Trade, specialization and inflation*

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Abstract

This paper studies how optimal inflation, from an optimal taxation perspective, depends on the level of trade and interacts with the degree of specialization. When specialization is fixed, an increase in trade leads to an increase in inflation. If the specialization choice is endogenous, increases in inflation leads to a decrease in the quantity of goods traded and, in turn, leads to less specialization. In equilibrium, the policymaker then reduces inflation in response to an increase in trade, since this increases the tax base and hence tax revenues. From a historical perspective, low inflation currencies tended to be associated with a high level of trade. We study this for the period starting just after the Black death and ending in the early modern era for several European currency areas. Using an IV approach, we find support of the model in the data, i.e., an increase in the degree of specialization leads to a lower inflation rate.

Keywords: Seigniorage, Specialization, Optimal inflation

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1 Introduction

That the amount and level of trade affect economic agents money holdings is not surprising. Also, changes in trade can affect the degree of specialization of the economy. This, in turn, affects the amount of goods that is traded, hence influencing money holdings. In Camera, Reed, and Waller (2003), money supply is analyzed in a model with specialization. However, only different levels of money supply is analyzed, and the optimal inflation rate is not studied in the paper.

Historically, currencies with low inflation were introduced in economies with a high level of trade, e.g., the Venetian ducato and the Florentine fiorino, see Spufford (1988). In other, in a trade perspective less developed areas of Europe, e.g., eastern Germany, Poland and parts of Scandinavia, the monetary tax was higher, see Svensson and Westermark (2020).

This paper studies how the inflation rate, optimally chosen from a fiscal perspective, and the degree of specialization in an economy depends on the level of trade or market size in an economy. In the model, an increase in specialization leads to a more cost efficient way of producing goods, but the goods produced face positive demand from a smaller set of consumers. The paper builds on the framework of Lagos and Wright (2005), as well as the paper by Rocheteau and Wright (2005). We model buyers and sellers as in Geromichalos and Simonovska (2014). Thus, sellers remain sellers and buyers remain buyers, and there is no probabilistic transition between being buyer and seller, as in e.g., Lagos and Wright (2005).

In the model, when specialization is fixed, an increase in trade leads to an increase in inflation, since the tax base increases. When specialization is endogenous, an increase in trade/market size leads to more specialization. This affects the incentives of the policymaker and, in contrast to the case when specialization is fixed, the resulting increase in specialization from the increase in trade leads to a fall in inflation. The reason is the following. If the specialization choice is endogenous, increases in inflation leads to a decrease in the quantity of goods traded. This, in turn, leads to less specialization and a further reduction in the amount traded, due to the increased production costs. Then, in equilibrium, in response to an increase in trade, the policymaker reduces inflation since this leads to an increase in the quantity of goods traded, in turn increasing the tax base.

We study whether the effects of specialization on inflation is important, from a cliometric perspective. Specifically, we look at several European currency areas for the period following the Black Death until the early modern period. As a proxy for specialization, we use the number of guilds. We use existing databases for Italy and the Low countries, and collect information about the number of guilds for cities located in the Holy Roman empire, as well as for Krakow in Poland. Using an IV approach, we find a negative relationship between the degree of specialization and the inflation rate, in line with the predictions of the model.

Several papers analyze the optimal inflation rate a social planner would choose in this class of models. These papers often finds support for the Friedman rule. Fewer papers analyze this in a setting focusing on a public finance perspective. In a model where lump-sum taxes are available to fund government expenditure, the Friedman rule is often optimal. When lump sum taxes are difficult to implement, an inflation tax can potentially alleviate distortions caused by other taxes. The paper Svensson and Westermark (2020) studies optimal policy in a framework with Gesell Taxes on money, where the policy normally entails a substantially positive inflation rate. Some other papers studying specialization in this framework, but not addressing the issue in this paper is Shi (1997) and Reed (1998).

The paper is organized as follows. Section 2 describes the model and in section 3 we analyze the properties of equilibria in the model. Section 4 describes the data and the empirical approach. Finally, section 5 delineates the conclusions.

2 The model

We first study the relationship between specialization and seigniorage or inflation in a money search model in the spirit of Lagos and Wright (2005), and then derive som empirical predictions from the theoretical analysis.

