History Dependence of Pension Systems

Sahber Ahmadi-Renani*
Tehran Institute for Advanced Studies,
Khatam University

First Version: January 23, 2023
This Version: February 16, 2025

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Abstract

In countries employing a defined-benefit system, pension benefits are typically governed by composition of two functions: the first summarises earnings history into a single variable (the "history of earnings" function), and the second computes benefits based on this variable (the "pension benefit" function). History dependence of a pension system affects the timing of retirement and the level of consumption insurance of a retiree by governing how her pension benefit is influenced by the profile of lifetime labor market shocks. This is particularly important for individuals with low attachment to the workforce, i.e. workers who experience long unemployment periods and women who leave the labor market at child-bearing ages. This is the first paper that develops and estimates a model to quantitatively access the incentive, insurance, and redistribution built in the design of the history of earnings function of public pension systems and study different counterfactual policies as well as its interaction with the (non-history-dependent) personal income tax system. First, I present new stylized facts regarding the lifetime earnings of workers over the life cycle and show how different ways to summarise this lifetime history of earnings would affect workers' pension benefits. Next, I develop a novel numerical algorithm based on deep neural nets to solve life cycle models with expanding state space. I utilize this algorithm to solve and estimate a rich dynamic model of labor supply, saving, and retirement with different labor market shocks. Then, I study various counterfactual policies including a change from the current US pension policy which uses the top 35 years of earnings to a pension system that accounts for the lifetime earnings, common in other OECD countries.

*This paper is based on the second chapter of my Ph.D. dissertation at the university of Wisconsin, Madison. I am grateful to my advisors Matthew Wiswall, Naoki Aizawa, and Corina Mommaerts. I thank Zahra Delbari for outstanding research assistant. I thank Jeffrey Smith, Christopher Taber, Jesse Gregory, and Yadollah Yaghoobzadeh for their helpful comments. All the errors are mine.

1 Introduction

The modest numbers on the bottom right of social security checks are the main sources of income for many American retirees and the last barrier for keeping them out of poverty. In addition, the incentive system that they build shape the leading public policy in influencing the retirement age of workers, as well as their labor supply in old age.

A retiree's public pension benefit is based on the history of her earnings before retirement. In the US, like most countries, two functions working in the composite determine the amount of pension benefits. One function summarizes all of the history of earnings into one outcome, and the other finds the amount of benefit based on that outcome. In the US, ignoring some details, the history of earnings will be summarized by taking the average of the top 35 years of earnings of a worker. Then, this result will yield the retirement benefit after passing through a progressive benefit function. Studying the ramifications of the design of the history-dependent part of the pension system and exploring the paths to improve it is the main focus of this paper.

It is crucial that this design be guided by how it affects the redistribution, insurance, and incentives system that it creates for workers. As it will be discussed in Section 2, different countries institute different sets of rules to calculate the pension benefits from the history of earnings. This creates a different balance of these often conflicting criteria in each country.

- **Redistribution:** The social security system is an integral part of the overall redistribution system that taxes and transfers construct. Workers who have the less steep trajectory of earnings over the life cycle would benefit more if years of the highest earnings (which mostly happen late in the life cycle) have less influence in determining their pension benefits. In addition, the workers with less attachment to the workforce who experience many years of no earnings would be overlooked if an extended number of years are counted in determining the eligibility and the amount of the pension benefit. Moreover, as the only part of the redistribution system that is history-dependent, pension benefits can be used to correct the negative redistribution toward workers with high fluctuations in their earnings caused by progressive taxation.

- **Insurance:** If negative earning shocks during working life heavily influence the workers' resources during their retirement, the system is not providing enough insurance for workers. More insurance can be achieved by lowering the weight of low-earning years on determining retirement benefits and providing a non-history dependent part for workers with limited earning history (i. e. a consumption floor).

- **Incentives:** In old age, the decisions on how many hours to work and how much longer to stay in the labor force are heavily influenced by how these decisions affect the amount of pension benefits. If the earnings of these last years are highly weighted in determining the benefits, the system is providing more work incentives. Moreover, the number of years required from individuals to be able to receive pension benefits (in full or in part) incentivizes individuals to build a history of earnings by not leaving the labor force in the case of the labor market and family shocks.

The history-dependent part of pension systems is vastly understudied. Although many papers have researched design of the benefit function, design of the summarizing function has been mostly ignored[1]

Understanding the consequences of the design of the history-dependent part of the pension system is the goal of this research. To the best of my knowledge, this is the first attempt to quantitatively study this part of the tax and transfer system. First, I utilize a long panel of data from the Panel Study of Income Dynamics (PSID) and introduce various stylized facts regarding retirement age, labor supply, and wages over the life cycle, and how different ways to summarise the history of earnings of individuals affect them. Some of the presented empirical regularities are updates of empirical facts in the literature with longer time series that span the whole life cycle of individuals, and some are new empirical regularities.

I show that when a smaller number the high-earning years are considered, the mean of average income increases and the standard deviation decreases. The more low-earning years are discarded, the system provides more insurance for the workers, as the negative years will not be considered in the function of the history of earnings. However, it also means more weight to the highest earning years which already represent more inequality than other years. The result is more inequality as fewer years are counted. On the other hand, workers' earnings fall for all workers late during their life cycle. Hence, when we use the latest years of earnings instead of the highest years of earnings, increasing the number of years decreases the mean of average earnings. As inequality is highest late in the life cycle, the concentration of average earnings also decreases by counting more years.

Next, I develop a novel numerical algorithm based on deep neural nets to solve life cycle models with expanding state space, specially suitable for environments where the whole history of a variable (here earnings) matter. I utilize this algorithm and solve a life cycle model of labor supply, saving, and retirement with various labor market shocks. I utilize the data from the PSID to estimate the quantitative model. I employ the quantitative model and evaluate the effects of an important counterfactual policy. The new policy changes the current US pension policy which uses the top 35 years of earnings to account for all of the lifetime earnings, common in other OECD countries. Moving to the new policy regime causes 43 % rise in consumption. Both college-educated and non-college-educated groups benefit from this policy. Part of this higher consumption comes from higher pension benefits as the PIA rules remain the same. It also results in 64 % increase in hours of work, 5.7 % decrease in the age of retirement.

1.1 Related Literature

This paper mainly contributes to the literature on the design of pension systems (Diamond and Mirrlees (1978), Diamond and Mirrlees (1986) Huggett and Parra (2010), Golosov et al. (2013), Shourideh and Troshkin (2017), Grochulski and Kocherlakota (2010) and Ndiaye (2020)). None

[1] It is noteworthy that this concern was raised by Peter Diamond in his Presidential Address on social security delivered during American Economic Association meeting in 2004 ((Diamond (2004)): "Depending on the nature of the underlying stochastic process of wage rates, both under weighting early years (relative to the use of interest rates) and not counting some low years may or may not help with insuring lifetime earnings. This is not an area that has received much research attention."

of these papers study the design of the history-dependence part of the pension systems. This paper also relates to the literature on labor supply and retirement literature decisions of workers (Fan et al. (2022), and Borella et al. (2023), O'Dea (2017), French (2005)). None of these papers consider the history-dependence of the pension systems or the optimal design of pension systems.

History-Dependent public transfer systems have been studied in the optimal taxation literature. Batzer (2021) studies optimal tax schedule when the tax function depends on all the history of income in a general equilibrium OLG model similar to Heathcote et al. (2017). Kapička (2022) studies the history dependence of taxation in a life cycle model and finds that a simple tax function that depends on the few past years can increase welfare. In the literature on income averaging and taxation on lifetime earnings, work of Vickrey (1947) is notwithstanding. Vickrey (1947) was concerned with the impact of progressive annual taxes on those with fluctuating incomes relative to those with constant incomes. However, using a longer period for determining taxes is likely to reduce the built-in stabilization from the income tax and lessen the easing of borrowing constraints. In Diamond (2009) absence of age and history-related tax rules counts as one of the reasons for a mandatory retirement pension System to redistribute based on lifetime between fluctuating and constant income individuals.

The remainder of this paper is organized as follows. In Section 2 I discuss history dependence of pension systems in the US and other countries. Section 3 describes the data. In Section 4, I present some empirical regularities. In Section 5, I develop the model and characterize the solution. In Section 6, I build the quantitative environment. In Section 7, I evaluate the effects of an important counterfactual policy. Section 8 concludes.

2 History Dependence of Pension Systems Around the World

In this section, I explain in more detail the history-dependence part of the US pension system and compare it to defined-benefit pension plans in other countries around the world.

2.1 History Dependence of Public Pension System in the US

In the US, the two functions that determine an individual's Social Security benefits are Average Indexed Monthly Earnings (AIME) and the Primary Insurance Amount (PIA). The top 35 years of earnings are used to calculate the AIME. At first, each year's earnings are indexed to reflect the growth in the economy and wage levels during workers' employment years. After the top 35 years of indexed earnings are determined, the indexed earnings will be summed and divided by the total number of months. Then, the average amount will be rounded down to the next lower dollar amount. The result is called AIME.

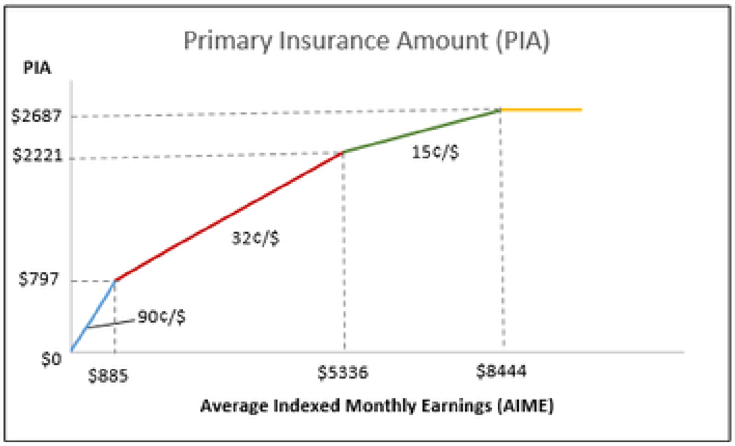
$$\underbrace{\text{AIME}}_{\substack{\text{Average Indexed} \\ \text{Monthly Earnings}}} = f(y_1, \cdots, y_T) = \frac{1}{12} \times \frac{1}{35} \times \sum_{y_i \in \left\{ \substack{\text{Top 35} \\ \text{years of} \\ \text{earnings}} \right\}} (y_i). \tag{1}$$

The second function takes AIME as given and determine the amount of monthly benefit (PIA).

$$\underbrace{\text{PIA}}_{\substack{\text{Primary Insurance}\\ \text{Amount}}} = g(AIME). \tag{2}$$

The current function g in the US is presented in 1.

Figure 1: Function g in the US: Determining PIA from AIME



2.2 History Dependence of Pension Around the world

There are three types of $f(.)$ that are most common across countries.

1. **Average Earnings of the Last L Years:** Although various OECD countries have moved from this method, this method is still employed among countries like France, Greece, Portugal, Spain, Norway, and Sweden. There are also numerous countries among developing countries that utilize this method. Thailand, Brazil, and Iran are some examples of developing counties.

2. **Average Life Time Earning, Excluding Some of the Lowest Earnings:** This method is used in the US in Canada. In the US, the highest 35 years determine the pension and in Canada, the lowest 15 % of earnings are excluded from the history.

3. **Average Life Time Earning:** In this method, the average of all of the earnings over the lifetime is used to calculate the pension benefit. Currently, this method is the most common method in OECD countries. Over the past decades, most OECD countries have moved from the second method (counting the last 10-25 years) to this method. Among developing countries, China, Indonesia, and Vietnam utilize all of their lifetime earnings.

Although the focus of this paper is public pension systems, it is also noteworthy that the first method is the most common method among private defined-benefit pension systems (like teacher unions)[2].

