilde{p}^i)_{i=1}^2$ cannot be a Bayes-Nash equilibrium. A profitable deviation would be for i to bid $\min\{\tilde{p}^i(\theta^i, y''), \tilde{p}^i(\theta^i, y') + \underline{\varepsilon}\}$ for all

$y \in [y', y'']$ with $\tilde{p}^i(\theta^i, y) \leq \tilde{p}^i(\theta^i, y') + \underline{\varepsilon}$. The contradiction can be analogously shown when $\mathcal{Y}_L$ has nonzero measure. $\square$

The proof of Lemma 6 now shows that strategies $(\tilde{p}^i)_{i=1}^N$ are actually global maximizers and therefore form a *locally optimal* equilibrium which is a contradiction. To apply Lemma 6 in the case with $N > 2$ I need the additional assumption of differentiability. With $N = 2$ differentiability follows from the proof of Lemma 3. Proposition 4 then follows as a corollary of Lemma K.1.

L Monotonicity of c and $\tilde{p}$

Consider some $0 < y < y' < \frac{1}{2}$ with marginal costs as below.

$$c(\theta, y) = \frac{6}{1.2\theta + 1.8} - 2$$

$$c(\theta, y') = 1.4 - \theta$$

$$c(\theta, 1 - y') = 1.7 - 1.2\theta$$

$$c(\theta, 1 - y) = 0.6(2 - \theta)^2$$

Note that $c(\theta, y) < c(\theta, y') < c(\theta, 1 - y') < c(\theta, 1 - y)$ for any $\theta \in [0, 1]$. Marginal costs satisfy the monotonicity condition. Consider now the equilibrium candidate given below.

$$\tilde{p}(\theta, y) = \frac{3}{1.2\theta + 1.8}$$

$$\tilde{p}(\theta, y') = 1.5 - 0.5\theta$$

$$\tilde{p}(\theta, 1 - y') = 1.6 - 0.6\theta$$

$$\tilde{p}(\theta, 1 - y) = 2 - \theta$$

Note that for $\theta < 0.5$ it holds that $\tilde{p}(\theta, y) > \tilde{p}(\theta, y')$. Monotonicity is violated in this example. Hence, the above is not an equilibrium as $\tilde{p}$ is not a valid strategy.

M Equilibrium example with three players

I take the offer curves $\tilde{p}(\theta, y) = 2 + \frac{1-\theta}{2-y}$ and plug into the characterization from Proposition 3 to verify whether they can supported as an equilibrium for some cost function. It follows that

$$\begin{aligned} Pr(y,p) = \mathbb{1}(p < 2.5)\left(\frac{1}{2}p^2 + 2y^2 + \frac{1}{2}p^2y^2 - 4p + 3yp^2 + 16y - 14py - 2py^2 + 7\right) \\ + \mathbb{1}\Bigg(p \geq 2.5)(-\frac{1}{2}(2y - p - py + 3)^2 + (2y - p - py + 3)(2 - (3 + y)(p - 2)) \\ + \frac{1}{2}(2p - 5)^2\Bigg). \end{aligned}$$

Then, the marginal cost function that gives rise to the above offer curves is given by

$$c(\theta, y) = \tilde{p}(\theta, y) + \frac{Pr(y, \tilde{p}(\theta, y))}{Pr_p(y, \tilde{p}(\theta, y))}.$$

Figure 5 depicts offer curves and marginal costs. Note that the region where bids are higher than marginal costs is the region of irrelevant units where bids are accepted with probability zero. The only remaining condition for equilibrium

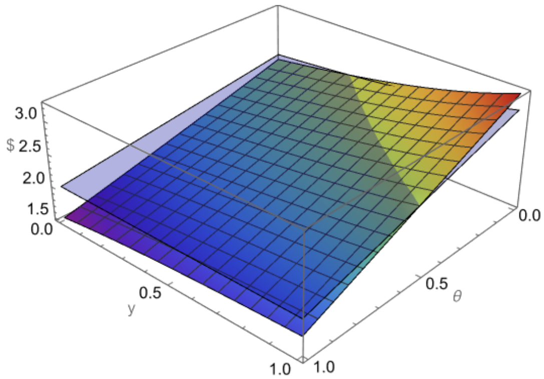


Figure 5: Offer curves and marginal costs

sufficiency of the above offer curve is to rule out profitable deviations to prices above the support of bids for quantities $y < 1/3$. By the argument from the proof of Lemma 6, it is sufficient to rule out such deviations for type $\theta = 0$. Figure 6 shows profits of type $\theta = 0$ from possible deviations. Prices above 2.6 do not need to be considered because $\tilde{p}(0, 1/3) = 2.6$. When offer curves in the irrelevant regions are assumed to be flat, bidding above 2.6 gives zero profit. The red line in Figure 6 indicates the profit from bidding $\tilde{p}(0, y)$. Deviations are not profitable.