2.1 The economic environment

In the model time is discrete and the horizon is infinite. Following along the lines of Lagos and Wright (2005), each time period is divided into two sub-periods. During the first sub-period, there is trade in decentralized markets (denoted by DM). A Walrasian market, denoted by CM, is open in the second sub-period. Following Geromichalos and Simonovska (2014), the identity as buyer and seller is permanent. We model specialization along the lines of Camera, Reed, and Waller (2003) Thus, there is a continuum of agents and good types distributed on a unit circle. An agent i has a positive payoff of consuming goods from point i to $i+x$ on the circle. If $q_j > 0$ then agent i derives payoff $u(q_j) > 0$ if $j \in [i, i+x]$ and zero otherwise. Here $u' > 0$, $u'' < 0$ and $u''' > 0$ and $u(0) = 0$. Specialization is modelled as follows. The sellers have a production location k on the circle. The seller chooses the length y, where $y \in [0,1]$, of the production interval around the point k so that it is $[k-y/2, k+y/2]$. Thus, when y increases, there is less specialization but a larger set of consumers find the goods produced attractive. Meetings are bilateral and random, following a Poisson process with an arrival rate ∂ where, by appropriately normalizing the time interval, $\partial = 1$. The probability that a buyer and seller that have met can trade is then $p(x,y) = x + y$. The timing of choices is as follows. The choice of technology is a more long-run decision, due to the e.g., investments required for technology modifications. Thus, it is made initially in a time period. Then the policymaker chooses inflation, which is followed by trade in the DM where quantities are chosen.[1] In the DM's buyers and sellers are randomly matched according to a matching function M. Buyer preferences are

$$\mathcal{U}(q, X, H) = u(q) + U(X) - H,$$

where q is the consumption of of good, X consumption in CM and H hours worked in the CM. Seller preferences are

$$\mathcal{V}(h, X, H) = -c(q, y) + U(X) - H,$$

where c is the cost of producing the good in the DM. We assume that c and U are C^2 and in addition $c(0, y) = 0$, $c_q > 0$, $c''_{qq} \geq 0$, $c_y > 0$, $U' > 0$ and $U'' \leq 0$. Also there is a $X^* \in (0, \infty)$ satisfying $U'(X^*) = 1$ and $U(X^*) > X^*$.

[1] Note that the first-order conditions are identical when the timing is modified so that the specialization choice is made simultaneously with seigniorage determination and then the quantity choice.

2.2 Value functions

We start to characterize the value functions in the CM. For a buyer, the value function is

$$W^B\left(m^B\right) = \max_{X,H,m^{B\prime}} \left\{U\left(X\right) - H + \beta V^B\left(m^{B\prime}\right)\right\},$$

subject to

$$X + \psi m^{B\prime} = H + \psi m^B + \Pi + T,$$

where m^B is money holdings, ψ the price of the currency. Finally, Π are firm dividends and T government transfers. Clearly, the buyer chooses $X = X^*$. Then, using the budget constraint, the value is

$$W^B\left(m^B\right) = U\left(X^*\right) - X^* + \psi m^B + \max_{m^{B\prime}} \left\{-\psi m^{B\prime} + \beta V^B\left(m^{B\prime}\right)\right\}. \quad (1)$$

Due to quasi-linearity, the choice of $m^{B\prime}$ is independent of m^B. Hence, we can write

$$W^B\left(m^B\right) = \psi m^B + K^B. \quad (2)$$

Now consider sellers. As equilibrium inflation will be above the Friedman rule (see below), sellers leave the CM with zero money holdings. Noting that sellers chooses $X = X^*$, we have $H = X^* - \psi m^S - \Pi - T$. We then have

$$W^S\left(m^S\right) = U\left(X^*\right) - X^* + \psi m^S + \beta V^S\left(0\right) + \Pi + T = \psi m^S + K^S.$$

2.3 Decentralized trades

We now consider bargaining in DM. The value of a buyer in the DM is then

$$V^B\left(m^B\right) = p\left[u\left(q\right) + W^B\left(m^B - \Delta\right)\right] + \left(1-p\right) W^B\left(m^B\right). \quad (3)$$