[2] For more information, see Ipp

3 Data

3.1 Data Sources

To demonstrate empirical regularities and parameterize the empirical model, I use data from the Panel Study of Income Dynamics (PSID). The PSID is the longest running longitudinal household survey in the world. It began as a nationally representative sample of 5,000 families in the United States. I use the data from 1968-2019 and only consider males who are head of their household.

3.2 Summary statistics

Table 1 reports the descriptive summary of the final sample. All monetary variables are in 2015 dollars and all time variables are in annual units.

Table 1: Descriptive Statistics of the Sample

	mean	25th percentile	Median	75th percentile	sd
Age	38.35	26.00	35.00	48.00	(15.62)
Years in School	13.17	12.00	13.00	16.00	(2.65)
College Degree	0.26	0.00	0.00	1.00	(0.44)
Work Hours (Before Retirement)	2068.78	1811.00	2058.00	2448.00	(741.24)
Labor Force Particpipation (Before Retirement)	0.89	1.00	1.00	1.00	(0.32)
Wage	24.93	13.42	20.61	31.18	(17.25)
Income (Before Retirement)	46188.68	20016.67	39400.19	63239.67	(38765.32)
Asset (Before Retirement)	234149.63	3129.88	43796.62	177682.78	(1124018.63)
Asset (After Retirement)	479893.73	28276.99	162900.42	489482.62	(1509194.48)
Social Security Income	13473.01	8682.77	12895.96	17435.62	(7085.43)
Age of Retirement (Based on Self Report)	50.01	39.00	59.00	63.00	(19.73)
Age of Retirement (Based on Receving SS Income)	57.94	41.00	65.00	72.00	(19.90)
Age of Retirement (Based on Stop Working)	60.07	55.00	61.00	65.00	(9.71)
Observations	394093				

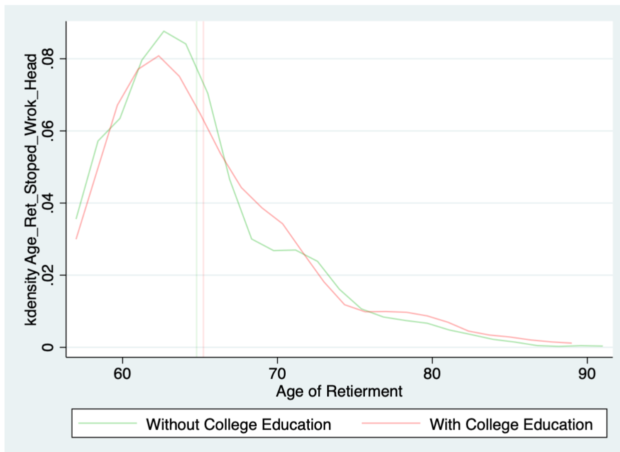
4 Empirical Regularities

In this section, I present some stylized facts regarding (1) retirement age, (2) labor supply and wages over the life cycle, and (3) how different ways to summarise the history of earnings of individuals would affect them. Sections 4.1 and 4.2 update some empirical regularities in the literature (i.e. Rupert and Zanella (2015)) with longer time series that span the whole life cycle of individuals. Section 4.3 shows some new empirical regularities.

4.1 Age of Retirement

Figure 2 shows the histogram of the age of retirement defined as when workers stopped working and never go back to work after that. If the whole history of earnings is not observed for an individual, I consider her retired at a given age if I observe at least a consensus 5 years of no income (at least one of them after age 55) after that year. The average age of retirement is 65.21 for college-educated workers and 64.78 for non-college-educated ones. There is noticeable heterogeneity in the age of retirement of workers. The standard deviation of the age of retirement of workers with a college degree is 6.05 years and for workers without a college degree is 5.69 years.

Figure 2: Histogram of Age of Retirement Conditional on Education level



Notes: The horizontal lines show the mean of variables.

4.2 Work Hours, Wage, and Earning over the Life Cycle

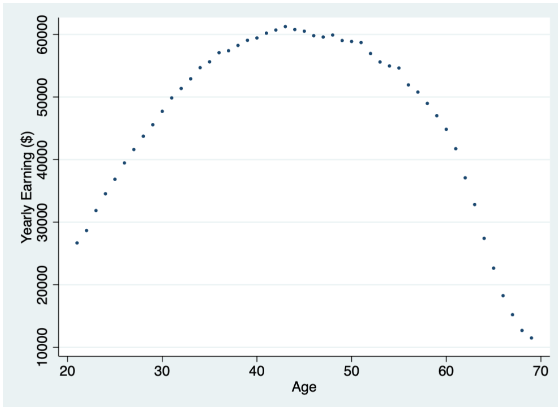
The figure in Figure 3 shows the earnings of workers over the life cycle from age 20 to 70. Earnings rise at the beginning of the life cycle, pick at age 42, and fall after that. To see how this pattern at the end of workers' careers is affected by workers leaving the workforce in old age, I only consider workers who stay in the labor market at least until age 70 next. For this sub-sample, the mean of earnings flattens from age 42 to 60, then falls. Hence, the early fall in earnings after age 42 can be explained by the lower earnings of workers leaving the labor market sooner.

In Figure 6, I show how the life cycle pattern has changed for different generations. I group individuals born in the same 5 years period into the same generation. Given the data, the workers are divided into 9 generations, where the youngest generation was born in 1963 and the oldest one was born in 1923. For some of these generations, I have their whole full history of income[3]. As can be seen, the fall in earnings after age 42 is influenced the most by the generations born before 1942. Means of earnings in all groups represent hump shape patterns and drop after age 55.

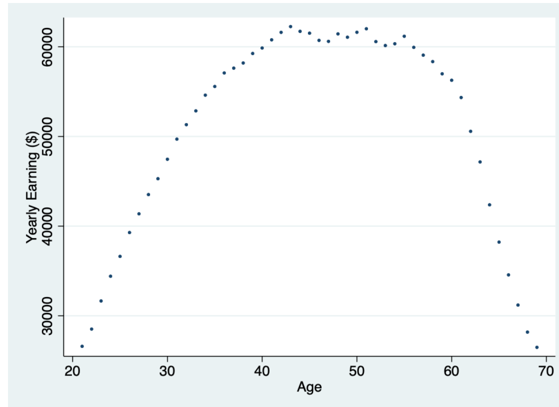
To see how the pattern of earnings is shaped by changes in wages and work hours over the life cycle, I focus on these two components of earnings next. Figure 4 shows the wage profile of workers. It rises and picks at around age 42, but it reveals a much flatter trajectory after that, compared to earnings. It stays mostly flat for all workers and the sub-sample of workers who

[3]After 1999, the PSID surveys biannually. For the outcomes between two survey years, I take the average of neighboring years.

Figure 3: Mean of Workers' Earning over the Life Cycle

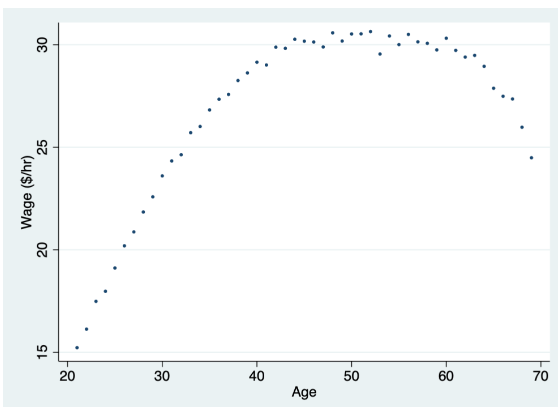


(a) All Workers

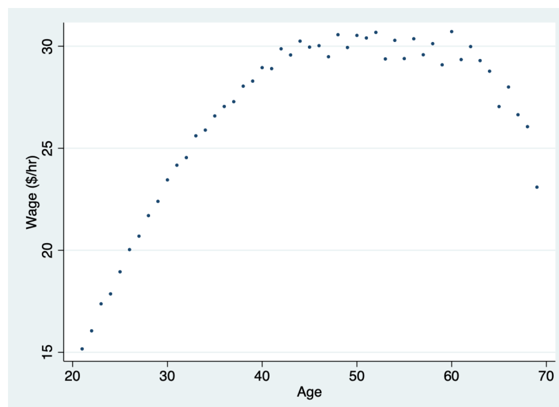


(b) Workers Who Retire After Age 70

Figure 4: Mean of Workers' Wages over the Life Cycle



(a) All Workers



(b) Workers Who Retire After Age 70

remain in the workforce at least until age 70 up until age 60. Then, it starts to fall for both samples. Figure 7 shows almost the same pattern for all cohorts.

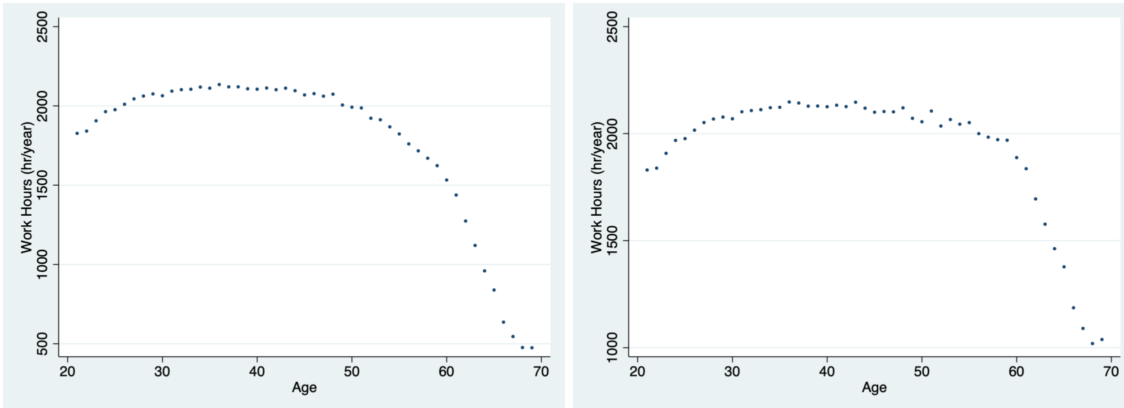
Figure 5 shows the hours of work of individuals. The workers increase their work hours until age 35 and work almost the same hours until age 48 when the mean of work hours starts to decrease. Focusing on a sample of late retirees, they work almost the same hours until age 55, then decrease their work hours gradually after that until age 60 when their work hours start to rapidly decrease.

Figure 8 shows the change in work hours for different cohorts. We can see that earlier generations start decreasing their work hour sooner.

4.3 Comparing Different Functions of History of Earnings

In this section, I discuss how different ways to calculate the history of earnings would generate different results for workers. How many and which years counted for pension benefit calculation

Figure 5: Mean of Workers' Work Hours over the Life Cycle



(a) All Workers

(b) Workers Who Retire After Age 70

affect inequality of outcomes of workers in several dimensions (1) inequality between groups with a steep trajectory of wages over the life cycle (i.e. college-educated workers) and workers with a less upward trajectory of wages (i.e. high schooled dropouts); (2) inequality between groups with higher fluctuation in their earnings (i.e. farmers) and groups with more stable earning profiles (i.e. public employees); and (3) inequality between groups who have experienced long spells of unemployment (i. e. temporary workers) or have been out of the workforce for a long time (i. e. stay at home parents) and the workers with steady work history.

4.3.1 Utilizing Highest Earnings Years

Here, I discuss how different ways to summarise history of earnings affect inequality in general. In Table 2, I show the mean and standard deviation of outcomes of different functions. As expected, when more of the low-earning years are ignored, the mean becomes higher. The fourth column in Table 2 illustrates that the standard deviation of these variables decreases as more high-earning years are counted. The more low-earning years are discarded, it would provide more insurance for the workers as those bad years will not be counted in the function of the history of earnings. However, it also means more weight to the highest earning years which already represent more inequality than other years. The result is more inequality as more years are counted.

