Price competition with zero consumer search costs and limited capacity*

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Abstract

The Diamond paradox says that competing firms offering perfect substitutes set monopoly prices if consumers have strictly positive search costs. By contrast, if consumers are fully informed, with constant marginal costs the perfectly competitive outcome prevails. In our setting, consumers have zero search costs and search sequentially for the best price. At least one firm is capacity-constrained and thus cannot serve all consumers at the competitive price. We provide conditions such that in duopoly, firms set the monopoly price. We provide further equilibrium characterization when these conditions are not satisfied.

Keywords: Price competition, consumer search, Diamond paradox, limited capacity, endogenous search cost

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1 Introduction

Price competition for homogeneous products is one of the foundational models of microeconomics. Suppose that duopolists have the same constant marginal costs of production and consumers observe all prices. Then, firms set prices equal to marginal costs (Bertrand, 1883). By contrast, if firms face capacity constraints (which is an extreme form of increasing marginal costs), there does not exist a pure-strategy price equilibrium when capacities are neither too small nor too large (Edgeworth, 1925). Instead of focusing on firms' limited volume to sell in the market, a different challenge to the undercutting logic inherent in the Bertrand model arises from limited consumer information: If consumers only observed the price of one of the firms and have to pay a search cost to learn the price of competitors, firms set prices equal to the monopoly price (Diamond, 1971). However, to obtain the monopoly outcome, search costs must be strictly positive.

In this paper, we assume that consumers search sequentially for prices, but at zero costs. Consumers are identical and have downward-sloping individual demand. If firms did not face any capacity constraints (constant marginal costs for any quantity up to market demand at the competitive price), the Bertrand result holds. However, if one of the two firms has insufficient capacity to sell to consumers at the competitive price, under some conditions, both firms set monopoly prices.

We consider two versions of our model. In the first version, the share of consumers who visit a particular firm first is an exogenous parameter. In this version, if the capacity-constrained firm cannot serve all its initial visitors (provided they all want to buy immediately), then the monopoly pricing emerges in the *unique* equilibrium. We also characterize a mixed-strategy equilibrium when this condition does not hold. Furthermore, when restricting attention to unit demand, we establish the uniqueness of equilibrium.

Our first version is motivated by the observation that, in many markets, consumers are steered to visit a particular firm first. For instance, in a physical retail environment, some consumers are located next to one shop and the others are located next to the competing shop – therefore, the share of consumers who visit any given firm is exogenous. Another instance is an e-commerce environment in which a platform steers consumers on its marketplace by showing firm 1's offer first to a share of consumers and firm 2's offer to the remaining share.[1] Consumers then have the option to buy the first offer they see or to search for a cheaper

[1] An example of such steering is the Amazon Buy Box which makes one of the sellers visible; this seller does not necessarily offer the lowest price, and Amazon can randomize over consumers and thus sets the fraction of consumers who visit the capacity constrained firm first.

option.

In the second version of the model, we endogenize the fraction of consumers who see firm 1's offer first and focus on the case of unit demand. In this case, we show that if the capacity of the restricted firm is sufficiently low, then all consumers prefer to start searching from the constrained firm. We show the existence of a unique equilibrium; this equilibrium features monopoly prices. Perhaps surprisingly, under unit demand assumption, monopoly pricing is the unique equilibrium outcome for all capacity levels up to almost three quarters of the market.

Related literature Limited capacity and costly consumer search are well-known market characteristics that lead to higher than perfectly competitive prices under price competition with homogeneous products. In duopoly, if one firm can not serve all consumer demand at marginal costs this firms can not deprive its competitor from all profits and allows the competitor (which may or may not be capacity constrained itself) to obtain some profit.

Costly consumer search leads to monopoly prices if all consumers have strictly positive search costs (Diamond, 1971). Consumer search theory has considered instances in which only some consumers have positive search costs and characterized the ensuing mixed-strategy equilibrium. By contrast, in our model consumers do not have any search costs. Monopoly pricing arises from a combination of limited capacity and sequential search, since this implies that consumers who return to an earlier offer may be rationed.

Classical papers on capacity constraints, see Levitan and Shubik (1972) and Kreps and Scheinkman (1983), often predict mixed strategy equilibria. Our equilibrium with all consumers starting with the unconstrained firm resembles results from this literature, with the only difference that in our model the firms are asymmetric. Equilibria with other first visit decisions are different, most notably the monopoly pricing pure strategy equilibrium does not exist in that literature if firms combined capacity is sufficient to cover the whole market.

In our model, limited capacity deters search by making recall uncertain. This mechanism is similar to the one in the model of search deterrence by Armstrong and Zhou (2016), where the seller can increase the price if the buyer returns. The capacity constraint naturally resolves the commitment power issue that arises in their paper. In the homogeneous goods version of their paper (see section 5.2.2), a monopoly pricing equilibrium may arise when all sellers prohibit recall and consumers expect the same price at all firms. In our model, only one seller needs to be capacity constrained. Moreover, our monopoly pricing equilibrium is robust in the sense that it remains an equilibrium even if a small fraction of consumers know both prices prior to search. The monopoly pricing equilibrium in Armstrong and Zhou

(2016) is not robust according to this definition. Furthermore, contrary to their results, the monopoly pricing equilibrium in our model is unique for sufficiently small k.

The paper proceeds as follows. In Section 2 we spell out the details of the model. In Section 3 we characterize the set of equilibria for general demand functions when initial consumer visits are exogenous. Then we establish uniqueness of equilibrium and obtain comparative statics results for unit demand case. In Section 4 we consider a model with endogenous consumer first visits.. In Section 5 discusses the role of assumptions and possible extensions of the model. Section **??** concludes.

2 Model

Two firms with constant marginal costs of production c up to their respective capacities compete in a homogeneous product industry. They sell to a unit mass of consumers. Each consumer demands $Q(p)$ units of the good at a price p. We assume that $Q(\cdot)$ is a weakly decreasing and differentiable function up to its finite choke price $\check{p}$. We assume that per-consumer profit, $\pi(p) \equiv (p-c)Q(p)$ is strictly quasi-concave on $[c, \check{p}]$. This implies that there is a unique maximizer, which we denote by p^m. We define consumer surplus of buying at a price p as $CS(p) \equiv \int_p^{\check{p}} Q(p')dp'$. As a special case, we consider unit demand with valuation $v > c$, implying that $p^m = v$. One of the two firms (firm 1) is unconstrained in the sense that its own capacity is sufficient to satisfy market demand at marginal costs – that is, to supply $Q(c)$ units of the product. The other firm (firm k) can not serve full demand at marginal cost and is constrained by its capacity $k < Q(c)$.

We present two models of sequential search. In the first model the order in which offers are presented is given (and may be different across consumers): fraction α of consumers are first steered to the constrained firm and the remaining fraction $1-\alpha$ to the unconstrained firm. In the second model, consumers freely decide with which firm to start leading to an endogenous α.

Consumers search over time, but they neither have a monetary search cost nor do they have an opportunity cost of time. Nevertheless, continued search may lead to rationing and search may turn out to be costly because by continuing search consumers may foregoes the possibility to buy the lower-priced product if it is sold out when the consumer returns.

At $t = 0$, firms simultaneously set prices, which can only be observed by consumers after search. At $t = 1$, consumers search one of the firms and decide whether to try to buy their preferred quantity of the product $Q(p_i)$. If they are unsuccessful or decide not to try, they enter the next period. At $t = 2$, remaining consumers visit the firm they did not yet visit. Upon observing the prices consumers decide whether they want to attempt to buy or come back. This process continues until all consumers made their purchases. When demand exceeds capacity we assume random consumer rationing: each consumer either is served and receives their preferred quantity $Q(p_i)$ or is not served.

3 Exogenous decision about first visit

In this section, we assume that an exogenous fraction of consumers α first visit the capacity-constrained firm.

3.1 General demand

Our first result establishes monopoly pricing when the capacity of the constrained firm is sufficiently small. The critical capacity level is such that the constrained firm can just satisfy demand of all first visits at the monopoly price.

Proposition 1. *Suppose that $\alpha Q(p^m) > k$. Then, there is a unique equilibrium outcome with $p_1 = p_k = p^m$.*

The intuition behind Proposition 1 is as follows. By contradiction, suppose that a non-monopoly price equilibrium existed. Let r be the reservation price associated with the price distribution of the unconstrained firm. Since the constrained firm faces demand larger than its capacity, it has no incentive to a charge price below r. Consequently, the unconstrained firm does not charge prices below r either, because charging a lower price would not attract additional demand. Given that the reservation price is weakly above the expected price of the unconstrained firm, we conclude that its price distribution is supported by a single price, r. Since the constrained firm cannot serve all consumers at this price, we must have that $r = p^m$.

Note that this equilibrium is different from the equilibrium with search costs in Diamond (1971). In our model, search costs are zero. Moreover, there is no price risk associated with the recall uncertainty on the equilibrium price path. The capacity constraint only affects consumer behavior in putative equilibria with price dispersion. Nonetheless, this force is strong enough to eliminate any such equilibria.

In this equilibrium, as in all search models, consumers immediately (attempt to) buy when indifferent between buying and searching. An attentive reader may notice that if such a tie-breaking rule were imposed in a standard Bertrand duopoly, then any pair of prices $p_i = p_j \in [c, p^m]$ would be an equilibrium. Such equilibria, however, are not robust in the following sense. Let λ be the fraction of consumers who know both prices before their first visit and let $p_i(\lambda)$ be corresponding equilibrium price.[2] We call a pair of prices (p_i, p_j) a robust equilibrium outcome if $p_i = \lim_{\lambda\to 0} p_i(\lambda)$. In Bertrand duopoly, the only robust equilibrium is $p_i = p_j = c$, as at any other p the firm can win the λ of informed consumers by slightly dropping its price. The monopoly pricing equilibrium from Proposition 1 is robust. Indeed, if $\lambda < k/Q(p^m) - \alpha$, then the constrained firm still sells out at the monopoly price, and hence the unconstrained firm would not increase its profit when setting a price less than $p_1 = p^m$.

We proceed with the case of "large" capacities, that is when the constrained firm has sufficient capacity to serve demand from all its first-time visitors at the monopoly price.

Lemma 1. *Suppose that $\alpha Q(p^m) \leq k$. Then there are no robust pure strategy pricing equilibria.*

We proceed with the characterization of the mixed-strategy equilibria for $\alpha Q(p^m) < k < Q(c)$. We denote the distribution functions of the unconstrained and constrained firms as $F_1(\cdot)$ and $F_k(\cdot)$ respectively. Corresponding upper and lower bounds are denoted by $\underline{p}_i, \overline{p}_i$, $i = 1, k$. We start with characterizing the boundaries of the supports of the equilibrium price distributions.

Lemma 2. *In any mixed strategy equilibrium $\underline{p} \equiv \underline{p}_1 = \underline{p}_k > c$ and $\max\{\overline{p}_1, \overline{p}_k\} \leq p^m$. Moreover, $\alpha Q(\underline{p}) < k$.*

Consumers who start their search from the unconstrained firm always weakly prefer checking the price of the competitor, implying that the reservation price equals to $\underline{p}_1$. Consumers who started their search from the unconstrained firm face a risk of being rationed out if they decide to visit the unconstrained firm and, upon discovery of a larger price, attempt to return. We denote by r the reservation price, i.e. the price at which consumers are indifferent between immediately buying from the constrained firm and searching the unconstrained competitor. Note that in any equilibrium, consumers must stop when indifferent (otherwise, a firm would prefer charging $r - \varepsilon$ to charging r). As all consumers visiting the constrained firm immediately purchase at $p_k = r$, a consumer contemplating search will not return: if the price in the unconstrained firm is lower than r, she will buy there; if the price is higher than r, the capacity will be sold out to searchers coming from the unconstrained firm. Hence,

$$CS(r) = \mathbb{E}\left[CS(p_1)\right]. \tag{1}$$

[2] Note that this notion of robustness naturally extends to mixed pricing strategies $F_i(p; \lambda)$.