We now derive the Euler equations for DM trade. We assume that the surplus is split according to Kalai Smorodinsky bargaining. The Kalai Smorodinsky solution implies that the buyer gets $u\left(q\right) - \Delta$ and the seller $\Delta - c\left(q\right)$ where $\Delta = W^S\left(m^S\right) - W^S\left(0\right) = \theta^B c\left(q\right) + \theta^S u\left(q\right) = \theta^B q\left(1+y\right) + \theta^S u\left(q\right)$. Here, the buyer gets the share θ^B of the surplus and the seller $\theta^S = 1 - \theta^B$. Using (3) in the next period in the expression for W in (1), we can write the (relevant part of the) objective for a buyer as

$$J^B\left(m^B\right) = \left[-\psi + \beta\psi'\right] m^B + \beta p\left[u\left(q\left(m^B\right)\right) - \Delta\right].$$

Then the first-order condition with respect to m_H^B for J_H^B is, using $\Delta = \theta^B c(q) + \theta^S u(q)$, $\Delta = \psi' m^{B\prime}$ and that $\frac{\partial \Delta}{\partial m^{B\prime}} = \psi'$,

$$\psi = \beta\psi' + \beta p\psi'\left[\frac{u'\left(q\left(m^{B\prime}\right)\right)}{\theta^B c'\left(q\left(m^{B\prime}\right)\right) + \theta^S u'\left(q\left(m^{B\prime}\right)\right)} - 1\right].$$

We assume that production satisfies constant returns, for a given degree of specialization y. Thus, $c(q,y) = qh(y)$ where h is normalized so that $h(0) = 1$ and also that $h_y > 0$. It is easily seen that the second-order condition is always satisfied. Using $\psi = (1+\pi)\psi'$, the first-order condition for the buyer simplifies to

$$u'\left(q\left(m^{B\prime}\right)\right) = \frac{1+\pi-\beta+\beta p}{\beta p - \theta^S\left(1+\pi-\beta+\beta p\right)}\theta^B h(y). \tag{4}$$

Note that, for marginal utility to be nonnegative, we require that the denominator is positive, i.e., $\beta p\left(1-\theta^S\right) > \theta^S\left(1+\pi-\beta\right)$.

Thus, an increase in inflation leads to an increase in the marginal utility and hence a decrease in the quantity consumed, since it is more costly to hold money. Also, an increase in p, i.e., an increase in the probability of matching, decreases the marginal utility and thus increases the quantity, since the increased probability of matching leads to an increase in money holdings. Finally, an increase in y, i.e., a fall in specialization, taking into account the effects through p, leads to an increase in marginal utility. Thus, the direct effect of a change in y dominates, i.e., the increase in marginal costs, over the increase in probability of matching.

2.4 Specialization choice

The seller chooses specialization to maximize the DM value, treating π as fixed. Seller DM value is

$$V^S(0) = \max_y p\left[-qh(y) + W^S\left(m^S\right) - W^S(0)\right] + W^S(0). \tag{5}$$

Also, using (5), seller CM value is

$$\begin{aligned} W^S\left(m^S\right) &= U\left(X^*\right) - X^* + \psi m^S \\ &\quad + \max_y \beta p\left[-qh\left(y\right) + W^S\left(m^S\right) - W^S\left(0\right)\right] + \beta W^S\left(0\right) + \Pi + T. \end{aligned}$$

Since sellers are atomistic, a change of y by a single seller do not affect buyer money holdings, the first-order condition can be derived, using that $\frac{\partial q}{\partial y}$ follows from differentiating $\Delta = \theta^B qh\left(y\right) + \theta^S u\left(q\right)$ with Δ fixed;

$$\frac{\partial q}{\partial y} = -\frac{\theta^B q}{\theta^B h\left(y\right) + \theta^S u'\left(q\right)} h_y$$

Then the first-order condition is, using Δ, where we define $v\left(q\right) = \theta^B qh\left(y\right) + \theta^S u\left(q\right)$,

$$\theta^S\left[\frac{u\left(q\right)}{q} - h\left(y\right)\right] - p\frac{\theta^S u'\left(q\right)}{v'\left(q\right)} h_y = 0. \tag{6}$$

Thus, increasing y increases of the probability of trade, leading to an increase in payoff of $W^S\left(m^S\right) - W^S\left(0\right) - qh\left(y\right)$, the first term in the expression above, but also increases the cost of producing the good.