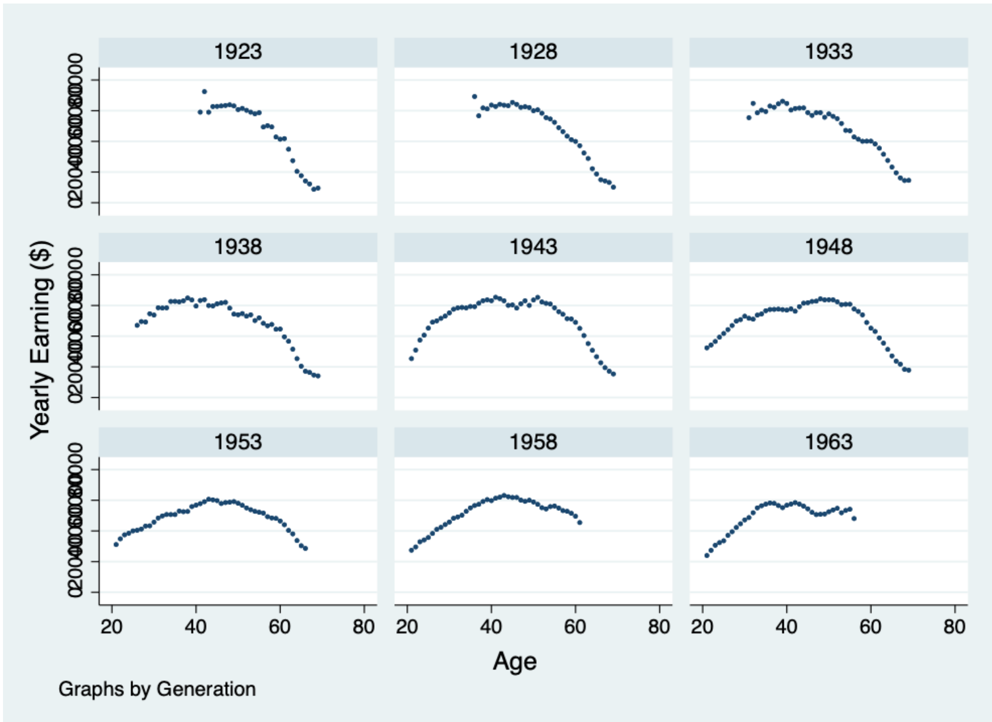
Table 2: Functions of History of Earnings Based on Highest Earnings

Mean of earning	Mean	Std. dev.
Highest 5 years	87830.09	42340.5
Highest 10 years	79764.62	38506.3
Highest 15 years	73878.37	36446.99
Highest 20 years	69272.62	34965.17
Highest 25 years	65449.93	33921.47
Highest 30 years	62401.4	33071.58
Highest 35 years	60402.91	32402.08
All	59815.76	32216.14

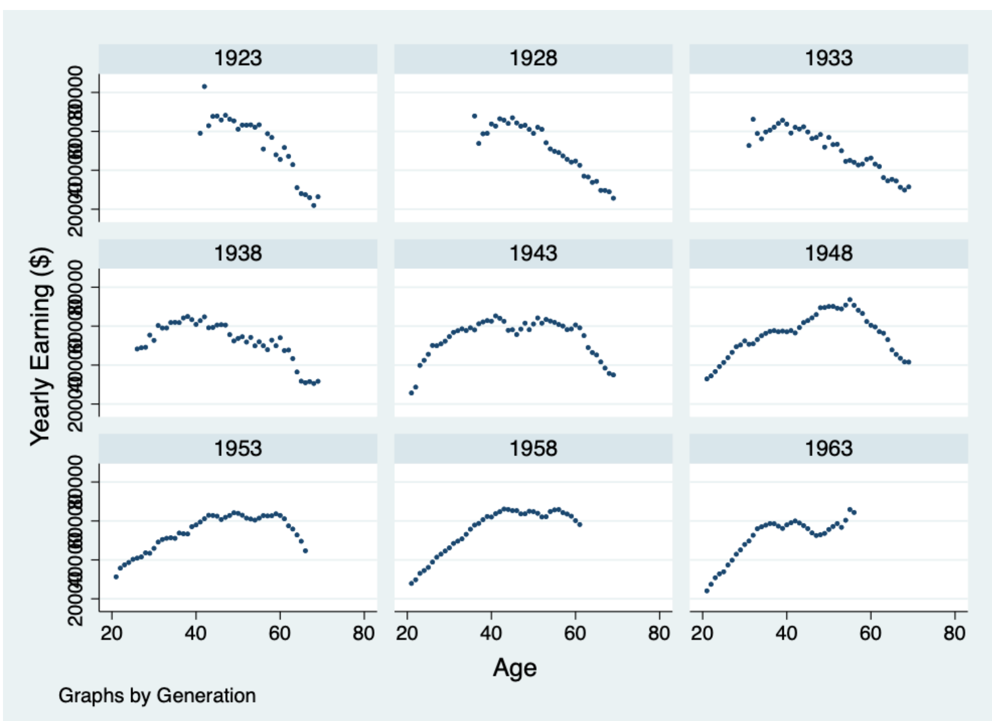
Notes: The number of observations are 814.

Looking at the histogram of outcomes of these functions in Figure 9, we can see more concentration of outcomes as the number of years increases.

Figure 6: Mean of Workers' Earning Over the Life Cycle Conditional on Generation



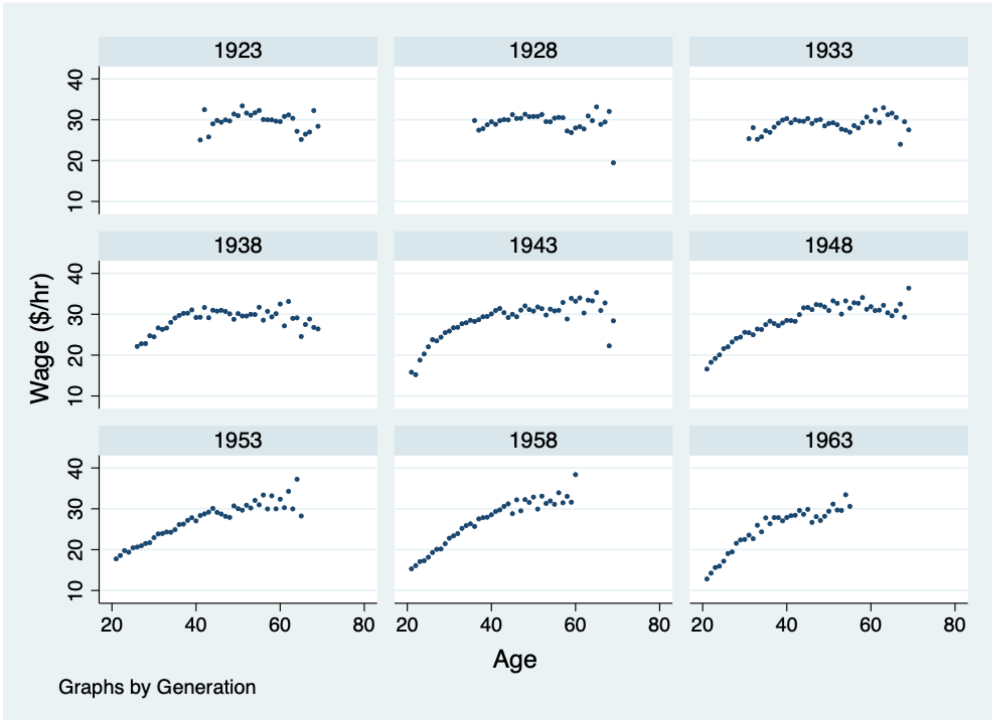
(a) All Workers



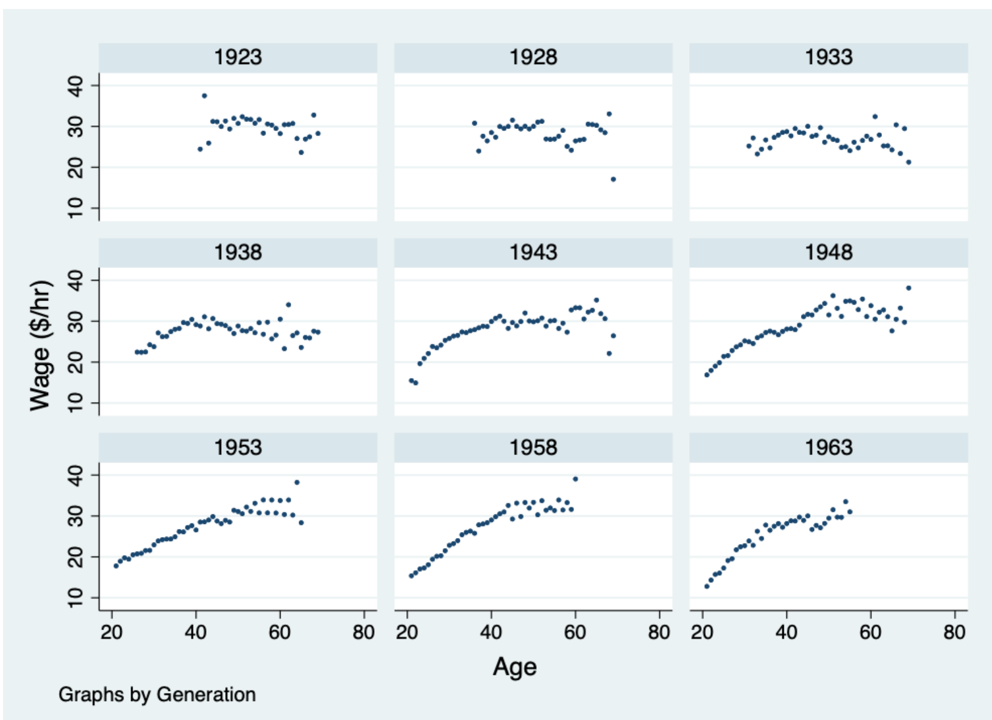
(b) Workers Who Retire After Age 70

Notes: Each generation is defined as workers who born on that year and the next 4 years.