Lemma 3. *Any mixed strategy equilibrium has the following properties:*

- *If $\alpha > 0$, the equilibrium price distribution F_1 has support $[\underline{p}, r] \bigcup [\hat{p}, p^m]$ with $\hat{p} \leq p^m$. If $\hat{p} < p^m$ then F_k has the same support, and if $\hat{p} = p^m$ then F_k has support on $[\underline{p}, r]$. F_k has an atom at r, F_1 has an atom at p^m, and both distribution functions are continuous at $p \neq r, p^m$.*

- *If $\alpha = 0$, the equilibrium price distribution functions have support $[\underline{p}, p^m]$. F_k is continuous on its support and F_1 is continuous on $[\underline{p}, p^m)$ and has an atom at p^m.*

Lemma 3 establishes the necessary conditions for the existence of equilibrium in mixed strategies.

We introduce notation

$$\psi(p) \equiv \mathbb{E}_k[1/Q(p_k)|p_k \leq p]. \tag{2}$$

That is, $k\psi(p)$ is the expected number of consumers the constrained firm serves, given that it charges a price below p. [3]

Consider the unconstrained firm. We start with the lower part of the support, $p \in [\underline{p}, r)$. The profit function of the unconstrained firm is given by

$$\Pi_1(p) = [1 - F_k(r)]\pi(p) + [F_k(r) - F_k(p)](1-\alpha)\pi(p) + F_k(p)[1 - k\psi(p)]\pi(p) \tag{3}$$

The first term corresponds to the case when the constrained firm charges the price above r, then the unconstrained firm serves the whole market as $p \in [\underline{p}, r)$. Note that if $F_k(r) = 1$, this term is zero, as serving the whole market is not possible because some consumers do not search beyond the constrained firm. The second term corresponds to the case when the constrained firm charges a price below r, but above the price of the unconstrained firm. In this case, the unconstrained firm serves everyone, except for α consumers who initially visited the constrained firm and did not search beyond it. The third term corresponds to the case when the unconstrained firm loses the price competition and serves the residual demand.

Now, we look at the case when $p > r$. In this case

$$\Pi_1(p) = [1 - F_k(p)]\pi(p) + F_k(p)[1 - k\psi(p)]\pi(p). \tag{4}$$

That is, if the price of the unconstrained firm charges a higher price, then the constrained firm serves the whole market (due to $p_k > p_1 > r$ all consumers search in this case); if the

[3]Note, that $k\psi(p) < 1$ for all prices. To see this, note that $\psi(p)$ is increasing in p. However, $k\psi(p^m) > 1$ would imply that the equilibrium profit of firm 1 is $\Pi_1(p^m) = 0$, but this cannot be the case due to $Q(c) > k$.

unconstrained firm charges a higher price, then it serves the residual demand. Plugging in $p = p^m$ to (4) we can calculate the equilibrium level of profit as

$$\Pi^m = [1 - k\psi(p^m)]\pi(p^m).$$

Plugging in $p = \underline{p}$ to (4) and equating the equilibrium profit to Π^m we obtain

$$F_k(r) = \frac{1}{\alpha}\frac{\pi(\underline{p}) - \Pi^m}{\pi(\underline{p})}.$$

This means that if $F_k(r) = 1$ we obtain $(1-\alpha)\pi(\underline{p}) = \Pi^m$, otherwise $(1-\alpha)\pi(\underline{p}) > \Pi^m$.

Now consider the constrained firm. For $p \leq r$ we obtain

$$\Pi_k(p) = F_1(p)\alpha\pi(p) + [1 - F_1(p)]k(p - c). \tag{5}$$

That is, if the competitor charges a higher price, the constrained firm sells to its α initial visitors. Otherwise, it sells up to full capacity.

For $p > r$ all consumers search, thus the firm sells only when its price is the lowest, i.e.

$$\Pi_k(p) = [1 - F_1(p)]k(p - c). \tag{6}$$

Plugging $p = \underline{p}$ into equation (5), we obtain the equilibrium level of profit for the constrained firm:

$$\Pi_k^* = k(\underline{p} - c).$$

Using the indifference conditions we can derive the equilibrium distribution functions, which we summarise in the following proposition.

Proposition 2. *Suppose $0 < \alpha Q(p^m) < k$. Then, there exist an equilibrium in mixed strategies such that*

$$F_1(p) = \begin{cases} \frac{k}{k-\alpha Q(p)}\left(1 - \frac{\underline{p}-c}{p-c}\right) & p \in [\underline{p}, r) \\ \frac{k}{k-\alpha Q(r)}\left(1 - \frac{\underline{p}-c}{r-c}\right) & p \in [r, \hat{p}) \\ 1 - \frac{\underline{p}-c}{p-c} & p \in [\hat{p}, p^m) \\ 1 & p \geq p^m \end{cases}, \qquad F_k(p) = \begin{cases} \frac{\Pi^m}{k\psi(p)-\alpha}\left(\frac{1}{\pi(\underline{p})} - \frac{1}{\pi(p)}\right) & p \in [\underline{p}, r) \\ \frac{1}{\alpha}\frac{\pi(\underline{p})-\Pi^m}{\pi(\underline{p})} & p \in [r, \hat{p}) \\ \frac{\pi(p)-\Pi^m}{k\psi(p)\pi(p)} & p \in [\hat{p}, p^m) \\ 1 & p \geq p^m \end{cases}$$

where r solves $CS(r) = \mathbb{E}[CS(p_1)]$, $\hat{p} = \min\{\hat{p}_1, p^m\}$ where $\hat{p}_1$ solves $\frac{k(r-\underline{p})}{(r-c)[k-\alpha Q(r)]} = \frac{\hat{p}_1-\underline{p}}{\hat{p}_1-c}$ and $\underline{p}$ solves $F_k(\hat{p}) = \frac{1}{\alpha}\frac{\pi(\underline{p})-\Pi^m}{\pi(\underline{p})}$.

To complete the analysis, we consider the case of $\alpha = 0$, i.e., when all consumers start their search from the unconstrained firm.

Proposition 3. *Suppose that $\alpha = 0$. Then, there exists a unique equilibrium in mixed strategies such that*

$$F_1(p) = \begin{cases} 1 - \frac{\underline{p}-c}{p-c} & p \in [\underline{p}, p^m) \\ 1 & p \geq p^m \end{cases}, \qquad F_k(p) = \begin{cases} \frac{\pi(p)-\Pi^m}{k\psi(p)\pi(p)} & p \in [\underline{p}, p^m) \\ 1 & p \geq p^m \end{cases}$$

where $\underline{p}$ and Π^m solve $\pi(\underline{p}) = \Pi^m$ and $\Pi^m = (1 - k\psi(p^m))\pi(p^m)$.

Note, that the probability distributions in this case correspond to the upper segments of the distributions from Proposition 2. The price distribution of the constrained firm is continuous, and the price distribution of the unconstrained firm has an atom at p^m. Although we do not establish uniqueness of equilibria in Proposition 2, the following Lemma establishes that any such equilibirium, should there be multiple, converges to the equilibrium in Proposition 3.

Lemma 4. *As $\alpha \to 0$, any equilibrium of the model converges to that with $\alpha = 0$ characterized in Proposition 3.*

For $\alpha Q(p^m) < k$ *any* pricing equilibrium is in mixed strategies and must have the structure described in Propositions 2 and 3. Whether variables $(\underline{p}, \hat{p}, r)$ are uniquely determined for a general demand function $Q(\cdot)$ remains an open question. We were able to establish such uniqueness for the case of unit demand (see the next section).

3.2 Unit Demand

In this section, we focus on the special case of unit demand. This allows us to obtain a precise characterization of the parameter set for which $\hat{p}$ is equal to p^m or the alternative set for which $\hat{p}$ is between 0 and p^m. Moreover, we establish the uniqueness of equilibrium and obtain comparative statics results.

Suppose that each consumer has unit demand and values the product at v. This implies that $Q(p) = 1$ for $p \leq v$ and $Q(p) = 0$ otherwise and $v = p^m$. Therefore, Proposition 1 implies that for $\alpha > k$ there is a unique equilibrium in which all firms charge $p = v$.

Moreover, as $\psi(p) = 1$ and $\pi(p) = p - c$ for all $p \leq v$, and $\Pi^m = (1 - k)(v - c)$, we can re-write Propositions 2 and 3 to obtain the following result.

Proposition 4. *For every $k > \alpha \geq 0$, there exists a unique mixed strategy equilibrium.Moreover, there exists a function $\hat{\alpha}(k) \in (0, k/2)$ such that:*

1. *For $\alpha \in [\hat{\alpha}(k), k)$, the equilibrium price distributions are given by:*

$$F_1(p) = \begin{cases} \frac{k}{k-\alpha}\frac{p-\underline{p}}{p-c} & p \in [\underline{p}, r) \\ \frac{k}{k-\alpha}\frac{r-\underline{p}}{r-c} & p \in [r, v) \\ 1 & p \geq v \end{cases}, \qquad F_k(p) = \begin{cases} \frac{1-k}{k-\alpha}\frac{v-c}{p-c}\frac{p-\underline{p}}{\underline{p}-c} & p \in [\underline{p}, r) \\ 1 & p \geq r, \end{cases}$$

where $r = \mathbb{E}[p_1]$ and $\underline{p} = \frac{1-k}{1-\alpha}(v-c) + c$;

2. *For $\alpha \in (0, \hat{\alpha}(k))$, the equilibrium price distributions are given by:*

$$F_1(p) = \begin{cases} \frac{k}{k-\alpha}\frac{p-\underline{p}}{p-c} & p \in [\underline{p}, r) \\ \frac{k}{k-\alpha}\frac{r-\underline{p}}{p-c} & p \in [r, \hat{p}) \\ 1 - \frac{\underline{p}-c}{p-c} & p \in [\hat{p}, v) \\ 1 & p \geq v \end{cases}, \qquad F_k(p) = \begin{cases} \frac{1-k}{k-\alpha}\frac{v-c}{p-c}\frac{p-\underline{p}}{\underline{p}-c} & p \in [\underline{p}, r) \\ \frac{1}{\alpha}\frac{p-c-(1-k)(v-c)}{p-c} & p \in [r, \hat{p}) \\ \frac{1}{k}\left(1-(1-k)\frac{v-c}{p-c}\right) & p \in [\hat{p}, v) \\ 1 & p \geq v, \end{cases}$$

where $r = \mathbb{E}[p_1]$; and $\underline{p} = \frac{1-k}{1-\alpha F_k(r)}(v-c) + c$ and $\hat{p}$ solves $\frac{\hat{p}-\underline{p}}{\hat{p}-c} = \frac{k}{k-\alpha}\frac{r-\underline{p}}{r-c}$;

3. *For $\alpha = 0$, the equilibrium price distributions are given by:*

$$F_1(p) = \begin{cases} 1 - \frac{\underline{p}-c}{p-c} & p \in [\underline{p}, v) \\ 1 & p \geq v \end{cases}, \qquad F_k(p) = \begin{cases} \frac{1}{k}\left(1-(1-k)\frac{v-c}{p-c}\right) & p \in [\underline{p}, v) \\ 1 & p \geq v, \end{cases}$$

where $\underline{p} = (1-k)(v-c) + c$.

When α is large, i.e. $\alpha > \hat{\alpha}(k)$, the constrained firm avoids charging a price above the reservation price. Charging such a price carries the risk of losing many consumers in return for the chance of selling at a higher price. When α is large the risk is not worth taking. When α is smaller, the charging such a price becomes more attractive and the constrained firm charges prices $p > r$ with positive probability.

Figure 1 depicts the regions of parameters when each type of equilibria exists. For $\alpha > k$ (blue region) the equilibrium is the monopoly price, as described in Proposition 1. The red region corresponds to part 1 of Proposition 4 , in this case the constrained firm does not charge prices above r and all its initial visitors buy immediately. In the green region, corresponding to part 2 of Proposition 4, the constrained firm sometimes charges prices above the reservation price. Finally, the axis $\alpha = 0$ corresponds to the third part of Proposition 4.