2.5 Seigniorage

The seigniorage is $\pi\psi m$ where $\psi m = \theta^B qh\left(y\right) + \theta^S u\left(q\right)$. The first-order condition is then, using that y is determined before (or simultaneously with) π (i.e., treated as given),

$$\left(\theta^B qh\left(y\right) + \theta^S u\left(q\right)\right) + \pi\left(\theta^B h\left(y\right) + \theta^S u'\left(q\right)\right)\frac{dq}{d\pi} = 0. \tag{7}$$

Thus, an increase in π increases the seigniorage from money holdings $\theta^B qh\left(y\right) + \theta^S u\left(q\right)$ but also reduce the quantity in the DM through (4) and hence money holdings, which in turn reduces the revenues. Using $\frac{dq}{d\pi}$ from expression (4) gives

$$\pi = -\theta^B h\left(y\right) u''\left(q\right) q \frac{\theta^B h\left(y\right) + \theta^S \frac{u(q)}{q}}{\left(v'\left(q\right)\right)^3} p = K\left(\theta, q, y\right) p \tag{8}$$

where $K\left(\theta, q, y\right) > 0$.

3 Equilibrium

We now analyze the equilibria of the model. The first-order conditions are

$$\begin{aligned} u'(q) &= \frac{\pi + p}{\left(p - \theta^S (\pi + p)\right)} \theta^B h(y) \\ \frac{u(q)}{q} - h(y) &= p \frac{u'(q)}{v'(q)} h_y \\ \frac{\pi}{\theta^B} &= -\frac{\theta^S u(q) + \theta^B q h(y)}{(v'(q))^3} h(y) u''(q) p. \end{aligned}$$

We first show that an equilibrium exists. Note first that, from the quantity choice (4), that optimal quantities are bounded in the interval $[0, 2 - x]$. Also, q is continuous in π and y, except when $\pi \to \frac{1-\theta^S}{\theta^S}(x+y)$ or equivalently, when $y \to \frac{\theta^S}{1-\theta^S}\pi - x$ along the sequence. In either case, as π or y converges in such a way so that the denominator of (4) is positive, we have $u'(q) \to +\infty$ and hence $q \to 0$. In the case when π or y converges in such a way so that the denominator of (4) is negative, we have $q = 0$. In either case, q is a continuous function of π and y at such a point. Also, from the specialization choice (6), y is continuous in q and π. By assumption, y is in the compact set $[0, 1]$. Finally, using (8) π is bounded in the interval $[0, \pi^{\max}]$. Since π is decreasing in u', the upper bound is determined when $u'(q) \to 1 = \frac{\pi+(x+y)}{(x+y)-\theta^S(\pi+(x+y))}\theta^B h(y)$, implying $h(y) = 1$. Letting q^1 denote the solution for q when $u'(q) \to 1$, then from (8) we have the upper bound as $\pi^{\max} = -\theta^B u''(q^1) q^1 \left(\theta^B + \theta^S \frac{u(q^1)}{q^1}\right)$. Clearly, we cannot have $\pi < 0$, since then seigniorage is negative. Note also that, from (4) and optimally chosen seigniorage, the denominator in the first-order condition (4) is positive and we must have $(x+y) > \theta^S(\pi + (x+y))$, since otherwise $q = 0$ for π which yields zero seigniorage revenues. Setting $\pi' = \varepsilon > 0$ but small so that the inequality holds instead yields a positive seigniorage, since then quantities and hence money holdings are positive. Hence, letting $X = [0, 2-x] \times [0, 1] \times [0, \pi^{\max}]$, the function $f : X \to X$, implicitly defined by the first-order conditions, is continuous and a fixed point exists.

To see the importance of specialization, we first focus on the case where y is fixed at some level $y = \bar{y}$, i.e., (4) and (7) are satisfied in equilibrium for $y = \bar{y}$. Thus, only the first order conditions for quantities and specialization, i.e., (4) and (8), are relevant.