Figure 7: Mean of Workers' Wages Over the Life Cycle Conditional on Generation



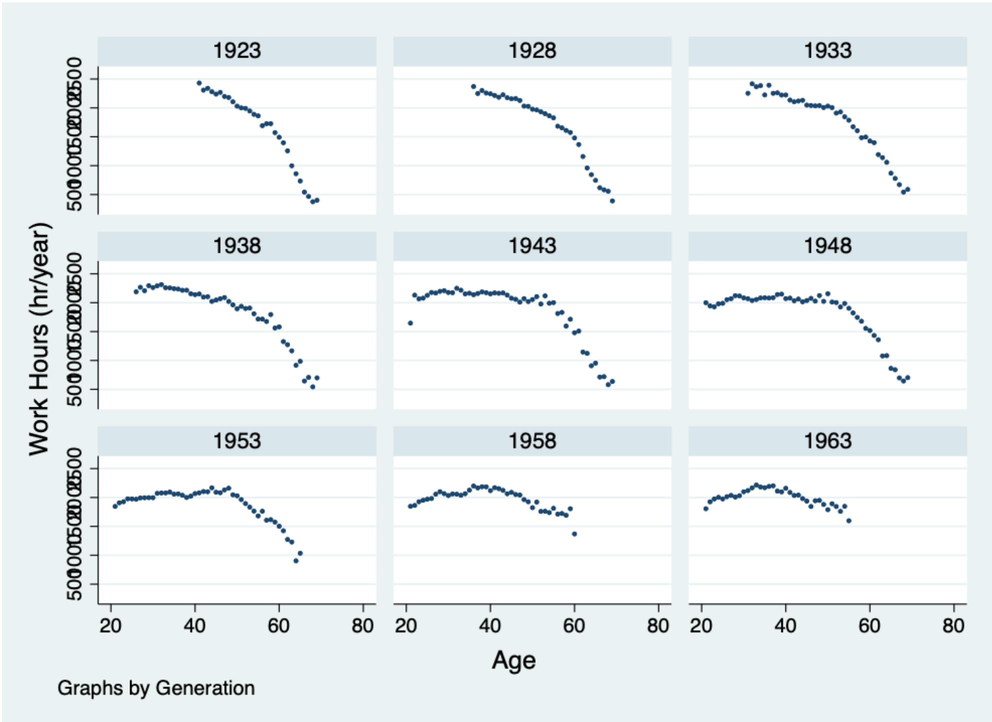
(a) All Workers



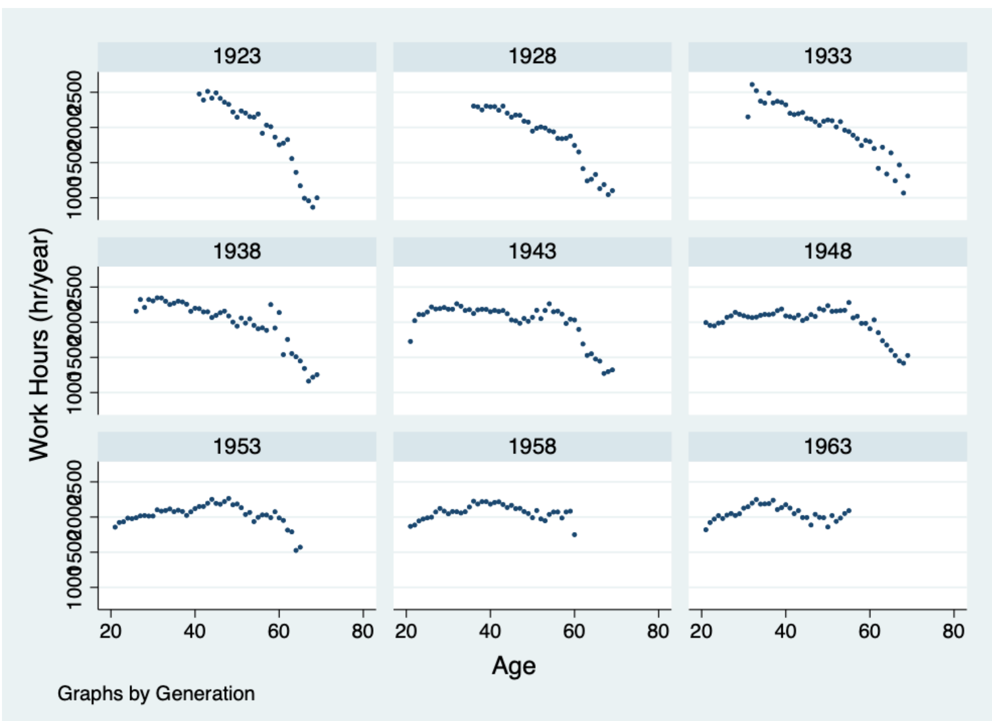
(b) Workers Who Retire After Age 70

Notes: Each generation is defined as workers who born on that year and the next 4 years.

Figure 8: Mean of Workers' Work Hours Over the Life Cycle Conditional on Generation



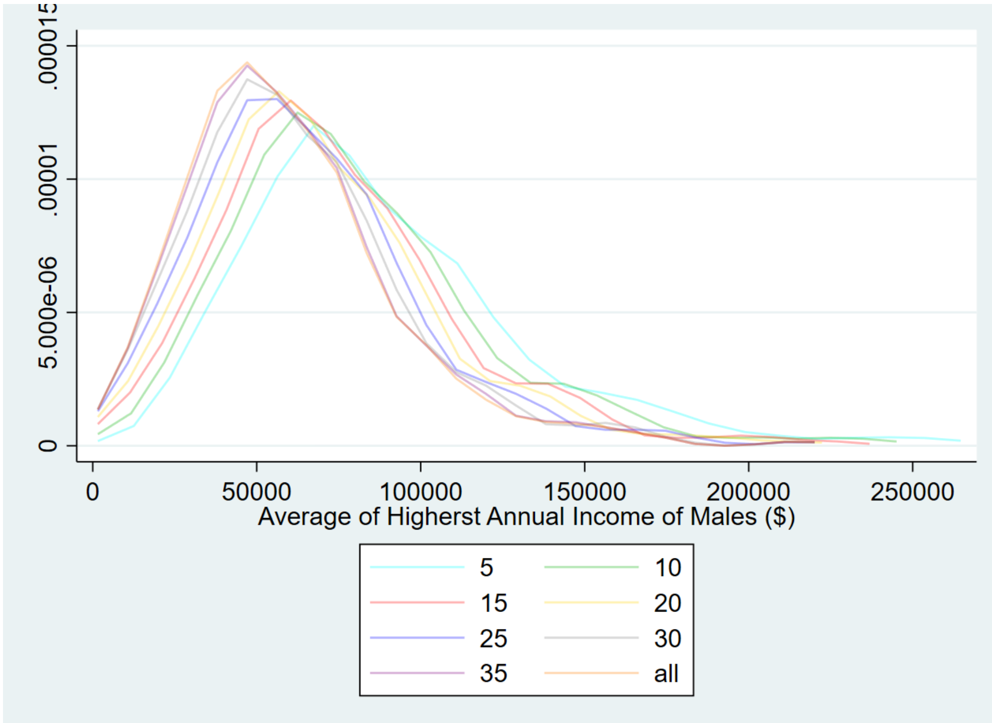
(a) All Workers



(b) Workers Who Retire After Age 70

Notes: Each generation is defined as workers who born on that year and the next 4 years.

Figure 9: Functions of History of Earnings Based on Highest Earnings



For the generations the full history of earnings are not observable, the highest earning years that we observe are not necessarily the highest earning years over all of the worker's life cycle.

4.3.2 Utilizing Latest Earnings Years

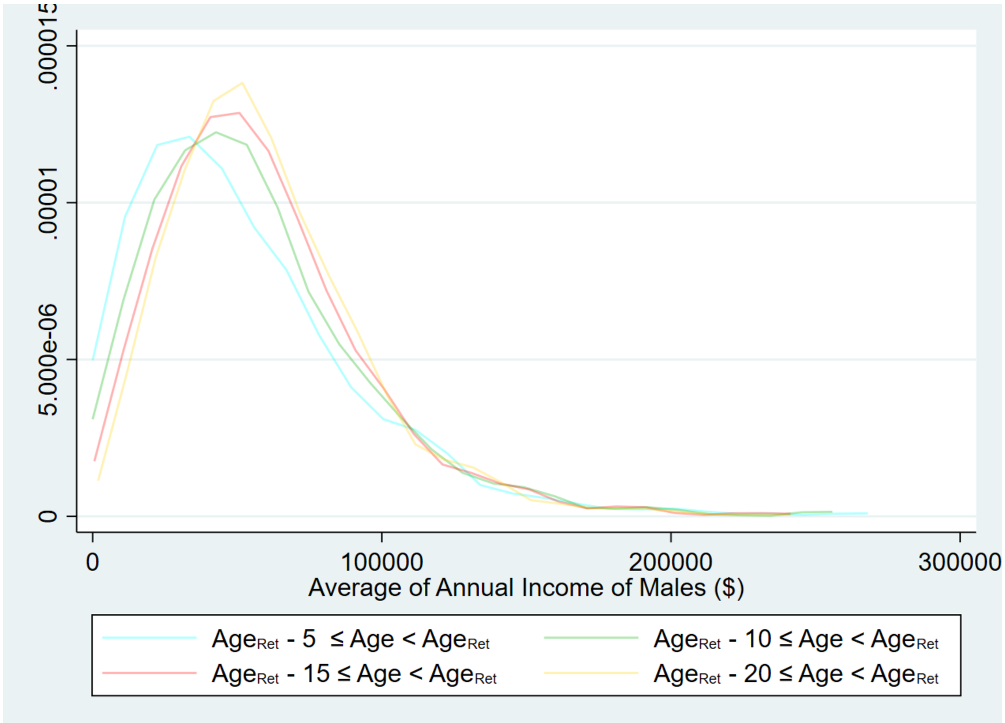
In Table 3 and Figure 10, I show the results for functions that consider the latest years of earnings using the first assumption. Similar to the highest earning method, the standard error of the outcomes decreases by increasing the number of years. Note as shown in Figure 3, the mean of earnings decreases late in the life cycle. Hence, the mean of average earnings will be higher as more years are taken into account. Note that I ignore the fact that the retirement age is endogenous and count the number of years backward from the observed age of retirement of the individual.

Table 3: Functions of History of Earnings Based on Latest Earnings from Actual Retirement Age

Mean of Earning	Mean	Std. dev.
Last 5 Years	52165.09	39807.09
Last 10 Years	56467.02	37854.27
Last 15 Years	59035.93	35301.48
Last 20 Years	60548.72	33644.44
All	61506.78	30177.98

Notes: The number of observations are 814.

Figure 10: Functions of History of Earnings Based on Latest Earnings from Actual Retirement Age



4.3.3 Inequality Among Different Groups with Different Methods and Rules

In this section, I discuss how the history-dependent part of the pension system shapes the inequality of pension benefits among different groups of workers. Table 4 illustrates the results of previous sections conditional on the education level of the workers. When the highest earning years are counted for the pension benefit, the more years counted, the less inequality between these two groups. This result can be explained by the fact that the difference between these two groups is not large when we consider the low-earning years of workers. However, the top-earning years of college-educated workers are much more than the top-earning years of non-college-educated workers. Similar results occur when we consider the latest years of earnings. As the number of years counted increases, the difference between the average earnings of no-college-educated workers and college-educated workers rises.

Table 4: Functions of History of Earnings Conditional on Education

	Without College Degree			With College Degree		
Mean of Earnings	Obs	Mean	Std. dev.	Obs	Mean	Std. dev.
Highest 5 Years		75895.08	33555.74		116374.6	47311.66
Highest 10 Years		68600.01	30392.11		106466.7	42590.16
Highest 15 Years		63215.37	28582.25		99380.73	40422.13
Highest 20 Years		58984.64	27461.92		93878.04	38581.33
Highest 25 Years		55630.34	26752.69		88935.12	37587.92
Highest 30 Year		53148.06	26284.26		84532.30	36969.00
Highest 35 Years		51604.20	25810.90		81446.47	36654.75
Latest 5 Years		44984	31628.65		69496.68	50776.2
Latest 10 Years		48468.04	29317.51		75564.58	47871.59
Latest 15 Years		50648.44	27480.97		79096.01	43075.57
Latest 20 Years		52031.33	26226.54		80919.48	40131.22
All	574	51169.77	25656.88	240	80494.08	36652.48

Table 5 presents the results based on the level of fluctuations in the earnings of workers. I divide the workers into two groups of high and low earning variability based on the median of the standard deviation of workers' earnings history. When only the top earnings of workers are cream skimmed the difference between the two groups is high, but as more years are considered the difference becomes smaller. This fact can be used to redistribute between these groups. The same pattern, to a lesser extent, applies to the latest years method.