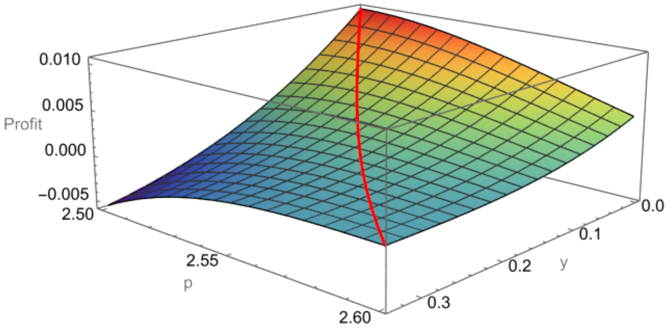


Figure 6: Profit of $\theta = 0$ from deviating.

References

Armantier, O. and Sbaï, E. (2006). “Estimation and comparison of treasury auction formats when bidders are asymmetric”. *Journal of Applied Econometrics* 21.6, pp. 745–779.

Athey, S. (2001). “Single crossing properties and the existence of pure strategy equilibria in games of incomplete information”. *Econometrica* 69.4, pp. 861–889.

Ausubel, L. M., Cramton, P., Pycia, M., Rostek, M., and Weretka, M. (2014). “Demand reduction and inefficiency in multi-unit auctions”. *The Review of Economic Studies* 81.4, pp. 1366–1400.

Chakraborty, I. (2006). “Characterization of equilibrium in pay-as-bid auctions for multiple units”. *Economic theory* 29.1, pp. 197–211.

Engelbrecht-Wiggans, R. and Kahn, C. M. (1998). “Multi-unit pay-your-bid auctions with variable awards”. *Games and Economic Behavior* 23.1, pp. 25–42.

Ewerhart, C., Cassola, N., and Valla, N. (2010). “Declining valuations and equilibrium bidding in central bank refinancing operations”. *International Journal of Industrial Organization* 28.1, pp. 30–43.

Fabra, N., von der Fehr, N.-H., and Harbord, D. (2006). “Designing electricity auctions”. *The RAND Journal of Economics* 37.1, pp. 23–46.

Federico, G. and Rahman, D. (2003). “Bidding in an electricity pay-as-bid auction”. *Journal of Regulatory Economics* 24.2, pp. 175–211.

Hästö, P. and Holmberg, P. (2006). “Some inequalities related to the analysis of electricity auctions”. *Applied Mathematics Letters* 19.8, pp. 814–819.

Holmberg, P. and Wolak, F. A. (2018). “Comparing auction designs where suppliers have uncertain costs and uncertain pivotal status”. *The RAND Journal of Economics* 49.4, pp. 995–1027.

Hortaçsu, A. and McAdams, D. (2010). “Mechanism choice and strategic bidding in divisible good auctions: An empirical analysis of the turkish treasury auction market”. *Journal of Political Economy* 118.5, pp. 833–865.

Kang, B.-S. and Puller, S. L. (2008). “The effect of auction format on efficiency and revenue in divisible goods auctions: A test using Korean treasury auctions”. *The Journal of Industrial Economics* 56.2, pp. 290–332.

Kastl, J. (2012). “On the properties of equilibria in private value divisible good auctions with constrained bidding”. *Journal of Mathematical Economics* 48.6, pp. 339–352.

Klemperer, P. D. and Meyer, M. A. (1989). “Supply function equilibria in oligopoly under uncertainty”. *Econometrica: Journal of the Econometric Society*, pp. 1243–1277.

Lamp, S., Samano, M., and Tiedemann, S. (2024). “Firms’ Bidding Behavior in a New Market: Evidence from Renewable Energy Auctions”. *Available at SSRN*.

Lebrun, B. (1997). *First Price Auctions in the Asymmetric N Bidder Case*. Tech. rep. Université Laval-Département d’économique.

— (2006). “Uniqueness of the equilibrium in first-price auctions”. *Games and Economic Behavior* 55.1, pp. 131–151.

Maskin, E. and Riley, J. (2000). "Equilibrium in sealed high bid auctions". *The Review of Economic Studies* 67.3, pp. 439–454.

— (2003). "Uniqueness of equilibrium in sealed high-bid auctions". *Games and Economic Behavior* 45.2, pp. 395–409.

McAdams, D. (2003). "Isotone equilibrium in games of incomplete information". *Econometrica* 71.4, pp. 1191–1214.

Pycia, M. and Woodward, K. (2023). "A Case for Pay-as-Bid Auctions".

Reny, P. J. (2011). "On the existence of monotone pure-strategy equilibria in Bayesian games". *Econometrica* 79.2, pp. 499–553.

Song, Z. and Zhu, H. (2018). "Quantitative easing auctions of treasury bonds". *Journal of Financial Economics* 128.1, pp. 103–124.

Vickrey, W. (1961). "Counterspeculation, auctions, and competitive sealed tenders". *The Journal of finance* 16.1, pp. 8–37.

Vives, X. (2011). "Strategic supply function competition with private information". *Econometrica* 79.6, pp. 1919–1966.

Wang, J. J. and Zender, J. F. (2002). "Auctioning divisible goods". *Economic theory* 19, pp. 673–705.

Wolfstetter, E. (1996). "Auctions: an introduction". *Journal of economic surveys* 10.4, pp. 367–420.

Woodward, K. (2016). "Strategic Ironing in Pay-as-Bid Auctions: Equilibrium Existence with Private Information."

Woodward, K. (2019). “Equilibrium with monotone actions”.