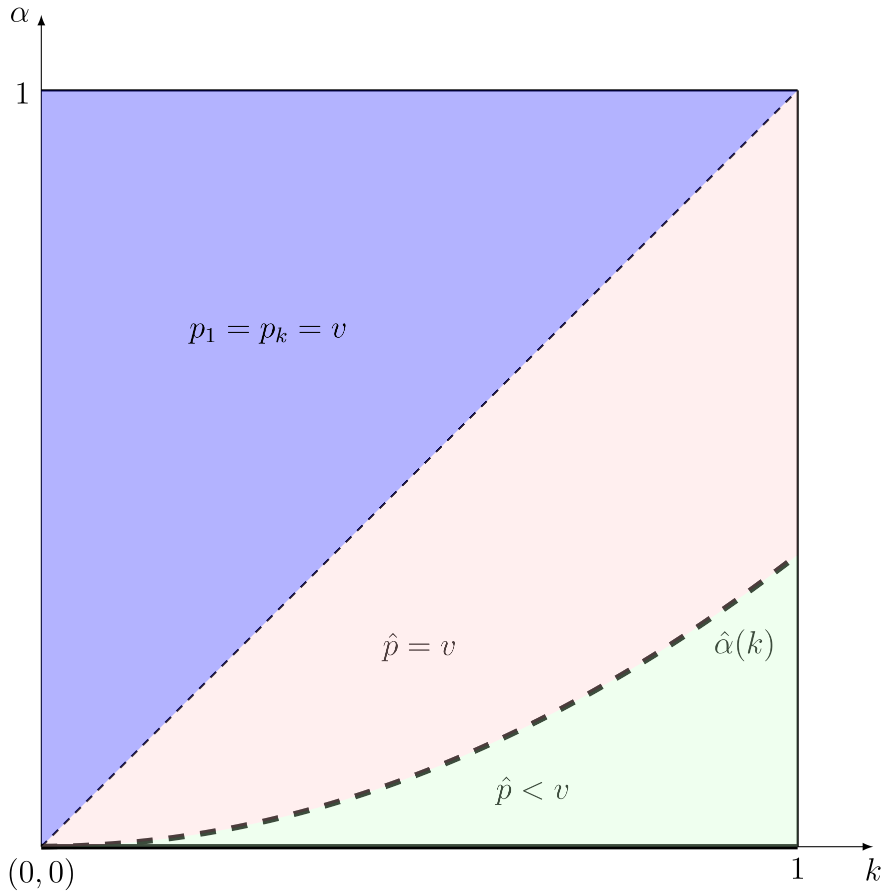


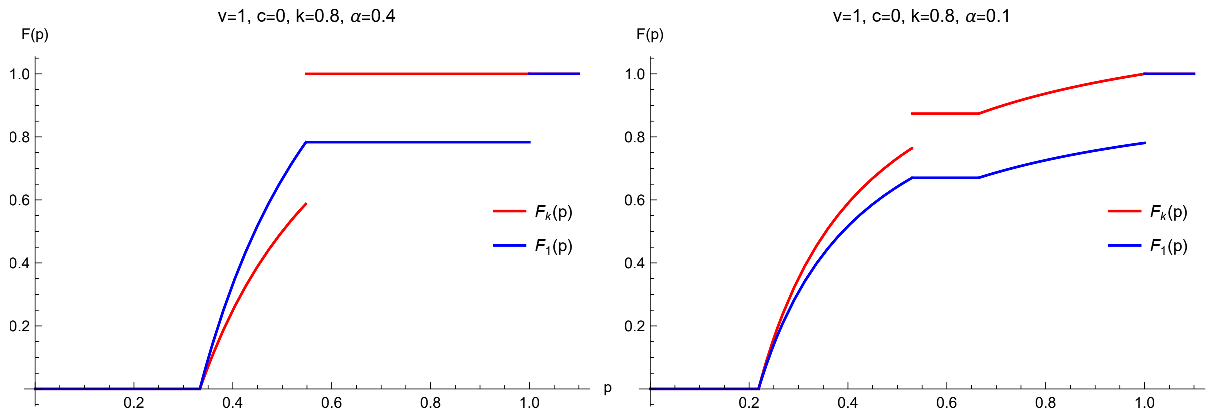
Figure 1: Pricing Equilibria

The equilibrium distributions functions, corresponding to red and green regions are depicted in Figure 2.

The equilibrium profit of the unconstrained firm equals to $\Pi_1(v) = (1-k)(v-c)$ in either case. This is the profit level the unconstrained firm can guarantee by setting $p = v$ and serving all the consumers who cannot be served by the constrained firm. In either case, the equilibrium profit of the constrained firm is equal to $\Pi_k(\underline{p}) = k(\underline{p} - c)$, since at $p = \underline{p}$ it sells out its capacity.

When $\alpha > \hat{\alpha}(k)$, consumers who visited the constrained firm do not search, and therefore $\Pi_1(\underline{p}) = (1-\alpha)(\underline{p} - c)$. Thus, $\underline{p} = \frac{1-k}{1-\alpha}(v-c) + c$ and $\Pi_k = k\frac{1-k}{1-\alpha}(v-c)$. The expected consumer surplus is given by

$$CS = v - \Pi_1 - \Pi_k - c = \left[1 - (1-k)\left(1 + \frac{k}{1-\alpha}\right)\right](v-c)$$



(a) $\alpha > \hat{\alpha}(k)$ (b) $\alpha < \hat{\alpha}(k)$

Figure 2: Price distributions

which is increasing in k for $k > \alpha$ and decreasing in α. This means that consumers are better off when the constrained firm has a larger capacity, which leads to stronger competition. Furthermore, as the number of consumers who start searching from the constrained firm, α, increases, the unconstrained firms risks to lose a smaller number of consumers when it increases its price, namely $k - \alpha$. As a result, it increases its prices. This increases the reservation price r, to which the unconstrained firm responds with its price increase. For $\alpha = k$, consumer surplus is equal to 0.

For $\alpha < \hat{\alpha}(k)$ we are getting

$$CS = v - \Pi_1 - \Pi_k - c = k\left(1 - \frac{k}{\frac{k-\alpha}{1-k} + \alpha\frac{v-c}{\hat{p}-c}}\right)(v - c).$$

This expression also increases in c, decreases in α, converges to the expression for $\alpha > \hat{\alpha}(k)$ as $\hat{p} \to v$; and simplifies to $CS = k(1-k)(v-c)$ when $\alpha = 0$, which is indeed the consumer surplus corresponding to case 3 of Proposition 4.

4 Consumers' endogenous choice of first visit

In the previous analysis, we fixed the consumers' decision on which firm they are visiting first. In this section, we consider the setting with endogenous α: consumers freely decide on the order in which they visit the firms. We assume that consumers can distinguish the constrained and the unconstrained firms prior to the initial search. Consumer strategy in this setting has two elements: α, the probability of starting their search with the constrained firm, and the reservation price r. When deciding where to start the search, consumers compare the expected price they will pay on each search path given the strategy of the other consumers and firms. In equilibrium, if $\alpha \in (0,1)$, then these expected prices must be equal to each other.

We start with a putative equilibrium with $\alpha Q(p^m) > k$. Proposition 1 implies that in this case $p_1 = p_k = p^m$ and hence consumers are indifferent about the starting point of their search. Therefore, any α such that $\alpha Q(p^m) > k$ constitutes an equilibrium strategy of a consumer.

Now we turn our attention to the mixed strategy region, i.e. $\alpha Q(p^m) < k$. We denote U_k the expected consumer surplus of the consumer who starts searching from the constrained firm and U_1 the expected surplus of the consumer who starts from the unconstrained firm. Then,

$$\begin{aligned} U_k = {} & F_k(r)\mathbb{E}_k\left[CS(p_k)|p_k \leq r\right] \\ & + [1 - F_k(r)]\mathbb{E}_k\left\{F_1(p_k)\mathbb{E}_1\left[CS(p_1)|p_1 < p_k\right]\right. \\ & \left. + [1 - F_1(p_k)]\left(\phi(p_k)CS(p_k) + [1-\phi(p_k)]\mathbb{E}_1\left[CS(p_1)|p_1 > p_k\right]\right)\right\}. \end{aligned}$$

That is, with probability $F_k(r)$ the consumer discovers the price below the reservation price and buys immediately. Otherwise, the consumer explores the unconstrained firm. If $p_1 \leq p_k$ then the consumer immediately buys at the unconstrained firm. Otherwise, they attempts to come back. With probability $\phi(p_k) = \max\{0, [k-(1-\alpha)Q(p_k)]/[\alpha Q(p_k)]\}$ they could buy from the constrained firm,[4] otherwise they have get back again to buy at the unconstrained firm.

Consumer who starts search from the unconstrained firm always checks the constrained one. Therefore,

$$\begin{aligned} U_1 = {} & F_k(r)\mathbb{E}_k\left\{\tilde{\psi}(p_k)\mathbb{E}_k\left[CS(p_k)|\right] + [1-\tilde{\psi}(p_k)]\mathbb{E}_1\left[CS(p_1)\right]|p_k \leq r\right\} \\ & + \mathbb{E}_1\left(F_k(p_1)\mathbb{E}_k\left\{\tilde{\phi}(p_k)\mathbb{E}_k\left[CS(p_k)|\right] + (1-\tilde{\phi}(p_k)]CS(p_1)|p_k \leq p_1\right\}\right) \\ & + \mathbb{E}_1\left\{[1-F_k(p_1)]CS(p_1)\right\}. \end{aligned}$$

That is, if $p_k \leq p_1$, then consumers try to buy at the constrained firm. If $p_k \leq r$ they can do

[4] The constrained firm will serve $(1-\alpha)$ consumers who started their search from the unconstrained firm. As $p_k > r$ the remaining capacity at this point is k. As $p_k > p_1$ all the visitors would attempt to buy. Then all consumers who visited the constrained firm first would be competing for the remaining capacity, in there is any.

it with probability $\tilde{\psi}(p_k) = \min\{1, [k - \alpha Q(p_k)]/[(1-\alpha)Q(p_k)]\}$.[5] If $p_k > r$ then consumers can buy with probability $\tilde{\phi}(p_k) = \min\{1, k/[(1-\alpha)Q(p_k)]\}$.

In any equilibrium with $\alpha \in (0,1)$ it must be the case that

$$U_k = U_1.$$

Whenever $U_k > U_1$ for all α we have a monopoly pricing equilibrium with all consumers starting their search from the constrained firm, i.e. $\alpha = 1$. Correspondingly, if $U_k < U_1$, then all consumers start their search from the unconstrained firm, i.e. $\alpha = 0$.

We numerically compute the full set of equilibria for unit demand, which is represented in Figure 3. Note, that perform the computation for each value of k, hence the only remaining parameters of the model are v and c, which we decided to normalize at 1 and 0 respectively.