Table 5: Functions of History of Earnings Conditional on Standard Error of Wages

	Below Median			Above Median		
Mean of Earnings	Obs	Mean	Std. dev.	Obs	Mean	Std. dev.
Highest 5 Years		57663.07	17601.06		117997.1	38150.68
Highest 10 Years		52130.44	17278.59		107398.8	33756.71
Highest 15 Years		47945.69	17492.57		99811.06	31713.2
Highest 20 Years		44763.66	17266.75		93781.58	30751.58
Highest 25 Years		41967.89	17073.76		88931.97	30118.52
Highest 30 Year		39800.83	16613.16		85001.96	29834.01
Highest 35 Years		38315.69	15804.46		82490.12	29571.96
Latest 5 Years		34505.15	19264.76		69956.5	46677.38
Latest 10 Years		37002.18	17860.3		75884.03	42318.53
Latest 15 Years		38425.02	16701.92		79646.85	36943.08
Latest 20 Years		39334.42	15605.36		81763.01	33479.08
All	407	37775.25	15414.81	407	81856.27	29440.41

Table 6 shows the results conditional on the level of workforce attachment of the workers. First, I calculate the share of the working history of each individual which they have zero earnings. Then, I divide them into two groups based on their position relative to the median of these shares. In the highest earning years method, the difference between these two groups becomes smaller, in general, as the number of years increases. As workforce attachment is highly associated with the level of earnings, the workers with lower workforce attachment do not benefit from counting less number of years. In the latest years method, the difference seems to not be affected greatly by the number of years. Note that the workforce attachment groups are categorized by working history during the whole life cycle of workers, which does not completely align with workforce attachment in the last 5 to 20 years of working life.

Table 6: Functions of History of Earnings Conditional on Share of Zero Work Hour Years

	Below Median			Above Median		
Mean of Earnings	Obs	Mean	Std. dev.	Obs	Mean	Std. dev.
Highest 5 Years		82300.88	41186.41		93386.53	42805.02
Highest 10 Years		74131.91	37435.28		85425.08	38782.09
Highest 15 Years		67983.74	35945.82		79802.04	36026.98
Highest 20 Years		63436.93	35083.38		75137.07	33889.18
Highest 25 Years		59935.63	34754.73		70991.4	32164.94
Highest 30 Year		58192.14	34599.45		66631.39	30931.23
Highest 35 Years		57594.57	34517.37		63225.08	29904.92
Latest 5 Years		52898.5	40443.33		51437.11	39201.66
Latest 10 Years		56067.85	39178.04		56867.17	36523.12
Latest 15 Years		57910.67	36541.25		60166.73	34017.72
Latest 20 Years		59251.38	34715.22		61852.44	32523.6
All	408	57594.57	34517.37	406	62047.9	29599.82

5 Model

5.1 Setup

The model starts at period 1, with a continuum of males (m) at age 26. Time is discrete and each time period (t) represents one year. Individuals are indexed by i which is excluded whenever it is not necessary for brevity. Each individual is either college-educated or not ($edu^i \in \{1, 0\}$). At each time period, individuals decide how much to consume versus save in a risk-free asset. During their working life, individuals also decide how many hours to work. Working individuals pay labor income and social security taxes. Each year after the early retirement age (age 62) until the full retirement age (age 66), they can decide to retire and receive their pension benefit (which depends on past incomes and retirement age and is determined according to pension rules) for the following years. Death is deterministic and happens at $T_L = 81$.

5.2 Preferences

The per-period utility of each individual is

$$u_t(c_t, h_t) = \frac{c_t^{1-\gamma}}{1-\gamma} - \psi \frac{{h_t}^{1+\frac{1}{\eta}}}{1+\frac{I}{\eta}} - \phi \mathbb{I}\{h_t > 0\} \tag{3}$$

The first term in $u_t(.)$ shows a person's utility from consumption (c_t). σ stands for the coefficient of relative risk aversion of consumption. The second term shows a person's disutility from work hours (h_t) in the intensive margin. ψ represents the disutility of work hours (h_t) and η is the Frisch elasticity of labor supply. The third term shows the fixed utility cost of working as determined by the coefficient ϕ.

5.3 Budget Constraint

The budget constraint before retirement age ($t < t_R$) is

$$c_t = y_t - T^{inc}(y_t) - T^{SS}(y_t) + a_t - \frac{1}{1+r}a_{t+1},\ y_t = w_t h_t, a_t > 0 \tag{4}$$

Where w_t, y_t, and a_t represent wage, income, and asset, respectively. $T^{inc}(y_t)$ and $T^{SS}(y_t)$ are income and social security taxes. For each individual i at period t, her wage follows

$$\log(w_t) = \mu(\boldsymbol{edu}, t) + \theta_t,\ \theta_t = \theta_{t-1} + \epsilon_t \tag{5}$$

$$\epsilon_t \sim N(0, {\sigma_\epsilon}^2),\ \theta_0 = 0. \tag{6}$$

Where $\mu(\boldsymbol{edu}, t)$ is the deterministic part of wage and has a quadratic form, and θ_t is the stochastic part of wage which follows an AR(1) process with normal shocks ϵ_t.

After an individual decided to retire, her budget constraint follows

$$c_t = b(t_R, \{y_s\}_{s=1}^{t_R}) + a_t - \frac{1}{1+r}a_{t+1}. \tag{7}$$

where $b(.)$ is the benefit function that determines the amount of annual pension payment according to social security pension rules based on the retirement age and past incomes.

5.4 Individual Problem

The value function of non-retired individuals is

$$V_t^W(\{y_s\}_{s=1}^{t-1}, a_t, \theta_t, \boldsymbol{edu}) = \max_{c_t, h_t} u_t(c_t, h_t) + \beta\mathbb{E}\Big[V_{t+1}^W(\{y_s\}_{s=1}^{t}, a_{t+1}, \theta_{t+1}, \boldsymbol{edu})\Big] \tag{8}$$

$$a_{t+1} = (1+r)\Big(a_t + w_t h_t - T^{inc}(y_t) - T^{SS}(y_t) - c_t\Big),\ \log(w_t) = \mu(\boldsymbol{edu}, t) + \theta_t \tag{9}$$

$$\theta_{t+1} = \theta_t + \epsilon_{t+1} \tag{10}$$

$$\{y_s\}_{s=1}^{t+1} = \Big(\{y_s\}_{s=1}^{t}, y_{t+1}\Big),\ y_{t+1} = w_{t+1}h_{t+1} \tag{11}$$

Where $\{y_s\}_{s=1}^{t}$ is vector history of earnings.

The value function of an individual after being retired is

$$V_t^R(AIME_{t_R}, a_t, t_R) = \max_{c_t} u_t(c_t, 0) + \beta V_t^R(AIME_{t_R}, a_{t+1}, t_R) \tag{12}$$

$$a_{t+1} = (1+r)\Big(a_t + PIA(AIME_{t_R}, t_R) - c_t\Big). \tag{13}$$

Where $AIME_{t_R}$ is the AIME at the time of retirement and $PIA(.)$ is the function that calculate the pension benefit based on AIME. Between the early retirement age and full retirement age, the value function follows

$$V_t^E(AIME_{t_R}, a_t, t_R) = \max_{Ret \in \{0,1\}} \Big\{ V_t^W(.), V_t^R(.) \Big\}, \tag{14}$$

as the individuals decides whether to retire $Ret = 1$ or not $Ret = 0$. The age when $Ret = 1$ is the retirement age (t_R).

6 Quantitative Environment (Preliminary)

Parametrization

6.0.1 Preference Parameters

The preference parameters taken from the literature are $\beta = 0.98803$ and $\gamma = 1.66$ (Huggett and Parra (2010)), $\eta = 0.5$ as it is common in the literature (i.e. **?**). I estimate $\psi = 0.04$ and $\phi = 0.0006$ to match mean working hours of men and labor supply participation of men above age 50 in the sample.

6.0.2 Budget Constraint Parameters

Total endowment hours in each year is assumed to be 8760. Furthermore, I am assuming $r = 0.042$ similar to Huggett and Parra (2010).

6.0.3 Wage Process

I estimate the wage function via GMM method. The wage function is assumed to take a quadratic form

$$\mu(Age, edu) = \beta_0^w + \beta_1^w Age + \beta_2^w Age^2 + \beta_3^w edu + \beta_4^w edu \cdot Age + \beta_5^w edu \cdot Age^2. \tag{15}$$

The coefficients of the wage function $(\beta_0^w, \beta_1^w, \beta_2^w, \beta_3^w, \beta_4^w, \beta_5^w)$ are $(1.6698, 0.0605, -.0006, -.3780, 0.03214, -0.0002)$ The difference between the observed and model-generated wages is assumed to measurement error. The variance of persistent shocks is estimated to $\sigma_\epsilon^2 = 0.02601$.

6.0.4 Tax and Social Security Rules

Similar to Heathcote et al. (2017), I assume the Income Tax takes the form of $T^{SS}(y_t) = y_t - \kappa {y_t}^{(1-\tau)}$ and estimate its parameters using TAXIM data: $\kappa = 2.716084$ and $\tau_T = 0.1029$. The social Security rules is assumed to be same as 2000 rules for all years. the socail security tax is $T^{SS}(y_t) = \tau_{SS} y_t = 0.106$ until the wage base $= 76200$. The PIA (primary insurance amount) has three bend points $a = \{6372, 38422, 76200\}$ and slopes are $b = \{0.9, 0.32, 0.15\}$ with the wage base of $= 76200$. The early retirement reeducation's are $re = \{0.75, 0.80, 0.867, 0.933\}$. AIME (Average Index monthly earning) is $\frac{1}{12}$ of mean of top 35 years of earnings.

6.1 Numerical Exercise

In this section, I utilize the presented quantitative model and evaluate the effects of a counterfactual policy that changes the current US pension policy which uses the top 35 years of earnings to account for the lifetime earnings, common in other OECD countries. Table 7 presents the result both separately for non-college-educated and college-educated workers and when they are pooled together. Accounting for the full history of earnings causes 42 % increase in consumption and 82 % increase in hours of work. Age of retirement decreases by 5.7 % as the age of retirement in the new regime will reach the lower constraints allowed in the model.

Table 7: Results of the Numerical Exercise

	Mean of Status Quo			Mean of Counterfactual			% change in Mean		
edu	0	1	all	0	1	all	0	1	all
c	30871	22884	29352	44167	47405	44783	43.06	107.15	52.56
h	1611	1236	1539	2872	2549	2811	78.30	106.16	82.56
a	223861	186657	216786	223126	308285	239321	-0.32	65.16	10.39
PIA	12203	10389	11858	11745	11627	11722	-3.75	11.92	-1.14
Age_R	65.38	65.95	65.49	62	62	62	-5.17	-5.99	-5.33

Notes: Consumption (c), assert (a), and PIA are in the unit of $, work hours (h) is in the unit of hours, and age of retirement (Age_R) is in the unit of years.

Moving to the pension system that considers the full earning history increases the marginal benefit of working in the years that are not among the top 35 years of earnings as now those years would affect the pension benefits. On the other hand, since the weight of each of the top 35 earning years is now lower, the marginal benefit of working during those years decreases. If the years between the early and normal retirement ages are among the top 35 years of earning (which is the case for most of the workers as it was shown in Figure 3) there is more incentive to retire earlier. If workers delay retirement, their pension benefits will become lower by the addition of the low-wage years of their 60s to their working history which lowers their average full-life earnings.

Note that due to the high number of continuous state variables, approximating the value function of working individuals is a major challenge to numerically solve this problem. Here, I utilize a quadratic polynomial approximation method to overcome this problem. However, this approximation method is only used for the current policy regime and not for the counterfactual one. Given the lower number of state variables needed to calculate pension benefits in the full history method, there is no longer a need for polynomial approximation of value functions. The use of different approximation methods can potentially distort the comparison of the results in these two cases. Moreover, the model parameters are calibrated for the status quo environment. Hence, these results should be seen as a numerical exercise and are not directly applicable to policymaking. Improving the approximation method is left for future research.