Noting that the first-order condition with respect to q, i.e., (4) is, when $\beta \to 1$,

$$\pi = p\left[\frac{u'(q)}{\theta^B(1+\bar{y}) + \theta^S u'(q)} - 1\right].$$

Using this in the inflation first-order condition (8)

$$\pi = -\theta^B h(\bar{y})\, u''(q)\, q \frac{\theta^B h(\bar{y}) + \theta^S \frac{u(q)}{q}}{(v'(q))^3} p$$

gives a unique solution for q (when $y = \bar{y}$), independent of π and p. This, in turn, implies that the ratio π/p is fixed in equilibrium. Hence, an increase in x leads to an increase in inflation. The intuition for the result is that an increase in x leads to an increase in the tax base, for a given inflation rate, in turn increasing the benefit from raising inflation. In equilibrium, the inflation rate is then higher.

Now consider the model where specialization is a choice variable. It turns out that it is easy to see the intuition of the model when $u = q^\alpha$, $h = 1 + y$ and $\beta \to 1$, since these assumptions simplifies the analysis of the model.[2] In this case, we claim that the quantity first-order condition (4) can be written as, letting $\theta = \theta^S$ and hence $\theta^B = 1 - \theta$,

$$u'(q) = k(\alpha, \theta)(1 + y).$$

To see this, note that the seigniorage first-order condition (8) can be written as, using the definition of $u(q)$ which implies $u(q)/q = \frac{1}{\alpha}u'(q)$ and $qu'(q) = (1-\alpha)\, u'(q)$,

$$\pi = (1-\theta)\frac{\left(\theta\frac{1}{\alpha}\tilde{k} + (1-\theta)\right)}{\left(\theta\tilde{k} + (1-\theta)\right)^3}(1-\alpha)\,\tilde{k}p. \tag{9}$$

Using this in the quantity first-order condition gives

$$\tilde{k} = \frac{(1-\theta)\frac{\left(\theta\frac{1}{\alpha}\tilde{k}+(1-\theta)\right)}{\left(\theta\tilde{k}+(1-\theta)\right)^3}(1-\alpha)\,\tilde{k} + 1}{1 - \theta\left(\frac{\left(\theta\frac{1}{\alpha}\tilde{k}+(1-\theta)\right)}{\left(\theta\tilde{k}+(1-\theta)\right)^3}(1-\alpha)\,\tilde{k}\right)}. \tag{10}$$

[2] This is assumed in Camera, Reed, and Waller (2003). Also, the literature on flexible manufacturing follows a similar approach, see Eaton and Schmitt (1994).

Thus, $\tilde{k}$ is independent of p and hence x and we can define $k(\alpha, \theta) = \tilde{k}$. From the definition of $k(\alpha, \theta)$, we have $k(\alpha, \theta) = \frac{\pi + p}{\pi + p - \frac{1}{1-\theta}\pi} \geq 1$. From the solution of $k(\alpha, \theta)$, (10) we get

$$(k(\alpha, \theta) - 1)(\theta k(\alpha, \theta) + (1 - \theta))^2 = \left(\theta \frac{1}{\alpha} k(\alpha, \theta) + (1 - \theta)\right)(1 - \alpha) k(\alpha, \theta).$$

Thus, if $k(\alpha, \theta) > 0$, we must have $k(\alpha, \theta) > 1$. Since $k(\alpha, \theta) \leq 0$ violates optimality, $k(\alpha, \theta) > 1$ is the case for any interior solution. Note that there is a cutoff for θ when the denominator in the solution (10) for $k(\alpha, \theta)$ is zero.

We finally derive conditions for the bounds on y to be nonbinding in equilibrium.