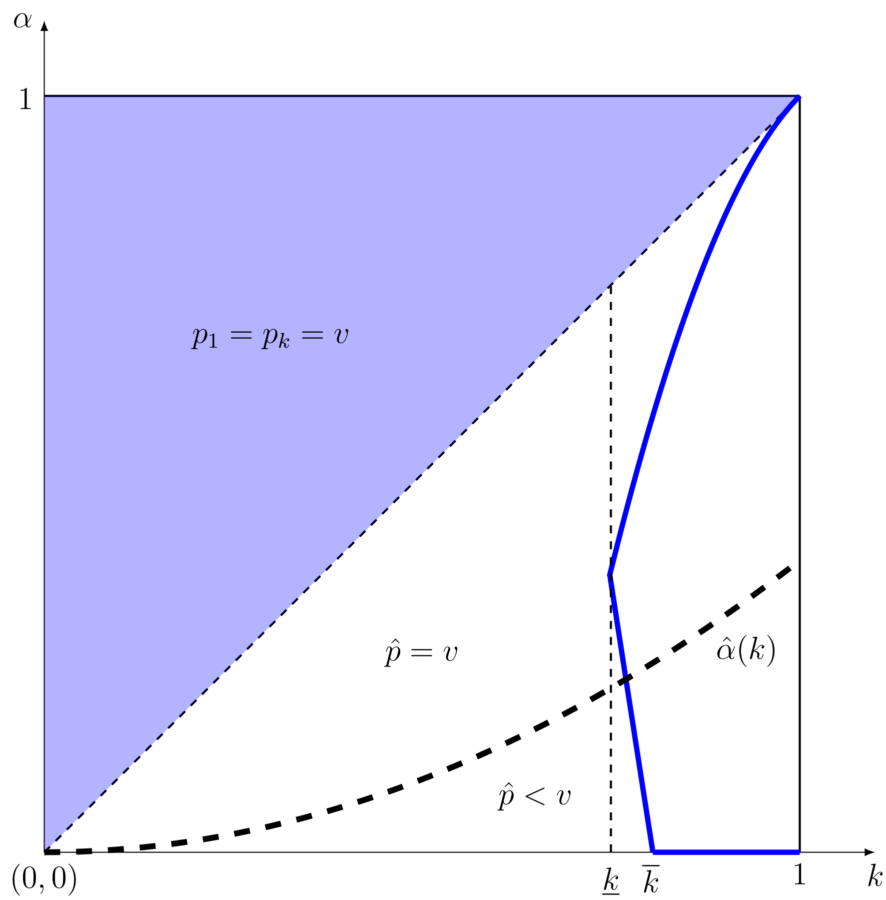


Figure 3: Equilibria marked in blue, $\underline{k} \approx 0.748$, $\overline{k} \approx 0.803$

In Figure 3, for $k > \overline{k}$, there are three equilibria: one where consumers coordinate on

[5] Note, that the numerator in this case is always non-negative.

searching the unconstrained firm first, and two where they randomize between the constrained and unconstrained firms. For such k, if many consumers search the unconstrained firm first, it makes sense to follow the same strategy. If k is large and α is small, a consumer starting a search from the unconstrained firm is very likely to purchase the product at the constrained firm if the price there turns out to be lower. However, if such a consumer deviates and starts searching with the constrained firm, they will not be able to come back: if the constrained firm charges the lowest price, the product will be sold out. Therefore, deviating to search the constrained firm first makes search without recall, which is not attractive. Hence, all consumers prefer to start searching from the unconstrained firm. The more consumers visit the constrained firm first, the weaker the incentives to start the search from the unconstrained firm. On the one hand, if the price at the constrained firm is below the reservation price, visitors from the unconstrained firm are more likely to be rationed out. On the other hand, if p_k is high enough to justify search, but eventually is larger than p_1, then consumers searching the constrained firm first get their chance for recall increasing in α: the probability of recall is $\frac{k-1-\alpha}{\alpha} = 1 - \frac{1-k}{\alpha}$. This makes searching from the constrained firm progressively more attractive, and at some point $U_1 = U_k$, so a mixed strategy equilibrium arises. When $\alpha > k$, then both price distributions collapse to a single point $p_1 = p_k = v$ and searching from either firm is equally profitable. For $k \in [\underline{k}; \overline{k}]$, the equilibrium with $\alpha = 0$ does not exist; instead, there are two equilibria with $\alpha \in (0, k)$.

If $k < \underline{k}$ it is too risky to visit the unconstrained firm first, and in any equilibrium we have $\alpha > k$. Therefore, for $k < \underline{k}$ monopoly pricing is the *unique* equilibrium outcome in the model with endogenous first visits. Note, that monopoly pricing does not require severe capacity restrictions, as $k < \underline{k} \approx 3/4$. That is, the monopoly pricing is the unique equilibrium even if the market has excess capacity of 74%.

This result has important implications. If one of the firms restricts its capacity to the level just below $\underline{k}$ (or a capacity constrained entrant enters a market with capacity unconstrained incumbent), such firm may earn higher profits that the unconstrained firm: $\underline{k}(v - c) > (1 - \underline{k})(v - c)$, i.e. the profit of the constrained firm may be almost three times as large as the profit of the unconstrained one. Thus, unilateral capacity restriction may be a profitable strategy.

5 Discussion

In our main analysis we focused on a duopoly case. However, the results of Proposition 1 can be extended to an oligopoly setting. That is, suppose that there are n capacity constrained firms with capacities $k_1, \ldots, k_n$ with $\sum_{i=1}^{n} k_i < 1$ and one firm without capacity constraints. Suppose that the initial distribution of consumers is such that $\alpha_i Q(p^m) > k_i$. In this case each firm in the market charges $p_i = p^m$, $i = 1, \ldots n+1$. The argument is analogous to Proposition 1. If there was another equilibrium, then let r_i be the reservation price associated with search upon visiting firm i in the first round. Then firm i has no incentives of charging prices below r_i. Then none of the firms charges prices below $\underline{r} = \min_i r_i$. This implies that $\underline{r}$ cannot be a reservation price, unless all the price distributions are degenerate with support on $\underline{r}$. Hence, all firms charge the monopoly price. Characterization of mixed strategy equilibria in the oligopoly setting is quire cumbersome and is beyond the scope of this paper.

Appendix

Proof for Proposition 1. We prove the statement of the proposition by ruling out all other equilibrium candidates, starting with pure strategy ones.

Note that in any equilibrium $p_1 \leq p^m$ as otherwise the constrained firm could increase its profit by decreasing the price. This implies that $p_k \leq p^m$ as otherwise all consumers who initially visited the constrained firm would leave for the incumbent.

On the way to a contradiction, suppose that $p_k < p_1 \leq p^m$. Then $p_k' = (p_k + p_1)/2$ is a profitable deviation of the constrained firm, as it will still sell up to the full capacity and because $p_k' < p^m$ we obtain that $\Pi_k(p_k') > \Pi_k(p_k)$.

On the way to a contradiction, suppose that $p_1 < p_k \leq p^m$. Then $p_1' = (p_k + p_1)/2$ is a profitable deviation by the incumbent firm: as $p_1' < p_k$ the total number of consumers served does not change and $p_1 < p_1' < p^m$.

Thus, it must be the case that in any pure strategy equilibrium $p_1 = p_k = p \in (c, p^m]$.

On the way to a contradiction, suppose that $p < p^m$. Because $\alpha Q(p^m) > k$ we obtain that the number of consumers who initially visited the constrained firm satisfies $\alpha > k/Q(p)$. Suppose that measure $\beta > \alpha - k/Q(p)$ decided to check the price at the incumbent firm and measure $\gamma \leq \beta$ attempted to return.

If $\gamma = 0$ (none of the consumers returns), then there exists $\varepsilon > 0$ such that the constrained firm has a profitable deviation to $p_k = p - \varepsilon$ as

$$(\alpha - \beta)Q(p)(p - c) < k(p - \varepsilon - c)$$

for ε small enough.

If $\gamma > 0$ (some consumers return), then then there exists $\varepsilon > 0$ such that the unconstrained firm has a profitable deviation to $p_1 = p - \varepsilon$ as

$$(1 - \alpha + \beta - \gamma)Q(p)(p - c) < (1 - \alpha + \beta)Q(p - \varepsilon)(p - \varepsilon - c)$$

for ε small enough. We conclude that it cannot be the case that $\beta > \alpha - k/Q(p)$, hence $\beta = \alpha - k/Q(p)$, i.e. the entrant sells its full capacity to the first-time visitors. Therefore, the incumbent's profit is $(1 - \alpha)Q(p)(p - c)$, thus has a profitable deviation from $p_1 = p$ to $p_1 = p^m$.

We conclude that the only pure strategy equilibrium is when $p_1 = p_k = p^m$ and the constrained firm sells out its full capacity in the first round.[6]

To establish uniqueness we need to verify that there are no (non-degenerate) mixed strategy equilibria. Suppose that such equilibrium exists. Denote the strategies as F_1 and F_k. Denote by $r \leq p^m$ the reservation price, i.e. a price at which consumer who started search at the constrained firm is indifferent between accepting this price immediately and checking the unconstrained firm.

First, note that for $p_k < r$ the profit of the constrained firm is given by $k(p_k - c)$, which is increasing in p_k. Thus, the lower bound of the support of F_k must weakly exceed r. This implies that the lower bound of the support of the unconstrained firm must weakly exceed r. Then r is a reservation price if and only if the distribution of the unconstrained firm is degenerate with support at $p = r$, which implies that the distribution of the constrained firm is also degenerate (as charging $p_k > r$ results in zero demand). Thus, the mixed strategy equilibrium reduces the pure strategy equilibrium case analysed earlier.

□

Proof of Lemma 1. Following the arguments in the proof of Proposition 1 if a pure strategy equilibrium exists, then it must be the case that $p_1 = p_k = p$.

Now we show that none of these candidate equilibria is robust. Denote $\tilde{p}$ solution to $\alpha Q(\tilde{p}) = k$, $\tilde{p} < p^m$. If $p > \tilde{p}$ then the constrained firm can increase its profit by charging

[6] Note, that if $\lambda < \alpha - k/Q(p^m)$ the monopoly price equilibrium is robust.

$p_k' = p - \varepsilon$ and attracting λ of consumers who sample both firms. If $p \leq \tilde{p} < p^m$ then the constrained firm sells out, and the unconstrained firm is better off by charging $p_1' = p^m$. $\square$

Proof of Lemma 2. We first prove that $\underline{p}_1 = \underline{p}_k$. On the way to a contradiction, suppose that $\underline{p}_i < \underline{p}_{-i}$. The profit of firm i is increasing on the interval $(\underline{p}_i, \underline{p}_{-i})$, leading to a contradiction.

Now we prove that $\max\{\overline{p}_1, \overline{p}_k\}$. Consider the unconstrained firm. If $p_1 > p^m$, then the firm can increase its profit by deviating to $p_1' = p^m$ as per-consumer profit increases, and the total number of served consumers does not decrease. Therefore $\overline{p}_1 \leq p^m$. This implies that for $p_k > p^m$ the constrained firm profit is $\Pi_k(p_k) = 0$. Hence, the constrained firm also does not charge prices above p^m.

Note that both firms make positive profits in any equilibrium. The unconstrained firm can earn strictly positive profits by setting price $c+\varepsilon$, where $\varepsilon > 0$ is sufficiently small. Since $k < Q(c)$, the constrained firm setting a price lower than $c + \varepsilon$ cannot serve all consumers, implying that some consumers always buy from the unconstrained firm. Thus, $\underline{p} > c$. The constrained firm can earn positive profits by setting price $\underline{p} - \varepsilon$.

$\square$

Proof of Lemma 3. The following lemma establishes that the support of the equilibrium price distribution may have at most one gap, and this gap must be above the reservation price.

Lemma 5. *Suppose that* $\underline{p} < a < b < \min\{\overline{p}_1, \overline{p}_k\}$.

1. *If* $F_i(b) - F_i(a) = 0$ *then there exits* $\varepsilon_0 > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0)$ *and* $F_{-i}(b - \varepsilon) - F_{-i}(a) = 0$.
2. *If* $F_i(b) - F_i(a) = 0$ *then* $a = r$.

Proof. Note that if firm i has a gap on some (a, b), then $\Pi_{-i}(p)$ is increasing on this interval, so firm $-i$ also must have a gap. This proves the first claim and allows us to focus on common gaps in the support.

Suppose that $a \neq r$. On the way to a contradiction, we consider two cases: when a firm has an atom at $p = a$ and when both distribution functions are continuous at $p = a$.

Suppose that both firms have an atom at $p = a$. If $a > \underline{p}$ then all consumers from the unconstrained firm search and hence, there exists $\varepsilon > 0$ such that $\Pi_i(a - \varepsilon) > \Pi_i(a)$, $i = 1, k$. If $a = \underline{p}$ and consumers who initially visited the unconstrained search, then the same undercutting argument applies. If these consumers search, then due to $a < r$ there exists $\varepsilon > 0$ such that $\Pi_k(a) < \Pi_k(a + \varepsilon)$ (as all consumers initially visiting the constrained firm stay, and all consumers who may come from the unconstrained firm observed $p_1 > \underline{p} + \varepsilon$).

Secondly, suppose that firm $-i$ does not have a mass point at $a \neq r$ and firm i charges $p_i = a$ with strictly positive probability. Then it must be the case that $\Pi_i(a) < \Pi_i(a + \varepsilon)$, as the probability of sale does not change when deviating from a to $a + \varepsilon$ and $a < p^m$.