6.2 Deep Learning Numerical Algorithm

6.2.1 Overview of Algorithm

To numerically solve the dynamic problem of individuals, I approximate policy functions using deep neural networks (DNNs). Deep neural networks have been successfully applied to solve various economic models. A particular challenge in this research is the expansion of the state space due to the history dependence of pension benefits, which renders existing methods in the literature unsuitable. One of the main contributions of this paper is to propose a novel method for solving high-dimensional life-cycle models with varying state spaces and control choices using deep neural networks.

I approximate the policy functions with DNNs and formulate the dynamic programming problem in its extensive form, where analytical integrals are replaced by Monte Carlo simulations of shock paths. To construct an auxiliary sample, I recursively simulate endogenous state variables based on exogenous state variables and approximated policy functions. The loss function consists of the negative ex-ante welfare of this sample, supplemented by regularization terms to enhance the efficiency and stability of the algorithm. The individual maximization problem is thus transformed into a loss minimization problem, where the objective is to estimate the parameters of the DNNs that best approximate the policy functions.

6.2.2 Problem

$$\max_{\{c_t\}_{t=1,T}\{R_t\}_{t=t_{ER},t_R}\{h_t\}_{t=0,t_R}} \mathbb{E}\,|_{\epsilon_1,\cdots,\epsilon_T} \left(\sum_{t=1}^{T} \beta^{t-1} \Big[\frac{(c_t)^{1-\gamma}}{1-\gamma} - \psi \frac{(h_t/\bar{h})^{1+\frac{1}{\eta}}}{1+\frac{I}{\eta}} - \phi \mathbb{I}\{h_t > 0\} \Big] \right) \tag{16}$$

$$s.t. \;\; (t < t_R)\; c_t = y_t - \tau^{inc}(y_t) - \tau^{SS}(y_t) + a_t - \frac{1}{1+r} a_{t+1},\; y_t = w_t h_t, a_t > 0, a_1 = 0 \tag{17}$$

$$(t \geq t_R)\; c_t = b(t_R, \{y_s\}_{s=1}^{t_R}) + a_t - \frac{1}{1+r} a_{t+1} \tag{18}$$

$$(t \leq t_R)\; \log(w_t) = \mu(edu, t) + \theta_t,\; \theta_{t+1} = \theta_t + \epsilon_{t+1},\; \epsilon_t \sim N(0, {\sigma_\epsilon}^2),\; \theta_0 = 0 \tag{19}$$

$$c_t \in (0, \inf), h_t \in \{0, 1300, 2080, 2860\}, \tag{20}$$

$$R_t \in \{0,1\}, t_R = \min\{argmin_t[R_t = 1], T_{LR}\} \tag{21}$$

6.2.3 Problem after Substituting the Approximated Policy Function

Defining M_t as the state space at time t of and $\pi_{\mathbf{t}}(\mathbf{M_t})$ the (multi-input,multi output) policy function, and $\pi_{\mathbf{t}}(\mathbf{M_t}, \mathbf{\Omega_t^{\lambda_t}})$ the approximated policy function from class of deep neural net architecture λ_t and weights $\Omega_t^{\lambda_t}$.

$$\Omega = \textbf{\textit{argmax}}_{\Omega} V(\Omega) = \mathbb{E}\,|_{(\epsilon_1,\cdots,\epsilon_T,\textbf{\textit{edu}})} \left(\sum_{t=1}^{T} \beta^{t-1} u(\pi_t(M_t,\omega_t)) \right) \tag{22}$$

$$u(\pi_t(M_t,\omega_t)) = \frac{[c_t(M_t,\omega_t)]^{1-\gamma}}{1-\gamma} - \psi \frac{(h_t(M_t,\omega_t)/\bar{h})^{1+\frac{1}{\eta}}}{1+\frac{I}{\eta}} - \phi \mathbb{I}\{h_t(M_t,\omega_t) > 0\} \tag{23}$$

$$(t < t_R)\;\; c_t(M_t,\omega_t) = y_t - \tau^{inc}(y_t) - \tau^{SS}(y_t) + a_t - \frac{1}{1+r} a_{t+1}(M_t,\omega_t),\; y_t = w_t h_t(M_t,\omega_t), a_1 = 0 \tag{24}$$

$$(t < t_R)\;\; M_t = \{\{y_l\}_{l=1}^{t-1}, a_t, \theta_t, \textbf{\textit{edu}}\} \tag{25}$$

$$(t < T_{ER})\;\; \pi_t(M_t,\omega_t) = \{c_t(M_t,\omega_t), h_t(M_t,\omega_t)\} \tag{26}$$

$$(T_{ER} \le t < t_R)\;\; \pi_t(M_t,\omega_t) = \{c_t(M_t,\omega_t), h_t(M_t,\omega_t), R_t \in \{0,1\}\} \tag{27}$$

$$(t \ge t_R)\;\; M_t = \{b_t, a_t\} \tag{28}$$

$$(t \ge t_R)\;\; c_t = B(t_R, \{y_s\}_{s=1}^{t_R}) + a_t - \frac{1}{1+r} a_{t+1} \tag{29}$$

$$s.t.\;\; \log(w_t) = \mu(\textbf{\textit{edu}}, t) + \theta_t,\; \theta_{t+1} = \theta_t + \epsilon_{t+1},\; \epsilon_t \sim N(0, {\sigma_\epsilon}^2),\; \theta_0 = 0 \tag{30}$$

$$c_t \in (0, \inf), h_t \in \{0, 1300, 2080, 2860\} \tag{31}$$

$$t_R = \min\{\textbf{\textit{argmin}}_t [R_t = 1], T_{LR}\} \tag{32}$$

$$B_{t_R} = b(t_R, S), S = s(\{y_l\}_{t=1}^{t_R-1}),\; B_t = B_{t-1}, t > t_R \tag{33}$$

6.2.4 Dealing with discrete Choice

One problem with discrete choice models is the non-differentiability of value and policy functions. A common approach to addressing this issue is to introduce smoothing via uncertainty, which enables the use of numerical differentiation. However, the challenge is that efficient algorithms developed for training deep learning models (e.g., PyTorch) rely on analytical differentiation through the chain rule.
[4]

In this paper, I present a method that allows for solving dynamic problems with discrete control and state variables while maintaining analytical differentiation, thereby significantly improving computational efficiency. The key idea is to replace the discrete state of retirement and the discrete control variable of retiring with their corresponding probabilities. To ensure that the decision remains discrete, I parameterize the probability function using a logistic function with a temperature parameter. This guarantees differentiability while ensuring that the probabilities remain close to 0 or 1. The difference between an actual 0 and a near-zero probability is negligible and is treated as an approximation error.

As the temperature parameter increases, the problem asymptotically converges to the discrete choice model. Given the regularization of the policy function and the inherent uncertainty in shocks, the probability of intermediate values is practically negligible. Importantly, this approximation is only applied during the training process. In the final simulation, discrete choices are

[4] The method I propose here is more general. Unlike existing approaches, it does not introduce additional constraints or depend on specific deep learning algorithms, thereby avoiding unnecessary complications.

used, and the discrepancy is treated as an approximation error that converges to zero. In other words, these two formulations are equivalent in the limit, and the proposed method ensures convergence (in probability?) to the original problem.

The approximated problem is as follows:

Defining M_t as the state space at time t of and $\pi_{\mathbf{t}}(\mathbf{M_t})$ the (multi-input,multi output) policy function, and $\pi_{\mathbf{t}}(\mathbf{M_t}, \mathbf{\Omega_t})$ the approximated policy function given the class of deep neural net architecture with weights Ω_t.

To have a better insight, lets first write down the problem in recursive form:

For $t = 1, \cdots, T_{ER} - 1$, the state variables are $M_t = \{\{y_l\}_{l=1}^{t-1}, a_t, \theta_t, \textit{edu}\}$ and the control variables are $\pi_t(M_t, \Omega_t) = \{a_{t+1}(M_t, \Omega_t), h_t(M_t, \Omega_t)\}$ and the individual solves

$$\begin{aligned}
\Omega_t^* &= argmax_{\Omega_t} \mathbb{E}\ |_{\theta_{t+1}}\ V_t\big(\pi_t(M_t, \Omega_t) \mid M_t\big) = \\
&\quad u^t(a_{t+1}(M_t, \Omega_t), h_t(M_t, \Omega_t)) + \beta \mathbb{E}\ |_{\theta_{t+1}}\ V_{t+1}\big(\pi_{t+1}(M_{t+1}, \Omega_{t+1}), \Omega_{t+1}) \mid M_{t+1}\big) \\
&\textit{s.t.} \\
M_{t+1} &= \{\{y_l\}_{l=1}^{t}, a_{t+1}, \theta_{t+1}, \textit{edu}\} \\
a_{t+1}(M_t, \Omega_t) &= (1+r)\Big(a_t + y_t - \tau^{inc}(y_t) - \tau^{SS}(y_t) - c_t\Big), y_t = \exp(\mu(\textit{edu}, t) + \theta_t) \cdot h_t(M_t, \Omega_t) \\
\{y_l\}_{l=1}^{t} &= \{\{y_l\}_{l=1}^{t-1}, \exp(\mu(\textit{edu}, t) + \theta_t) \cdot h_t(M_t, \Omega_t)\} \\
\theta_{t+1} &= \theta_t + \epsilon_{t+1},\ \epsilon_{t+1} \sim N(0, {\sigma_\epsilon}^2)
\end{aligned} \tag{34}$$

For $t = T_{ER}$, the state variables are $M_{T_{ER}} = \{\{y_l\}_{l=1}^{T_{ER}-1}, a_{T_{ER}}, \theta_{T_{ER}}, \textit{edu}\}$ and the control variables are $\pi_{T_{ER}}(M_{T_{ER}}, \Omega_{T_{ER}}) = \{\mathbb{P}^R_{T_{ER}}(M_{T_{ER}}, \Omega_{T_{ER}}), a^W_{T_{ER}+1}(M_{T_{ER}}, \Omega_{T_{ER}}), a^R_{T_{ER}+1}(M_{T_{ER}}, \Omega_{T_{ER}}), h^W_{T_{ER}}(M_{T_{ER}}, \Omega_{T_{ER}})\}$, where $a^W_{T_{ER}+1}$ and $a^R_{T_{ER}+1}$ are the assets left for period $T_{ER}+1$ in case of working or retiring at period T_{ER}, respectively and $\mathbb{P}^R_{T_{ER}}$ is the probability of choosing to be retired at period T_{ER}. The individual solves

$$
\begin{aligned}
\Omega^*_{T_{ER}} &= argmax_{\Omega_{T_{ER}}} \mathbb{E} \mid_{\theta_{T_{ER}+1}} V_{T_{ER}}\big(\pi_{T_{ER}}(M_{T_{ER}}, \Omega_{T_{ER}}) \mid M_{T_{ER}}\big) = \\
&\quad \mathbb{P}^R_{T_{ER}}(M_{T_{ER}}, \Omega_{T_{ER}}) \cdot u^{T_{ER}}\big(a^R_{T_{ER}+1}(M_{T_{ER}}, \Omega_{T_{ER}}), b(T_{ER}, s(\{y_l\}_{t=1}^{T_{ER}-1}))\big) \\
&\quad + \Big(1 - \mathbb{P}^R_{T_{ER}}(M_{T_{ER}}, \Omega_{T_{ER}})\Big) \cdot u^{T_{ER}}\big(a^W_{T_{ER}+1}(M_{T_{ER}}, \Omega_{T_{ER}}), h^W_{T_{ER}+1}(M_{T_{ER}}, \Omega_{T_{ER}})\big) \\
&\quad + \beta \mathbb{E} \mid_{\theta_{T_{ER}+1}} V_{T_{ER}+1}\big(\pi_{T_{ER}+1}(M_{T_{ER}+1}, \Omega_{T_{ER}+1}), \Omega_{T_{ER}+1}) \mid M_{T_{ER}+1}\big) \\
&\textit{s.t.} \\
M_{T_{ER}+1} &= \{\mathbb{P}^R_{l<T_{ER}+1}, \{y_l\}_{l=1}^{T_{ER}}{}_{|_W}, a_{T_{ER}+1|_W}, a_{T_{ER}+1|_R}, B_{T_{ER}+1|_R}, \theta_{T_{ER}+1}, \textit{edu}\} \\
\mathbb{P}^R_{l<T_{ER}+1} &= \mathbb{P}^R_{T_{ER}}(M_{T_{ER}}, \Omega_{T_{ER}}) \\
\{y_l\}_{l=1}^{T_{ER}}{}_{|_W} &= \{\{y_l\}_{l=1}^{T_{ER}-1}, \exp(\mu(\textit{edu}, T_{ER}) + \theta_{T_{ER}}) \cdot h^W_{T_{ER}}(M_{T_{ER}}, \Omega_{T_{ER}})\} \\
a_{T_{ER}+1|_W} &= a^W_{T_{ER}+1}(M_{T_{ER}}, \Omega_{T_{ER}}) = (1+r)\Big(a_{T_{ER}} + y^W_{T_{ER}} - \tau^{inc}(y^W_{T_{ER}}) - \tau^{SS}(y^2_{T_{ER}}) - c^W_{T_{ER}}\Big), y^W_{T_{ER}} = \\
&\quad \exp(\mu(\textit{edu}, T_{ER}) + \theta_{T_{ER}}) \cdot h^W_{T_{ER}}(M_{T_{ER}}, \Omega_{T_{ER}}) \\
a_{T_{ER}+1|_R} &= a^R_{T_{ER}+1}(M_{T_{ER}}, \Omega_{T_{ER}}) = (1+r)\Big(a_{T_{ER}} + b(T_{ER}, s(\{y_l\}_{t=1}^{T_{ER}-1})) - c^R_{T_{ER}}\Big) \\
B_{T_{ER}+1|_R} &= b(T_{ER}, s(\{y_l\}_{t=1}^{T_{ER}-1})) \\
\theta_{T_{ER}+1} &= \theta_{T_{ER}} + \epsilon_{T_{ER}+1},\ \epsilon_{T_{ER}+1} \sim N(0, {\sigma_\epsilon}^2)
\end{aligned}
\tag{35}
$$

For $t = T_{ER}+1, \cdots, T_{FR}-1$, the state variables are $M_t = \{\mathbb{P}^R_{l<t}, \{y_l\}_{l=1}^{t-1}{}_{|_W}, a_{t|_W}, a_{t|_R}, B_{t|_R}, \theta_t, \textit{edu}\}$, where $\mathbb{P}^R_{l<t}$ is the probability of entering period t while being retired (becoming retired at one time in the past). All the variables that are conditional on being in the working stage ($\{y_l\}_{l=1}^{t-1}{}_{|_W}$ for history of earning at t and $a_{t|_W}$ for asset at t) show state variables in t conditional on entering period t in working stage (not being retired). All the variables that are conditioned on being on retirement stage ($B_{t|_R}$ for the yearly pension benefit at t and $a_{t|_R}$ for asset at t) show state variables in t conditional on entering period t while in retirement stage (getting retired at $s < t$ in the past).

The control variables are

$$
\pi_t(M_t; \Omega_t) = \{\mathbb{P}^R_t(M_t, \Omega_t), a^W_{t+1|_W}(M_t, \Omega_t), a^R_{t+1|_W}(M_t, \Omega_t), a^R_{t+1|_R}(M_t, \Omega_t), h^W_{t}{}_{|_W}(M_t, \Omega_t)\}, \tag{36}
$$

where $a^W_{t+1|_W}(M_t$ and $\Omega_t), a^R_{t+1|_W}(M_t$ are the assets left for period $t+1$ conditional on being in working stage in t in case of working or retiring at period t, respectively. $h^W_{t}{}_{|_W}(M_t, \Omega_t)$ is the amount of work conditional on being still in working stage. $a^R_{t+1|_R}(M_t, \Omega_t)$ is the asset left for period $t+1$ conditional on being retired in age t. $\mathbb{P}^R_t(M_t, \Omega_t)$ is the probability of choosing to be retired at period t. The individual solves

$$
\begin{aligned}
\Omega_t^* &= argmax_{\Omega_t}\mathbb{E}\mid_{\theta_{t+1}} V_t\big(\pi_t(M_t,\Omega_t)\mid M_t\big) = \\
&\mathbb{P}^R_{l<t}\cdot u^t\big(a^R_{t+1|_R}(M_t,\Omega_t);a_{t|_R},B_{t|_R}\big) \\
&+\big(1-\mathbb{P}^R_{l<t}\big)\cdot\Big[\mathbb{P}^R_t(M_t,\Omega_t)\cdot u^t\big(a^R_{t+1|_W}(M_t,\Omega_t),b(t,s(\{y_l\}^{t-1}_{l=1}{}_{|_W}));a_{t|_W}\big) \\
&+(1-\mathbb{P}^R_t(M_t,\Omega_t))\cdot u^t\big(a^W_{t+1|_W}(M_t,\Omega_t),h^W_t{}_{|_W}(M_t,\Omega_t);a_{t|_W}\big)\Big] \\
&+\beta\mathbb{E}\mid_{\theta_{t+1}} V_{t+1}\big(\pi_{t+1}(M_{t+1},\Omega_{t+1}),\Omega_{t+1})\mid M_{t+1}\big) \\
\textit{s.t.} \\
M_{t+1} &= \{\mathbb{P}^R_{l<t+1},\{y_l\}^t_{l=1|_W},a_{t+1|_W},a_{t+1|_R},B_{t+1|_R},\theta_{t+1},\boldsymbol{edu}\} \\
\mathbb{P}^R_{l<t+1} &= \mathbb{P}^R_{l<t}+(1-\mathbb{P}^R_{l<t})\cdot\mathbb{P}^R_t(M_t,\Omega_t) \\
\{y_l\}^t_{l=1|_W} &= \{\{y_l\}^{t-1}_{l=1},\exp(\mu(\boldsymbol{edu},t)+\theta_t)\cdot h^W_t{}_{|_W}(M_t,\Omega_t)\} \\
a_{t+1|_W} &= a^W_{t+1|_W}(M_t,\Omega_t) = (1+r)\Big(a_{t|_W}+y^W_t-\tau^{inc}(y^W_t)-\tau^{SS}(y^W_t)-c^W_t{}_{|_W}\Big), y^W_t = \\
&\exp(\mu(\boldsymbol{edu},t)+\theta_t)\cdot h^W_t{}_{|_W}(M_t,\Omega_t) \\
a_{t+1|_R} &= \mathbb{P}^R_{l<t}\cdot a^R_{t+1|_R}(M_t,\Omega_t)+(1-\mathbb{P}^R_{l<t})\cdot a^R_{t+1|_W}(M_t,\Omega_t), a^R_{t+1|_R}(M_t,\Omega_t) = \\
&(1+r)\Big(a_{t|_R}+B_{t|_R}-c^R_t{}_{|_R}\Big), a^R_{t+1|_W}(M_t,\Omega_t) = (1+r)\Big(a_{t|_W}+b(t,s(\{y_l\}^{t-1}_{l=1|_W}))-c^R_t{}_{|_W}\Big) \\
B_{t+1|_R} &= \mathbb{P}^R_{l<t}\cdot B_{t|_R}+(1-\mathbb{P}^R_{l<t})\cdot b(t,s(\{y_l\}^{t-1}_{l=1|_W})) \\
\theta_{t+1} &= \theta_t+\epsilon_{t+1},\ \epsilon_{t+1}\sim N(0,{\sigma_\epsilon}^2)
\end{aligned}
\tag{37}
$$

For $t = T_{FR}$, the state variables are $M_{T_{FR}} = \{\mathbb{P}^R_{l<T_{FR}}, \{y_l\}^{T_{FR}-1}_{l=1}{}_{|_W}, a_{T_{FR}|_W}, a_{T_{FR}|_R}, B_{T_{FR}|_R}\}$, where $\mathbb{P}^R_{l<T_{FR}}$ is the probability of entering period T_{FR} while being retired (becoming retired at one time in the past). All the variables that are conditional on being in the working stage ($\{y_l\}^{T_{FR}-1}_{l=1}{}_{|_W}$ for history of earning at T_{FR} and $a_{T_{FR}|_W}$ for asset at T_{FR}) show state variables in T_{FR} conditional on entering period T_{FR} in working stage (not being retired). All the variables that are conditioned on being on retirement stage ($B_{T_{FR}|_R}$ for the yearly pension benefit at T_{FR} and $a_{T_{FR}|_R}$ for asset at T_{FR}) show state variables in T_{FR} conditional on entering period T_{FR} while in retirement stage (getting retired at $s < T_{FR}$ in the past).

The control variables are

$$
\pi_{T_{FR}}(M_{T_{FR}};\Omega_{T_{FR}}) = \{a_{T_{FR}+1|_W}(M_{T_{FR}},\Omega_{T_{FR}}), a_{T_{FR}+1|_R}(M_{T_{FR}},\Omega_{T_{FR}})\}, \tag{38}
$$

where $a_{T_{FR}+1|_W}(M_{T_{FR}}$is the asset left for period $T_{FR}+1$ conditional on being in working stage in $t = T_{FR}$ in case of working or retiring at period T_{FR}. $a_{T_{FR}+1|_R}(M_{T_{FR}},\Omega_{T_{FR}})$ is the asset left for period $T_{FR}+1$ conditional on being retired in age T_{FR}. The individual solves

$$
\begin{aligned}
\Omega^*_{T_{FR}} &= argmax_{\Omega_{T_{FR}}} V_{T_{FR}}\big(\pi_{T_{FR}}(M_{T_{FR}}, \Omega_{T_{FR}}) \mid M_{T_{FR}}\big) = \\
&\quad \mathbb{P}^R_{l<T_{FR}} \cdot u^{T_{FR}}\big(a_{T_{FR}+1|_R}(M_{T_{FR}}, \Omega_{T_{FR}}); a_{T_{FR}|_R}, B_{T_{FR}|_R}\big) \\
&\quad + \big(1 - \mathbb{P}^R_{l<T_{FR}}\big) \cdot u^{T_{FR}}\big(a_{T_{FR}+1|_W}(M_{T_{FR}}, \Omega_{T_{FR}}), b(T_{FR}, s(\{y_l\}_{l=1}^{T_{FR}-1}{}_{|_W})); a_{T_{FR}|_W}\big) \\
&\quad + \beta V_{T_{FR}+1}\big(\pi_{T_{FR}+1}(M_{T_{FR}+1}, \Omega_{T_{FR}+1}), \Omega_{T_{FR}+1}) \mid M_{T_{FR}+1}\big) \\
&\textit{s.t.} \\
M_{T_{FR}+1} &= \{a_{T_{FR}+1}, B_{T_{FR}+1}\} \\
a_{T_{FR}+1} &= \mathbb{P}^R_{l<T_{FR}} \cdot a_{T_{FR}+1|_R}(M_{T_{FR}}, \Omega_{T_{FR}}) + (1 - \mathbb{P}^R_{l<T_{FR}}) \cdot a_{T_{FR}+1|_W}(M_{T_{FR}}, \Omega_{T_{FR}}), a_{T_{FR}+1|_R}(M_{T_{FR}}, \Omega_{T_{FR}}) = \\
&\quad (1+r)\Big(a_{T_{FR}|_R} + B_{T_{FR}|_R} - c^R_{T_{FR}|_R}\Big), a_{T_{FR}+1|_W}(M_{T_{FR}}, \Omega_{T_{FR}}) = \\
&\quad (1+r)\Big(a_{T_{FR}|_W} + b(T_{FR}, s(\{y_l\}_{l=1}^{T_{FR}-1}{}_{|_W})) - c^R_{T_{FR}|_W}\Big) \\
B_{T_{FR}+1|_R} &= \mathbb{P}^R_{l<T_{FR}} \cdot B_{T_{FR}|_R} + (1 - \mathbb{P}^R_{l<T_{FR}}) \cdot b(T_{FR}, s(\{y_l\}_{l=1}^{T_{FR}-1}{}_{|_W}))
\end{aligned} \tag{39}
$$

For $t > T_{FR}$, the state variables are $M_t = \{a_t, B_t\}$ and the control variables is $\{a_{t+1}\}$. The individual solves

$$
\begin{aligned}
\Omega^*_t &= argmax_{\Omega_t} V_t\big(\pi_t(M_t, \Omega_t) \mid M_t\big) = \\
&\quad u^t\big(a_{t+1}(M_t, \Omega_t); a_t, B_t\big) \\
&\quad + \beta V_{t+1}\big(\pi_{t+1}(M_{t+1}, \Omega_{t+1}), \Omega_{t+1}) \mid M_{t+1}\big) \\
&\textit{s.t.} \\
M_{t+1} &= \{a_{t+1}, B_{t+1}\} \\
a_{t+1} &= (1+r)\Big(a_t + B_t - c_t\Big) \\
B_{t+1|_R} &= B_t
\end{aligned} \tag{40}
$$

6.2.5 Network Architecture

The deep neural net architecture is consisted of

- T deep neural nets.
- 4 layers, each with 10 neuron.
- Relu activation function between layers.
- The first 2 layers are general, the third layer is output specific, and the forth layer is output and year specific.