Finally, if both distribution functions are continuous in the lower neighborhood of a, then there exists $\varepsilon_1, \varepsilon_2 > 0$ such that for any $p \in (a - \varepsilon_1, a]$ we have that $\Pi_i(p) < \Pi_i(a + \varepsilon_2)$. That is, a cannot be the infimum of the gap for firm i leading to a contradiction. □

Lemma 6. *$F_i(p)$ is continuous at any $p \in [\underline{p}, \overline{p}_i)$ and $p \neq r$. Moreover, if $\lim_{\varepsilon \to 0} F_i(p) - F_i(p - \varepsilon) > 0$ then $\lim_{\varepsilon \to 0} F_{-i}(p) - F_{-i}(p - \varepsilon) = 0$.*

Proof. Suppose that $\lim_{\varepsilon \to 0}[F_i(p_0) - F_i(p_0 - \varepsilon)] > 0$. Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ we have $\Pi_{-i}(p_0 + \varepsilon) < \Pi_{-i}(p_0)$, which implies that $F_{-i}(p_0 + \varepsilon_0) - F_{-i}(p_0) = 0$. Then, from Lemma 5 it follows that either $p_0 = r$ or $p_0 = \overline{p}_{-i}$. Moreover, if $\lim_{\varepsilon \to 0}[F_i(p_0) - F_i(p_0 - \varepsilon)] > 0$ then there exists such $\varepsilon > 0$ that $\Pi_{-i}(p_0 - \varepsilon) < \Pi_{-i}(p_0)$, implying that firm $-i$ cannot charge p_0 with positive probability. □

Lemma 6 states that if either firm has a price played with positive probability, then it must be either the reservation price, of the upper bound of the support of the firm's pricing distribution. Moreover, if one firms plays such price with a positive probability, then another firm cannot do the same.

Lemma 7. *Suppose that $F_i(r) < 1$ for any $i \in \{1, k\}$. Then $\overline{p}_1 = \overline{p}_k = p^m$.*

Proof. Suppose that $\overline{p}_i > \overline{p}_{-i}$. As $F_i(r) < 1$, then from Lemma 5 it follows that there exists ε such that distribution function F_i is continuous on $(\overline{p}_i - \varepsilon, \overline{p}_i)$. However, $\overline{p}_i > \overline{p}_{-i}$ implies that $\Pi(p)$ is strictly increasing on $(\overline{p}_i - \varepsilon, \overline{p}_i)$ leading to a contradiction. Denote the common upper bound $\overline{p}$.

Now, from Lemma 6 it follows at most one firm charges $\overline{p}$ with positive probability. Denote this firm i. Then, $\Pi_i(p)$ is increasing on $p \in [\overline{p}, p^m]$, as the competitor charges a lower price than p with probability 1. Hence, if $\overline{p} < p^m$ firm i has a profitable deviation. □

Lemma 8. *Suppose that $\alpha > 0$. Then, there exists $\varepsilon_0 > 0$ such that $F_i(r) = F_i(r + \varepsilon_0)$. Moreover, $\lim_{\varepsilon \to 0}[F_k(r) - F_k(r - \varepsilon)] > 0$.*

Proof. First, notice that as all consumers search at $p_k > r$ we have that there exists $\varepsilon > 0$ such that $\Pi_k(r+\varepsilon) < \Pi_k(r)$, i.e. F_k has a gap above r, hence, by Lemma 5, F_1 also must have a gap above r.

Second, suppose that $\lim_{\varepsilon\to 0}[F_k(r) - F_k(r-\varepsilon)] = 0$, i.e. firm k does not have atom at r. Then, there exists small enough $\varepsilon > 0$ such that $\Pi_1(r+\varepsilon) > \Pi(r)$ as the expected demand for the unconstrained is the same at $r+\varepsilon$ and r. □

Lemma 9. $\lim_{\varepsilon\to 0}[F_1(p^m) - F_1(p^m-\varepsilon)] > 0$.

Proof. On the way to a contradiction suppose that $\lim_{\varepsilon\to 0}[F_1(p^m) - F_1(p^m-\varepsilon)] = 0$. We consider two cases: $F_k(r) < 1$ and $F_k(r) = 1$.

Suppose that $F_k(r) < 1$. Then from Lemmata 5 and 7 there exists an interval $(p^m - \varepsilon, p^m]$ such that F_1 is continuous on this interval. Then $\lim_{\varepsilon\to 0}\Pi_k(p^m-\varepsilon) = 0$, hence the constrained firm must have a gap below p^m, which contradicts Lemma 5.

Suppose that $F_k(r) = 1$. Note, that $\overline{p}_1 > r$ by the definition of reservation price. $F_k(r) = 1$ implies that $\Pi_1(p^m) > \Pi_1(p)$ for any $p \in (r, p^m)$, hence F_1 has an atom at p^m. □

Proof Lemma 3. From Lemma 5 we know that supports of the distributions may have at most one gap. I a gap exists, it is the same for both F_1 and F_k and starts at $p = r$.

Lemma 6 states that both distribution functions are continuous in their support except possibly for either $p = r$ or $p = \overline{p}_i$. If one distribution has an atom at such p, then another is continuous.

Lemma 2 states that if both firms charge prices above r, they must charge prices all the way up to p^m.

Lemma 8 states that if the constrained firm is initially visited by some consumers, then it must have an atom at $p = r$ (hence by Lemma 6 the unconstrained firm does not have an atom there). Moreover, either both firms have a gap above r or $F_i(r) = 1$.

Lastly, Lemma 9 states that the unconstrained firm has at atom at p^m.

Summing up, both firms play a continuous distribution on $[\underline{p}, r)$ (Lemmata 2 and 6). If $\alpha > 0$, then the constrained firm has an atom at r (Lemma 8), the unconstrained firm does not (Lemma 6). Now we consider two cases: $F_k(r) = 1$ and $F_k(r) < 1$.

If $F_k(r) = 1$, then $\Pi_1[p_1]$ is increasing on $p_1 \in (r, p^m]$, hence the unconstrained firm does not charge prices in this interval but charges p^m with strictly positive probability (Lemma 9). Hence, F_k has support $[\underline{p}, r]$ with an atom at r and F_1 has support $[\underline{p}, r]\bigcup\{p^m\}$ with an atom at p^m.

If $F_k(r) < 1$, then by Lemmata 8 and 5, both distributions have a single gap, with the lower bound at r. We denote the upper bound of the gap as $\hat{p}$. By Lemma 7, the upper bounds of the supports coincide and are equal to p^m. Hence, both distributions have support $[\underline{p}, r] \bigcup [\hat{p}, p^m]$ with F_k having atom at r and F_1 having atom at p^m (Lemmata 8 and 9).

For $\alpha = 0$ we prove that support is convex. From Lemma 5 we know that if gaps in supports exist, they must coincide. Denote this gap (a, b). As all consumers start with the unconstrained firm, they always check the constrained one. As the constrained firm does not have an atom at a (Lemma 6) we get that $\Pi_k(p)$ is increasing on $[a, b)$ leading to a contradiction. Thus, support is convex. Convexity of support and continuity of price distributions in the interior of the supports (Lemma 6) imply that supports coincide, which, in turn, implies that $F_k(r) < 1$. Hence, by lemma 7, the common upper bound equals p^m. Thus, both distributions are continuous on their common support $[\underline{p}, p^m)$ and F_1 has an atom at p^m (Lemma 9).

Proof of Proposition 2. The distribution functions are obtained from solving equations $\Pi_1(p) = \Pi^m$ and $\Pi_k(p) = k(\underline{p} - c)$. In what follows, we show that the functions are well-defined; that is, densities are positive on the support.

We start with the function F_k. For the segment $p \in [\underline{p}, r)$ we get

$$F_k(p)[\psi(p) - \alpha] = \Pi^m \left(\frac{1}{\pi(\underline{p})} - \frac{1}{\pi(p)} \right)$$

or, equivalently,

$$\int_{\underline{p}}^{p} \frac{k f_k(p_k)}{Q(p_k)} dp_k - \alpha F_k(p) = \Pi^m \left(\frac{1}{\pi(\underline{p})} - \frac{1}{\pi(p)} \right).$$

Differentiating both sides with respect to p gives

$$f_k(p) \left(\frac{k}{Q(p)} - \alpha \right) = \frac{\Pi^m}{[\pi(p)]^2} \pi'(p)$$

which implies that the density is positive for $p < r < p^m$.

For the segment $p \in [\hat{p}, p^m]$ we get

$$F_k(p)\psi(p) = 1 - \frac{\Pi^m}{\pi(p)}.$$

Differentiating both sides with respect to p gives

$$\frac{kf_k(p)}{Q(p)} = \frac{\Pi^m}{[\pi(p)]^2}\pi'(p),$$

which implies that the density is positive for $p < p^m$ and is zero for $p = p^m$.

Now consider the distribution function F_1. On the segment $p < r$ applying the Implicit Function Theorem to (5) gives

$$f_1(p) = \frac{F_1\alpha\pi'(p) + [1 - F_1(p)]k}{k(p-c) - \alpha\pi(p)} > 0.$$

On the segment $p \in [\hat{p}, p^m)$ differentiation gives

$$f_1(p) = \frac{\underline{p} - c}{(p-c)^2} > 0.$$

Thus, the distribution functions in the proposition are well-defined.

Finally, note that the constrained firm is indifferent between charging r and charging $\hat{p}$, and strictly prefers r to any $p \in (r, \hat{p})$. Also, note that

$$\lim_{\hat{p}\to p^m} \frac{\pi(\hat{p}) - [1 - k\psi(p^m)]\pi(p^m)}{k\psi(\hat{p})\pi(\hat{p})} = 1.$$

Therefore, if the pricing distribution has a gap on (r, p^m) then deviation to any point in this gap is not profitable for firm k.

To show the existence of the equilibrium, it is sufficient to show that there exist $(\underline{p}, r, \hat{p}_1)$ and $\hat{p} = \min\{\hat{p}_1, p^m\}$ such that

$$CS(r) = \mathbb{E}[CS(p_1)] \tag{7}$$

$$\frac{\hat{p}_1 - \underline{p}}{\hat{p}_1 - c} = \frac{r - \underline{p}}{r - c}\frac{k}{k - \alpha Q(r)} \tag{8}$$

$$(1 - k\psi(p^m))\pi(p^m) = [1 - \alpha F_k(\hat{p})]\pi(\underline{p}) \tag{9}$$

As the first step, we show that for any $\underline{p} \in (c, p^m)$ there exist continuous functions $r(\underline{p})$ and $\hat{p}(\underline{p})$ such that they solve equations (7) and (8) with $\underline{p} < r(\underline{p}) < \hat{p}(\underline{p}) \leq p^m$.

Fix some $\underline{p} \in (c, p^m)$. Consider $r = \underline{p}$. Equation (8) implies that $\hat{p} = \underline{p}$, which together with (7) implies that $CS(\underline{p}) > \mathbb{E}[CS(p_1)]$.

Consider $r = p^m$. Then, equation (8) implies that $\hat{p}_1 > r = p^m$ and therefore $\hat{p} = p^m$.

$CS(p^m) < \mathbb{E}[CS(p_1)]$.

Now we define r' as the value of r such that $\hat{p}_1 = p^m$:

$$\frac{p^m - \underline{p}}{p^m - c} = \frac{r' - \underline{p}}{r' - c} \frac{k}{k - \alpha Q(r')}.$$

For any $r \in (\underline{p}, r')$, equation (8) implies that $\hat{p}(r) < p^m$. This allows us to rewrite equation (7) as

$$G_1(\underline{p}, r) \equiv CS(r) - \int_{\underline{p}}^{r} CS(p)dF_1(p) - \int_{\hat{p}(r)}^{p^m} CS(p)dF_1(p) - [1 - F_1(p^m)]CS(p^m) = 0.$$

Taking the derivative with respect to r gives:

$$\frac{\partial G_1(\hat{p}, r)}{\partial r} = -Q(r) - CS(r)f_1(r) + CS(\hat{p})f_1(\hat{p})\frac{\partial \hat{p}}{\partial r}.$$

As $F_1(r) = F_1(\hat{p}(r))$, we have $f_1(r) = f_1(\hat{p})\frac{\partial \hat{p}}{\partial r}$ and therefore

$$\frac{\partial G_1(\hat{p}, r)}{\partial r} = -Q(r) - [CS(r) - CS(\hat{p})]f_1(r) < 0.$$

Now consider $r \in (r', p^m)$. Then, in the neighbourhood of r, equation (8) implies that $\hat{p} = p^m$. This allows us to rewrite equation (7) as

$$G_2(\underline{p}, r) \equiv CS(r) - \int_{\underline{p}}^{r} CS(p)dF_1(p) - [1 - F_1(r)]CS(p^m) = 0$$

and

$$\frac{\partial G_2(\hat{p}, r)}{\partial r} = -Q(r) - [CS(r) - CS(p^m)]f_1(r) < 0.$$

Finally, at $r = r'$, negative left- and right-derivatives exist. Hence, $G(\underline{p}, r)$ is strictly decreasing in $r \in (\underline{p}, p^m)$. Using continuity of $G(\underline{p}, r)$ and the facts that $G(\underline{p}, \underline{p}) > 0$ and $G(\underline{p}, p^m) < 0$ and applying the Intermediate Value Theorem implies that for any $\underline{p}$ there is a pair $(\hat{p}, r)$ which solves equations (7) and (8). As $G(\underline{p}, r)$ is continuous and strictly monotone, we can apply the Implicit Function Theorem and obtain that the solution $r(\underline{p})$ of $G(\underline{p}, r) = 0$ is unique and continuous in $\underline{p}$. As $\hat{p}(r)$ is continuous in r, we obtain that $\hat{p}$ is continuous in $\underline{p}$.

Now, consider equation (9). Note, that if $\underline{p} = p^m$ then $(r, \hat{p})$ solving (7) and (8) equals (p^m, p^m). Plugging these values into (9), we obtain

$$(1 - \alpha)\pi(p^m) - [1 - k/Q(p^m)]\pi(p^m) > 0.$$

On the other hand,

$$[1-\alpha F_k(\hat{p})]\pi(\underline{p}) - [1-k\psi(p^m)]\pi(p^m) < [1-\alpha F_k(\hat{p})]\pi(\underline{p}) - [1-kQ(p^m)]\pi(p^m)$$

and

$$\lim_{\underline{p}\to c}[1-\alpha F_k(\hat{p})]\pi(\underline{p}) - [1-kQ(p^m)]\pi(p^m) = -[1-kQ(p^m)]\pi(p^m) < 0.$$

Therefore, there exists $\underline{p}$ which solves equation (9), and hence there exists a triplet $(\underline{p}, r, p^m)$ with solves equations (7)-(9).

$\square$

Proof of Proposition 3. From Lemma 3 we know that for $\alpha = 0$ both distribution functions have convex support $[\underline{p}, p^m]$, F_k is continuous on its support, and F_1 is continuous for $p < p^m$ and has an atom at p^m.

The indifference condition for the unconstrained firm gives:

$$\Pi_1(p) = [1 - F_k(p)]\pi(p) + F_k(p)[1 - k\psi(p)]\pi(p).$$

Note that $\Pi_1(\underline{p}) = \pi(\underline{p})$ and $\Pi_1(p^m) = \Pi^m$, implying $\pi(\underline{p}) = \Pi^m$. Using the latter boundary condition we obtain

$$F_k(p) = \frac{\pi(p) - \Pi^m}{k\psi(p)\pi(p)}.$$

In the proof of Proposition 2 we have shown that this is an increasing function. Now, we consider the constrained firm:

$$\Pi_k(p) = [1 - F_1(p)]k(p - c)$$

with the boundary condition $\Pi_k(\underline{p}) = k(\underline{p} - c)$. This implies that

$$F_1(p) = 1 - \frac{\underline{p} - c}{p - c},$$

which is increasing in p.

To prove the existence, we need to show that equation $\pi(\underline{p}) = \Pi^m$ has a solution. Note, that

$$f_k(p) = \frac{\Pi^m}{k}\frac{\pi'(p)}{\pi(p)(p - c)}.$$

Using the fact that $F_k(p)$ is continuous at p^m, we obtain

$$\frac{\Pi^m}{k}\int_{\underline{p}}^{p^m}\frac{\pi'(x)dx}{\pi(x)(x-c)}=1.$$

Plugging in $\Pi^m=\pi(\underline{p})$ and rearranging, we have

$$H(\underline{p})\equiv\pi(\underline{p})\int_{\underline{p}}^{p^m}\frac{\pi'(x)dx}{\pi(x)(x-c)}=k.$$

Note that $H(\underline{p})$ tends to 0 when $\underline{p}$ tends to p^m. Moreover, note that the integral

$$\int_{\underline{p}}^{p^m}\frac{\pi'(x)dx}{\pi(x)(x-c)}>\frac{1}{p^m-c}\int_{\underline{p}}^{p^m}\frac{\pi'(x)dx}{\pi(x)}=\frac{1}{p^m-c}\log\frac{\pi(p^m)}{\pi(\underline{p})},$$

and, therefore, tends to $+\infty$ as $\underline{p}\to c$. Thus, by L'Hospital's rule, we have that:

$$\lim_{\underline{p}\to c}H(\underline{p})=\lim_{\underline{p}\to c}\frac{\int_{\underline{p}}^{p^m}\frac{\pi'(x)dx}{\pi(x)(x-c)}}{1/\pi(\underline{p})}=\lim_{\underline{p}\to c}\frac{\frac{-\pi'(\underline{p})}{\pi(\underline{p})(\underline{p}-c)}}{-\frac{\pi'(\underline{p})}{\pi^2(\underline{p})}}=Q(c)>k,$$

implying that $H(\underline{p})$ is greater than k for $\underline{p}$ sufficiently close to c. The existence follows from the Intermediate Value Theorem.

Next, we show uniqueness. The derivative of H with respect to $\underline{p}$ is

$$\frac{dH}{d\underline{p}}=\pi'(\underline{p})\left[\int_{\underline{p}}^{p^m}\frac{\pi'(x)dx}{\pi(x)(x-c)}-\frac{1}{\underline{p}-c}\right].$$

Note that for any $\underline{p}\in(c,p^m]$, we have

$$\frac{d}{d\underline{p}}\left[\int_{\underline{p}}^{p^m}\frac{\pi'(x)dx}{\pi(x)(x-c)}-\frac{1}{\underline{p}-c}\right]=-\frac{\pi'(\underline{p})}{\pi(\underline{p})(\underline{p}-c)}+\frac{1}{(\underline{p}-c)^2}=-\frac{Q'(\underline{p})}{\underline{p}-c}>0,$$

implying that the expression in square brackets strictly increases on $(c,p^m]$ and attains its maximal value $-1/(p^m-c)$. Therefore, for any $\underline{p}\in(c,p^m)$,

$$\frac{dH}{d\underline{p}}<-\frac{\pi'(\underline{p})}{p^m-c}<0,$$

implying that $H(\underline{p})$ strictly decreases on (c,p^m).

Since $H(\underline{p})-k$ is monotone and changes its sign on (c,p^m), we have that there is a unique $\underline{p}\in(c,p^m)$ that solves $H(\underline{p})-k=0$.

$\square$

Proof of Lemma 4. For every $\alpha\in(0,k/Q(p^m))$, let r be an equilibrium reservation price of consumers who start their search at the constrained firm. Define also $\underline{p}$, $\hat{p}$, $\hat{p}_1$ and Π^m as the associated parameters of the equilibrium distribution functions $F_k(\cdot)$ and $F_1(\cdot)$, characterized in Proposition 2.

Define $\underline{p}_0$ as the lower bound of the equilibrium support in the model with $\alpha=0$. By Proposition 3, $\underline{p}_0$ is uniquely defined and $\underline{p}_0\in(c,p^m)$.

Assume by contradiction that the equilibrium $\underline{p}$ does not converge to $\underline{p}_0$ as α tends to 0. Then, there exists a sequence $(\alpha^n)_{n\geq 0}$ that tends to zero and the associated sequence $(\underline{p}^n)_{n\geq 0}$ remains bounded away from $\underline{p}_0$. Since this sequence is bounded, we can extract a subsequence converging to some $\underline{p}_0'\neq\underline{p}_0$. In the following, all limits are taken along that converging subsequence.

Note that $\underline{p}_0'>c$. Otherwise, the corresponding equilibrium profits of firm 1, given in equation (3), tend to 0 as $n\to\infty$, contradicting the fact that firm 1 can always guarantee positive profits of $(1-k)\pi(p^m)$. Moreover, we must have that $\underline{p}_0'<p^m$. Otherwise, if $\underline{p}_0'=p^m$, then for sufficiently high n, the constrained firm could achieve a discrete jump in demand by setting a price slightly below $\underline{p}^n$ but still sufficiently close to p^m.

Rewriting the equation for $\hat{p}_1^n$, we have that

$$\frac{\hat{p}_1^n-r^n}{\hat{p}_1^n-c}=\alpha^n\frac{Q(r^n)}{k-\alpha^nQ(r^n)}\frac{r^n-\underline{p}^n}{\underline{p}^n-c}\underset{n\to\infty}{\longrightarrow}0,$$

since $\underline{p}^n-c$ is bounded away from zero for sufficiently high n ($\underline{p}_0'>c$). Moreover, since r^n is bounded, we have that $\hat{p}_1^n$ is also bounded (from the definition of $\hat{p}_1$), implying that $\hat{p}_1^n-r^n\to 0$ as $n\to\infty$. It follows that $\hat{p}^n-r^n\to 0$.

Next, we show that F_1^n weakly converges to

$$F_1(p,\underline{p}_0')=\begin{cases}1-\frac{\underline{p}_0'-c}{p-c} & p\in[\underline{p}_0',p^m)\\ 1 & p\geq p^m.\end{cases}$$

Let $p<\underline{p}_0'$. Then, for n sufficiently large, $\underline{p}^n>p$ and $F^n(p)=0$, which converges to 0 as $n\to\infty$. Next, let $p\in[\underline{p}_0',p^m)$. Since $\hat{p}^n-r^n$ tends to 0, we have that the gap $(r^n,\hat{p}^n)$ in

$F_1^n(p)$ vanishes as $n \to \infty$. Moreover, we have that $F_1^n(p) \underset{n\to\infty}{\longrightarrow} 1 - \frac{\underline{p}'_0 - c}{p-c} = F_1(p)$. Therefore, F_1^n weakly converges to F_1.

In the following step of the proof, we show that r^n tends to $r'_0 < p^m$, where r'_0 solves

$$CS(r'_0) - CS(p^m) = \int_{\underline{p}'_0}^{p^m} Q(p)F_1(p, \underline{p}'_0)dp.$$

The equation determining r^n can be rewritten as

$$CS(r^n) - CS(p^m) = \int_{\underline{p}^n}^{p^m} Q(p)F_1^n(p)dp.$$

Rearranging terms and taking absolute values, we have that

$$|CS(r^n) - CS(r'_0)| = \left| \int_{\underline{p}}^{\underline{p}'_0} Q(p)F_1^n(p)dp + \int_{\underline{p}'_0}^{p^m} Q(p)(F_1^n(p) - F_1(p, \underline{p}'_0))dp \right|.$$

As n goes to ∞, the first integral in the right-hand side tends to 0 as the integrand is bounded. By Lebesgue's dominated convergence theorem, the second integral on the right-hand side also tends to 0, as the integrand is bounded and converges pointwise to 0 on $[\underline{p}'_0, p^m]$. This implies that r^n tends r'_0 as $n \to \infty$.