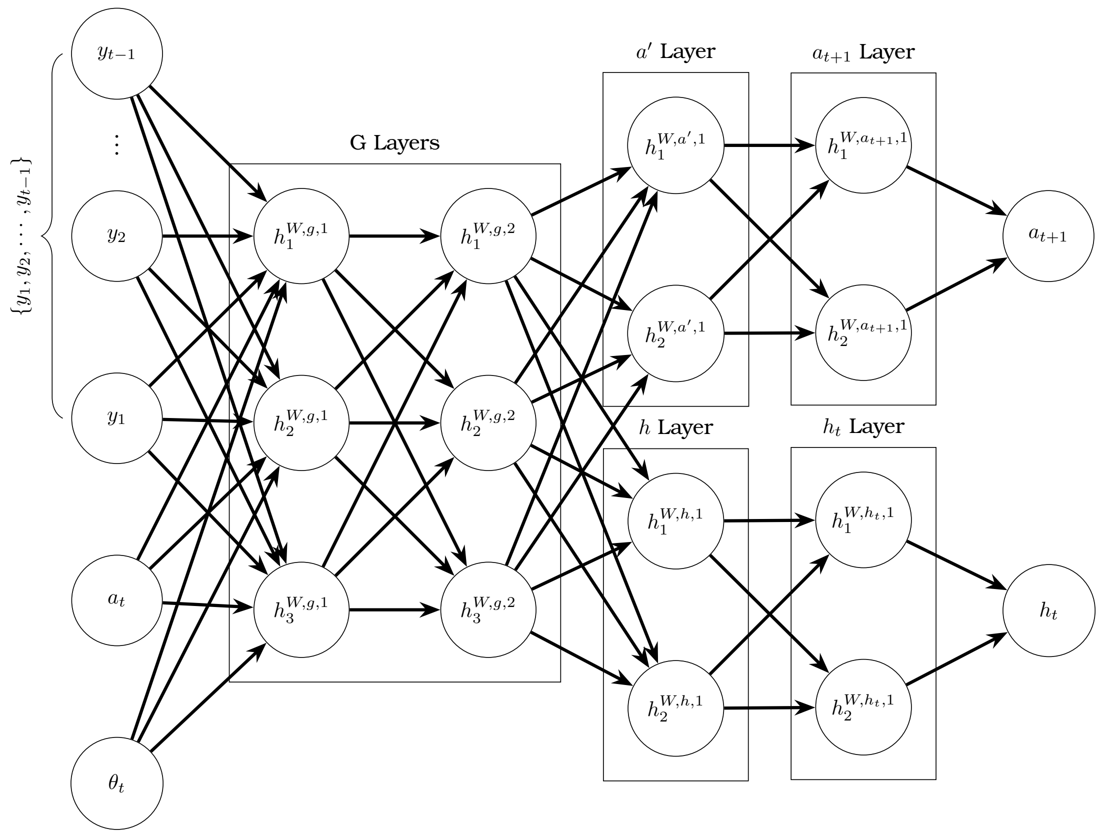


Figure 11: The schematic diagram of the deep neural net for policy function at year t. For clarity of the graph, not all the neurons in each layer are drawn.

- The DNN architecture is drawn for $t = T$. For each $t-1$, the y_t and its relative weights are dropped until there are only 2 inputs of (θ, a) for $t = 1$.

6.2.6 Sudo-Algorithm

1. Simulate path of exogenous state variables for weighted nodes according to 6.2.7.
2. Initials random wights (Ω^{Init}) for the DNN described in 6.2.5.
3. Simulate the sample created using the method described in 6.3.
4. Calculate the loss functions in 6.3.1 for the sample.
5. Update the parameters of DNN according to ADAM algorithm.

6.2.7 Calculation of Expectations

Use Monto Carlo method with $N_\epsilon = 100000$ to calculate the integral in Equation (16).

1. Simulate $N_\epsilon \times T$ normal draws from the distribution of ϵ to simulate $J = N_\epsilon$ possible paths for $\{\epsilon_t\}_{t=\{1,\cdots,T\}}$.
2. Using Equation (19) to calculate the path $\{\theta_t\}^j_{t=\{1,\cdots,T\}}$ for each $j = 1, \cdots, J$.
3. For each j, also draw $\textit{edu}_j$ form the Bernoulli distribution of education types.

6.3 Sample

In each iteration (for finding the parameters of approximated policy function (DNNs)):

Starting from $t = 1$ recursively calculate the path endogenous state variables $\{h^j_t, a^j_{t+1}\}_{t=\{1,\cdots,T\}}$ for each $j = 1, J$ using policy functions ($a_{t+1}(\Omega)$ and $h_t(\Omega)$) and the simulated exogenous state variables $\textit{edu}_j$ and $\{\theta_t\}^j_{t=\{1,\cdots,T\}}$ until $t = T_{ER}$, where Ω is the set of weights and biases of the deep neural net approximating the policy function $a_{t+1}(\{y_s\}^{t-1}_{s=1}, a_t, \theta_t)$ and $h_t(\{y_s\}^{t-1}_{s=1}, a_t, \theta_t)$ at this iteration, for each j: Given the θ^j_1 and $a^j_1 = 0$ using the DNN approximated policy functions for $t = 1$, a^j_2 and h^j_1 is calculated. Next, using θ^j_2 and the simulated a^j_2 and h^j_2, a^j_3 and h^j_3 are calculated from DNN approximated policy functions for $t = 2$. We continue to simulate h^j_t and a^j_t until $t = T$ using the DNN approximated policy functions and previous states.

6.3.1 Loss Function

Consolidating the budget constraint in the objective function, replacing integrals with sum with numerical weights, approximating policy functions with deep neural nets, taking into account of policy variable constraints using network architecture. We can write the the loss function as

$$\mathbb{L} = -\, l_1 \times \sum_{j=1}^{J} \left(\sum_{t=1}^{T} \beta^{t-1} \left[\frac{(c_t^j)^{1-\gamma}}{1-\gamma} - \psi \frac{(h_t(M_t^j;\Omega_t^{\lambda_t^j})/\bar{h})^{1+\frac{1}{\eta}}}{1+\frac{I}{\eta}} - \phi I\{h_t(M_t^j;\Omega_t^{\lambda_t^j}) > 0\} \right. \right. \tag{41}$$

$$+ \sum_{t=1}^{T_{FR}-1} \frac{1}{(T-t)} \left(l_g \times \sum_{o=1}^{N_t^{\omega_g}} \left[\omega_{T_{FR}-1}^{g,o} - \omega_t^{g,o}\right]^2 + l_a \times \sum_{o=1}^{N_t^{\omega_a}} \left[\omega_{T_{FR}-1}^{a,o} - \omega_t^{a,o}\right]^2 + l_h \times \sum_{o=1}^{N_t^{\omega_h}} \left[\omega_{T_{FR}-1}^{h,o} - \omega_t^{h,o}\right]^2 \right) \tag{42}$$

where

$$c_t^j = a_t(M_{t-1}^j;\Omega_{t-1}^{\lambda_{t-1}^j}) - \frac{1}{1+r} a_{t+1}(M_t^j;\Omega_t^{\lambda_t^j}) + 1\{t < t_R^j\}\left[y_t^j - \tau^{inc}(y_t^j) - \tau^{SS}(y_t^j)\right] + 1\{t \geq t_R^j\} B_t^j \tag{43}$$

$$y_t^j = w_t^j \cdot h_t(M_t^j;\Omega_t^{\lambda_t}) \tag{44}$$

$$t_R^j = \min\{\textit{argmin}_t[R_t^j = 1], T_{FR}\} \tag{45}$$

$$B_{t_R}^j = b(t_R^j, S^j), S = s(\{y_l^j\}_{t=1}^{t_R^j-1}),\ B_t^j = B_{t-1}^j, t > t_R \tag{46}$$

$$\lambda_t^j = \begin{cases} W & \textit{if}\, t < t_R^j, \\ R & \textit{if}\, t \geq t_R^j. \end{cases} \tag{47}$$

$$M_t^j = \begin{cases} \{\{y_l^j\}_{l=1}^{t-1}, a_t^j, \theta_t^j, \textit{edu}^j\} & \textit{if}\, t < t_R, \\ \{B_t^j, a_t^j\} & \textit{if}\, t \geq t_R. \end{cases} \tag{48}$$

$$\tag{49}$$

Notes:

- $\Omega_t^{\lambda_t}$ shows the weighs of the neural net architecture type $\lambda_t \in \{W, R\}$ at time t.
- The weights of general layers are regularized by l_G and the output specific layers are regularized by l_h and l_a. The output-year specific layer is not regularized. All the weights are regularized relative to the network at the last year (T). The weights of each regulation term is relative to its time distance from the $t = T$.
- $\{\omega^G, \omega^h, \omega^a\}$ are the weights of general and specific layer series. $\{n_{w_G}, n_{w_a}, n_{w_h}\}$ are the number of parameters of each layer series.

7 Counterfactual Policies (In Progress)

In this section, I utilize the presented quantitative model and evaluate the effects of a counterfactual policy that changes the current US pension policy which uses the top 35 years of earnings to account for the lifetime earnings, common in other OECD countries.

8 Conclusion

In most countries that use a defined-benefit system, two functions working in the composite determine the amount of pension benefits. One function summarizes all of the history of earnings into one outcome, and the other finds the amount of benefit based on that outcome. In the US, ignoring some details, the history of earnings will be summarized by taking the average of the top 35 years of earnings of a worker. Then, this result will yield the retirement benefit after passing through a progressive benefit function. History dependence of pension systems influences labor supply at old age and the retirement age, the level of redistribution among workers based on their full history of earnings, and consumption insurance of retirees. Studying the effects of the design of the history-dependent part of the pension system and ways to improve it (an understudied area of research) is the main goal of this paper. In this research, I introduce some new stylized facts and show how utilizing different methods to calculate pension benefits from the history of earnings would affect different workers. Next, I examine how different ways to summarize the history of earnings affect workers. I develop a dynamic model of labor supply, saving, and retirement with labor market shocks and evaluate a counterfactual policy that changes the current US pension policy which uses the top 35 years of earnings to account for lifetime earnings.

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