Next, we show that F_k^n weakly converges to

$$F_k(p, \underline{p}'_0) = \begin{cases} \frac{\pi(\underline{p}'_0)}{k} \int_{\underline{p}'_0}^{p} \frac{\pi'(x)dx}{\pi(x)(x-c)} & p \in [\underline{p}'_0, p^m) \\ 1 & p \geq p^m. \end{cases}$$

From the definition of $\underline{p}^n$, we have that

$$\pi(\underline{p}^n) - \Pi^{m,n} = \alpha^n \pi(\underline{p}^n) F_k(r^n) \underset{n\to\infty}{\longrightarrow} 0,$$

implying that $\Pi^{m,n}$ tends to $\pi(\underline{p}'_0)$ as $n \to \infty$. Using the expression for F_k in the proof of Proposition 2, we have that

$$F_k^n(p) = \begin{cases} \Pi^{m,n} \int_{\underline{p}^n}^{p} \frac{1}{k - \alpha^n Q(x)} \frac{\pi'(x)dx}{\pi(x)(x-c)} & p \in [\underline{p}^n, r^n) \\ \frac{\Pi^{m,n}}{k} \int_{\underline{p}^n}^{\hat{p}^n} \frac{\pi'(x)dx}{\pi(x)(x-c)} & p \in [r^n, \hat{p}^n) \\ \frac{\Pi^{m,n}}{k} \int_{\underline{p}^n}^{p} \frac{\pi'(x)dx}{\pi(x)(x-c)} & p \in [\hat{p}^n, p^m) \\ 1 & p \geq p^m. \end{cases}$$

By Lebesgue's dominated convergence theorem, we have that the first and the third integrals tend to $\frac{\pi(\underline{p}'_0)}{k}\int_{\underline{p}'_0}^{p}\frac{\pi'(x)dx}{\pi(x)(x-c)}$ (as the integrands are bounded and converge pointwise to $\pi'(x)/(\pi(x)(x-c))$). Since $\hat{p}^n, r^n$ tends to r'_0, the gap $(r^n, \hat{p}^n)$ in $F^n_k(p)$ vanishes as $n \to \infty$. It follows that, F^n_k weakly converges to $F_k(p, \underline{p}'_0)$.

Since $\hat{p}^n \to r'_0 < p^m$ as $n \to \infty$, we have that $F_k(p, p'_0)$ is continuous at $p = p^m$, implying that

$$\frac{\pi(\underline{p}'_0)}{k}\int_{\underline{p}'_0}^{p^m}\frac{\pi'(x)dx}{\pi(x)(x-c)} = 1.$$

In Proposition 3, we have established that there exists a unique solution to this equation, which is equal to p_0. Thus, $\underline{p}'_0 = \underline{p}_0$, a contradiction. Hence, the equilibrium $\underline{p}$ converges to $\underline{p}_0$ as α tends to 0.

Following the same steps, we can show that the equilibrium $F_k(p)$ and $F_1(p)$, given in Proposition 2, weakly converge to the equilibrium distributions given in Proposition 3. Moreover, $\hat{p}, r$ tend to r_0 that solves

$$CS(r_0) - CS(p^m) = \int_{\underline{p}_0}^{p^m} Q(p)\left(1 - \frac{\underline{p}_0 - c}{p - c}\right) dp.$$

$\square$

5.1 Proof of Proposition 4

The existence of equilibrium and equations defining the distribution functions follow from Propositions 2 and 3. It remains to be shown the uniqueness of equilibrium and existence of $\hat{\alpha}(k)$. The proof follows a sequence of Lemmata. Lemma 10 establishes uniqueness of solution to search equation when $\hat{p} = p^m$. Lemma 12 shows that equilibrium with $\hat{p} = p^m$ exists if and only if $\alpha > \hat{\alpha}(k)$, and, together with the previous Lemma establishes uniqueness. Lemma 13 characterises the properties of function $\hat{\alpha}(k)$. Finally, Lemma 14 establishes uniqueness of equilibrium for $\alpha < \hat{\alpha}(k)$.

We define the function

$$b(r) = r - \int_{\underline{p}}^{r} p dF_1(p) - (1 - F_1(r))v,$$

that is, $b(r) = r - \mathbb{E}_1 p_1$ in the case when $\hat{p} = p^m$.

Lemma 10. *Suppose that $\alpha \in [0, k)$ and $\overline{p}_k = r$. Then, there exists a unique $r \in (\underline{p}, v)$ that solves $b(r) = 0$.*

Proof. Note that $b(\underline{p}) = \underline{p} - v < 0$ and $b(v) = \int_{\underline{p}}^{v}(v-p)dF_1(p) > 0$. Moreover, since $b' = 1 - rf_1(r) + vf_1(r) > 0$ we have that there exists $r \in (\underline{p}, v)$ that solves $b(r) = 0$. □

We define $\tau = \frac{r-c}{\underline{p}-c}$ and define a function

$$b(x;\alpha,k) = \frac{k-\alpha}{k}x - \log x - \left(\frac{1}{x} - \frac{\alpha}{k}\right)\frac{1-\alpha}{1-k}. \tag{10}$$

Note, that $b(\tau;\alpha,k) = 0$. The equilibrium with $\hat{p} = p^m$ exists when the constrained firm prefers to set any price from its support to setting a price just below v, i.e. whenever $k(\underline{p} - c) \geq k[1 - F_1(r)](v - c)$. We define $\hat{\alpha}(k)$ as a solution of

$$1 = [1 - F_1(r)]\frac{v-c}{\underline{p}-c},$$

were r is determined by $b(r) = 0$. Then denoting $\hat{\tau}$ a solution to $b(\cdot;\hat{\alpha}(k),k) = 0$ we obtain

$$1 = \hat{\tau} - \frac{k}{k - \hat{\alpha}(k)}\log\hat{\tau}, \tag{11}$$

Lemma 11. *The ratio $\tau = (r-c)/(\underline{p}-c)$ strictly decreases in α on $[0,k)$ and strictly increases in k on $(\alpha,1)$.*

Proof. We start by exploring the derivative of τ with respect to α, where $\alpha \in [0,k)$. This derivative can be computed as $\tau'_\alpha = -\frac{\partial b}{\partial \alpha}/\frac{\partial b}{\partial x}$. We separately analyze the sign of the partial derivatives of $b(x;\alpha,k)$ at $x = \tau$. The partial derivatives of $b(x;\alpha,k)$ with respect to x and α are respectively given by

$$\begin{aligned}\frac{\partial b}{\partial x} &= \frac{k-\alpha}{k} - \frac{1}{x} + \frac{1-\alpha}{1-k}\frac{1}{x^2};\\ \frac{\partial b}{\partial \alpha} &= -\frac{x}{k} + \frac{1}{x}\frac{1}{1-k} + \frac{1-2\alpha}{k(1-k)}.\end{aligned}$$

It is easy to see that the partial derivative of b with respect to x is positive at $x = \tau$ since $\frac{\partial b}{\partial x}(\tau;\alpha,k) > \frac{1}{\tau}\left(\frac{1-\alpha}{1-k}\frac{1}{\tau} - 1\right) = \frac{1}{\tau}\left(\frac{v-c}{\underline{p}-c}\frac{\underline{p}-c}{r-c} - 1\right) = \frac{1}{\tau}\left(\frac{v-c}{r-c} - 1\right) > 0$.

Next, we show that $\frac{\partial b}{\partial \alpha}(\tau;\alpha,k) > 0$. Note that $\frac{\partial b}{\partial \alpha}(x;\alpha,k)$ strictly decreases in x. First, suppose that $\frac{\alpha}{k} \leq \frac{1-k}{1-\alpha}$ (note that $\alpha < k < 1$). Then,

$$\frac{\partial b}{\partial \alpha}(\tau;\alpha,k) > \frac{\partial b}{\partial \alpha}\left(\frac{1-\alpha}{1-k};\alpha,k\right) = \frac{1}{1-k}\left(\frac{1-k}{1-\alpha} - \frac{\alpha}{k}\right) \geq 0.$$

Second, suppose that $\frac{\alpha}{k} > \frac{1-k}{1-\alpha}$. This implies that $\alpha > 0$ and k/α is well defined. Note that

$$b(k/\alpha; \alpha, k) = \frac{k}{\alpha} - 1 - \log\frac{k}{\alpha} > 0,$$

as $\log z < z - 1$ for $z > 1$. Thus, since function $b(\cdot; \alpha, k)$ strictly increases in x and is strictly positive at $x = k/\alpha$ we have that $\tau < \frac{k}{\alpha}$. Therefore, the partial derivative of b with respect to α can be evaluated from below as follows

$$\frac{\partial b}{\partial \alpha}(\tau; \alpha, k) > \frac{\partial b}{\partial \alpha}(k/\alpha; \alpha, k) = \frac{1}{k}\left(-\frac{k}{\alpha} + \frac{1-\alpha}{1-k}\right) > 0.$$

This implies that $\tau'_\alpha < 0$.

Next, we explore the behavior of τ with respect to k. The partial derivative of $b(\cdot)$ with respect to k is given by

$$\frac{\partial b}{\partial k} = \frac{\alpha}{k^2}x - \frac{1-\alpha}{(1-k)^2}\frac{1}{x} + \frac{\alpha(1-\alpha)(2k-1)}{k^2(1-k)^2}.$$

Note that $\frac{\partial b}{\partial k}$ strictly increases in x. First, suppose that $\frac{\alpha}{k} \leq \frac{1-k}{1-\alpha}$. Then,

$$\frac{\partial b}{\partial k}(\tau; \alpha, k) < \frac{\partial b}{\partial k}\left(\frac{1-\alpha}{1-k}; \alpha, k\right) = -\frac{1-\alpha}{(1-k)^2}\left(\frac{1-k}{1-\alpha} - \frac{\alpha}{k}\right) \leq 0.$$

Second, suppose that $\frac{\alpha}{k} > \frac{1-k}{1-\alpha}$. Then, since $k/\alpha > \tau$, we have that

$$\frac{\partial b}{\partial k}(\tau; \alpha, k) < \frac{\partial b}{\partial k}(k/\alpha; \alpha, k) = \frac{1-\alpha}{k(1-k)}\left(\frac{1-k}{1-\alpha} - \frac{\alpha}{k}\right) < 0.$$

Therefore, we can conclude that $\tau'_k > 0$. This establishes the result of the lemma. $\square$

Lemma 12. *There exists a unique mixed strategy equilibrium in which $\hat{p} = p^m$ if and only if $\alpha \in [\hat{\alpha}(k), k)$, where $\hat{\alpha}(k) \in (0, k/2)$ for all $k \in (0, 1)$.*

Proof. Let us define the function

$$g(\alpha) \equiv \frac{k-\alpha}{\tau - 1}\left(1 - (1 - F_1(r))\frac{v-c}{\underline{p}-c}\right),$$

where r solves $b(r) = 0$ and $\alpha \in [0, k)$. Note that firm k does not find it optimal to deviate to price $v - \varepsilon$ if and only if $g(\alpha) \geq 0$. We show that $g(\alpha)$ strictly increases on $[0, k)$ and there exists a unique $\hat{\alpha}(k) \in (0, k/2)$ that solves $g(\hat{\alpha}(k)) = 0$.

By plugging $F_1(\cdot)$ in the formula for $g(\alpha)$ and using the fact that $b(\tau;\alpha,k)=0$, we obtain that $g(\alpha)$ can be represented in the following way,

$$\begin{aligned} g(\alpha) &= \frac{k-\alpha}{\tau-1}\left(1-\frac{k}{k-\alpha}\left(\frac{1}{\tau}-\frac{\alpha}{k}\right)\frac{1-\alpha}{1-k}\right) \\ &= \frac{k-\alpha}{\tau-1}\left(1-\tau+\frac{k}{k-\alpha}\log\tau\right) \\ &= k\frac{\log\tau}{\tau-1}-(k-\alpha). \end{aligned}$$

By taking the derivative we obtain that

$$g'(\alpha) = k\frac{1-\frac{1}{\tau}-\log\tau}{(\tau-1)^2}\tau'_\alpha + 1.$$

Note that $1-\frac{1}{z}-\log z$ is equal to zero at $z=1$ and strictly decreases for any $z>1$, implying that $1-\frac{1}{\tau}-\log\tau<0$. By lemma 11, we have that $\tau'_\alpha<0$ which implies that $g'(\alpha)>0$ for any $\alpha\in[0,k)$.

To prove the existence and uniqueness of $\hat{\alpha}(k)$ that solves $g(\alpha)=0$ it remains to show that $g(\alpha)$ is negative at $\alpha=0$ and positive for $\alpha=k/2$. Note that the sign of $g(0)$ is determined by the sign of $1-[1-F_1(r)]\frac{v-c}{\underline{p}-c}=1-\frac{v-c}{r-c}<0$. At $\alpha=k/2$ we have that the corresponding τ satisfies $b(\tau;k/2,k)=0$ which is equivalent to

$$\frac{1}{2}\tau-\log\tau-\left(\frac{1}{\tau}-\frac{1}{2}\right)\frac{1-k/2}{1-k}=0.$$

Note that for any k we have that this $\tau\in(1,2)$. Therefore, by plugging this into $g(\cdot)$ at $\alpha=k/2$ we obtain

$$\begin{aligned} g(k/2) &= \frac{k/2}{\tau-1}\left(1-2\left(\frac{1}{\tau}-\frac{1}{2}\right)\frac{1-k/2}{1-k}\right) \\ &= \frac{k/2}{\tau-1}(1-\tau+2\log\tau) > \frac{k/2}{\tau-1}(1-1+\log 1)=0, \end{aligned}$$

where we used the fact that function $1-x+2\log x$ is a strictly increasing function on $[1,2)$.

Therefore, we showed that $g(\alpha)$ strictly increases in α, is negative at $\alpha=0$ and is positive at $\alpha=k/2$. This implies that it intersects with zero at exactly one point that we refer to as $\hat{\alpha}(k)\in(0,k/2)$. For $\alpha\geq\alpha(k)$, we have that $\underline{p}-c\geq[1-F_1(r)](v-c)$ and firm k does not find it profitable to deviate to a price slightly lower than v. Otherwise, if $\alpha<\hat{\alpha}(k)$, then the deviation is profitable and such an equilibrium does not exist.

$\square$

Lemma 13. *Function $\hat{\alpha}(k)$ strictly increases in k on $(0,1)$ and converges to $\hat{\alpha}_1$, that solves $1 + \frac{\hat{\alpha}_1 \log \hat{\alpha}_1}{(1-\hat{\alpha}_1)^2} = 0$, when k tends to 1.*[7]

Proof. Recall that $\hat{\alpha}(k)$ is determined by the following equation

$$1 = \hat{\tau} - \frac{k}{k - \hat{\alpha}(k)} \log \hat{\tau},$$

where $\hat{\tau} = \tau(\hat{\alpha}(k), k)$ and $\hat{\tau}$ solves $b(\cdot, \hat{\alpha}(k), k) = 0$. By taking the derivative of this equation with respect to k we have that

$$\left(1 - \frac{k}{k - \hat{\alpha}(k)} \frac{1}{\hat{\tau}}\right)\left(\tau'_\alpha \frac{d\hat{\alpha}(k)}{dk} + \tau'_k\right) - \log \hat{\tau} \left(\frac{k}{(k - \hat{\alpha}(k))^2} \frac{d\hat{\alpha}(k)}{dk} - \frac{\hat{\alpha}(k)}{(k - \hat{\alpha}(k))^2}\right) = 0.$$

Solving for the derivative of $\hat{\alpha}(k)$ with respect to k we obtain

$$\frac{d\hat{\alpha}(k)}{dk} = \left(-\phi(k)\tau'_k - \frac{\hat{\alpha}(k)}{(k - \hat{\alpha}(k))^2} \log \hat{\tau}\right) \Big/ \left(\phi(k)\tau'_\alpha - \frac{k}{(k - \hat{\alpha}(k))^2} \log \hat{\tau}\right), \tag{12}$$

where

$$\phi(k) = 1 - \frac{k}{k - \hat{\alpha}(k)} \frac{1}{\hat{\tau}}.$$

To determine the sign of the derivative of $\hat{\alpha}(k)$ with respect to k we show that $\phi(k) > 0$ for any $k \in (0,1)$. Note that by plugging the distribution function $F_1(r)$ into the definition of $\hat{\alpha}(k)$ we have that

$$1 = \frac{k}{k - \hat{\alpha}(k)} \left(\frac{1}{\hat{\tau}} - \frac{\hat{\alpha}(k)}{k}\right) \frac{1 - \hat{\alpha}(k)}{1 - k}.$$

By rearranging and solving for ϕ we show that ϕ is positive – that is,

$$\begin{aligned}\phi(k) = 1 - \frac{\hat{\alpha}(k)}{k - \hat{\alpha}(k)} - \frac{1 - k}{1 - \hat{\alpha}(k)} &= \frac{k^2 - 2k\hat{\alpha}(k) + \hat{\alpha}(k)}{(k - \hat{\alpha}(k))(1 - \hat{\alpha}(k))} \\ &= \frac{(k - \hat{\alpha}(k))^2 + \hat{\alpha}(k)(1 - \hat{\alpha}(k))}{(k - \hat{\alpha}(k))(1 - \hat{\alpha}(k))} > 0.\end{aligned}$$

Since $\phi(k) > 0$ and by Lemma 11 the partial derivatives $\tau'_\alpha < 0$ and $\tau'_k > 0$ we have that the numerator and the denominator of (12) are both strictly negative implying that $d\hat{\alpha}(k)/dk > 0$ for all $k \in (0,1)$. Therefore, $\hat{\alpha}(k)$ strictly increases on $(0,1)$.

Next, we explore the limiting behavior of $\hat{\alpha}(k)$ when k tends to 1. First, we compute the

[7] The root is approximately equal to 0.394229.

limit of τ for a given $\alpha \geq \hat{\alpha}(k)$ when k approaches 1. Note that since $\alpha > \hat{\alpha}(k) > 0$ (by Lemma 12), we have that k/α is well defined. In the proof of Lemma 11 we have shown that $\tau < k/\alpha$. Thus, τ is bounded from above by $1/\alpha$. Since τ strictly increases in k, then we can conclude that the limit there exists a finite limit of τ when k tends to 1. The equation $b(\tau; \alpha, k) = 0$ can be rewritten as

$$(\underline{p} - c)\tau = \frac{k}{k-\alpha}(\underline{p} - c)\log\tau + \left(\frac{k}{k-\alpha}\frac{1}{\tau} - \frac{\alpha}{k-\alpha}\right)(v - c).$$

By taking the limit on both sides when k tends to 1 and using the fact that $\underline{p}$ converges to 0, we obtain that

$$\lim_{k\to 1^-}\tau = \frac{1}{\alpha}.$$

Since $\hat{\alpha}(k)$ is an increasing function on $(0,1)$ and is bounded by $1/2$ (by Lemma 12), we have that $\hat{\alpha}(k)$ has a finite limit when $k \to 1$ that we denote as $\hat{\alpha}_1$. Note, that $\hat{\tau}$ converges to $1/\hat{\alpha}_1$ when k goes to 1. By taking the limit of equation (11) that determines $\hat{\alpha}(k)$ when $k \to 1$, we obtain

$$1 = \lim_{k\to 1^-}\left(\hat{\tau} - \frac{k}{k-\hat{\alpha}(k)}\log\hat{\tau}\right) = \frac{1}{\hat{\alpha}_1} + \frac{1}{1-\hat{\alpha}_1}\log\hat{\alpha}_1.$$

Therefore, the limit of $\hat{\alpha}_1$ solves

$$1 + \frac{\hat{\alpha}_1\log\hat{\alpha}_1}{(1-\hat{\alpha}_1)^2} = 0$$

and is approximately equal to 0.394229. $\square$

Lemma 14. *Suppose that $\alpha < \hat{\alpha}(k)$. Then there exists a unique mixed strategy equilibrium, and in this equilibrium $\hat{p} < p^m$.*

Proof. Note, that the existence of equilibrium follows from Proposition 2. Moreover, from Lemma 11 it follows that in this equilibrium $\hat{p} < p^m$. Thus, it remains to establish the uniqueness of equilibrium.

We define $\eta = \frac{\hat{p}-c}{\underline{p}-c}$ and $\zeta = \frac{v-c}{\hat{p}-c}$. Since $F_1(p)$ is continuous at $p = r$ we obtain that

$$1 - \frac{1}{\eta} = \frac{k}{k-\alpha}\left(1 - \frac{1}{\tau}\right).$$

Next, since $F_k(\hat{p}) = F_k(r)$, we have that $\pi_1(\underline{p}) = \pi_1(\hat{p})$ implies that $[1 - \alpha F_k(r)](\underline{p} - c) = [1 - kF_k(r)](\hat{p} - c)$. Dividing this equation by $(\underline{p} - c)$ and solving for $F_k(r)$, we obtain that $F_k(r) = (\eta - 1)/(k\eta - \alpha)$. Then, we equate the profits of firm 1 at $p = \hat{p}$ and $p = v$,

$[1-kF_k(r)](\hat{p}-c)=(1-k)(v-c)$ to obtain

$$\zeta = \frac{k-\alpha}{1-k}\frac{1}{k\eta-\alpha}.$$

By rearranging, we have that

$$1-\frac{1}{\zeta}=\frac{k(1-k)}{k-\alpha}\left(\frac{1-\alpha}{1-k}-\eta\right).$$

The reservation price is given by $r=\mathbb{E}[p_1]$:

$$r=\int_{\underline{p}}^{r}\frac{k}{k-\alpha}\frac{\underline{p}-c}{(p-c)^2}pdp+\int_{\hat{p}}^{v}\frac{\underline{p}-c}{(p-c)^2}pdp+\frac{\underline{p}-c}{v-c}v,$$

which simplifies to

$$\tau=\frac{k}{k-\alpha}\log\tau+\log\zeta+1.$$

To sum up, for any k,α such that $\alpha<k$, we have that τ,η,ζ solve the following system of equations:

$$\begin{cases}1-\frac{1}{\eta}=\frac{k}{k-\alpha}\left(1-\frac{1}{\tau}\right)\\ 1-\frac{1}{\zeta}=\frac{k(1-k)}{k-\alpha}\left(\frac{1-\alpha}{1-k}-\eta\right)\\ \log\zeta=\tau-1-\frac{k}{k-\alpha}\log\tau\end{cases}. \tag{13}$$

Since

$$\begin{aligned}\frac{1}{\zeta}&=1-\frac{k(1-k)}{k-\alpha}\left(\frac{1-\alpha}{1-k}-\eta\right)=1-(1-k)\left(\frac{k}{k-\alpha}\frac{1-\alpha}{1-k}-\frac{1}{\frac{1}{\tau}-\frac{\alpha}{k}}\right)\\&=(1-k)\left(\frac{1}{\frac{1}{\tau}-\frac{\alpha}{k}}-\frac{\alpha}{k-\alpha}\right),\end{aligned}$$

we obtain that the system can be simplified to

$$L(\tau;k,\alpha)\equiv(1-k)\left(\frac{1}{\frac{1}{\tau}-\frac{\alpha}{k}}-\frac{\alpha}{k-\alpha}\right)=\exp\left(1-\tau+\frac{k}{k-\alpha}\log\tau\right)\equiv R(\tau;k,\alpha)$$

We now show that this equation has a unique solution for $\alpha<\hat{\alpha}(k)$. Note that $\tau>1$, and because $\pi_k=k(\underline{p}-c)>\alpha(r-c)$ we have that $\tau<k/\alpha$. Both sides of this equation are continuous in τ on $\tau\in[1,k/\alpha)$.

Note that $L(1;k,\alpha) = 1 - k < 1 = R(1,k,\alpha)$, $R(k/\alpha;k,\alpha) = \exp\left(\frac{k\log(k/\alpha)}{k-\alpha} - \frac{k-\alpha}{\alpha}\right)$ and

$$\lim_{\tau\to k/\alpha} L(\tau;k,\alpha) = \infty.$$

Hence solution to equation exists.

Solving for $\frac{\partial^2 R}{\partial \tau^2} = 0$ gives

$$\tau_{1,2} = \frac{k}{k-\alpha} \pm \sqrt{\frac{k}{k-\alpha}}.$$

As the maximum of R is attained at $\tau = \frac{k}{k-\alpha}$, we have that $R(\tau;k,\alpha)$ is increasing at first infliction point and decreasing at second. Moreover, if $\tau_1 < 1$, then R is concave in τ on its increasing part, and therefore solution to equation is unique. Solving $\tau_1 < 1$ gives $\alpha^2 < k(k-\alpha)$ or $\alpha < \frac{\sqrt{5}-1}{2}k$. Note, that $\frac{\sqrt{5}-1}{2}k > \hat{\alpha}(k)$, hence $L(\tau;k,\alpha)$ is convex and $R(\tau;k,\alpha)$ is concave for $\alpha < \hat{\alpha}(k)$ and therefore solution is unique.

□

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