Lemma 1 *There is an interior equilibrium if*

$$x < \frac{1}{\alpha} - 1 \tag{11}$$

and, for $\alpha \geq \bar{\alpha}(\theta^B)$ *where* $\bar{\alpha}(\theta^B)$ *is decreasing in* θ^B *with* $\bar{\alpha}(0) = 0.75$ *and* $\bar{\alpha}(1) = 2 - \sqrt{2}$ *where the function* $\bar{\alpha}(\theta^B)$ *is defined in the proof.*

Proof: See the appendix. ■

Note that, to rule out equilibria at the upper bound $y = 1$, the proof uses that there is an upper bound on π in that case and derives the function $\bar{\alpha}(\theta^B)$ given that an equilibrium at the boundary potentially can attain that inflation rate (see equation (A.3) in the proof). In practice, equilibrium inflation rates are lower, and hence the set of values of α that rules out equilibria at the upper bound is larger than the set defined by $\bar{\alpha}(\theta^B)$. Similarly, regarding the condition (11) that rules out equilibria at the lower bound, this condition implies that the specialization first-order condition is positive for any inflation rate $\pi \geq 0$.

We have the following proposition.

Proposition 1 *In an interior equilibrium, we have* $\frac{dy}{dx} < 0$, $\frac{dp}{dx} < 0$ *and* $\frac{d\pi}{dx} < 0$.

Proof: We have, using the specialization first-order condition (6),

$$\frac{p}{(1+y)} = \left(\frac{1}{\alpha} k(\alpha, \theta) - 1\right) \frac{\theta k(\alpha, \theta) + (1 - \theta)}{k(\alpha, \theta)}.$$

Hence, using that $\frac{x+y}{1+y}$ depend only on α and θ it follows that y is decreasing in x;

$$\frac{dy}{dx} = -\frac{1+y}{1-x}$$

and hence

$$\frac{dp}{dx} = 1 - \frac{1+y}{1-x} = \frac{1-x-(1+y)}{1-x} = -\frac{x+y}{1-x} < 0$$

implying, using the seigniorage first-order condition (8),

$$\frac{d\pi}{dx} = (1-\theta) \frac{\left(\theta \frac{1}{\alpha} k(\alpha,\theta) + (1-\theta)\right)}{\left(\theta k(\alpha,\theta) + (1-\theta)\right)^3} (1-\alpha) k(\alpha,\theta) \frac{dp}{dx} < 0.$$

Hence, an increase in trade through x leads to an increase in specialization and a decrease in inflation. ■

Note that the effect of an increase in inflation on quantities is affected by allowing for an endogenous specialization choice. Equilibrium quantities is affected not only by the direct effect of an increase in inflation through (4), but also via indirect effects leading to less specialization. This leads to a reversal in the effect from an increase in trade on the optimal level of seigniorage. The intuition is that an increase in x leads to a decrease in y, thus increasing the degree of specialization in the economy. This, in turn, increases the quantity in each transaction in the decentralized market, increasing the tax base. Since inflation has larger negative effects on the tax base when specialization is endogenous, this leads the policymaker to prefer a lower inflation rate.

3.1 The general case

In this section, we study a generalization of the model above, with general functional forms of u and c, given constant returns in production. Hence the cost function is $c(q,y) = qh(y)$

and the first-order conditions are

$$
\begin{aligned}
u'(q) &= \frac{\pi+p}{p-\theta^S(\pi+p)}\theta^B h(y) \\
\frac{u(q)}{q} - h(y) &= p\frac{u'(q)}{v'(q)}h_y \\
\frac{\pi}{\theta^B} &= -\frac{\theta^S\frac{u(q)}{q}+\theta^B h(y)}{(v'(q))^3}h(y)\,qu''(q)\,p.
\end{aligned}
$$

We have, letting $\theta = \theta^S$ and hence $\theta^B = 1-\theta$,

$$
u'(q) = g(y,x,\theta)\,h(y) \Rightarrow q = u'^{-1}(g(y,x,\theta)\,h(y)).
$$

The following Lemma gives conditions for when y and p is decreasing in x.

Lemma 2 *In an interior equilibrium, if* $g(y,x,\theta) > \frac{1-\theta}{\theta}$*,*

$$
\frac{\frac{u(q)}{q}-u'(q)}{\frac{u(q)}{q}-h}\frac{\theta u'(q)+(1-\theta)h(y)}{(1-\theta)h(y)} < \sigma(q)
$$

and

$$
\frac{h_y}{h}\frac{1}{\frac{u(q)}{q}-h}\left(\frac{\frac{u(q)}{q}-u'(q)}{\sigma(q)}-h\right) > \frac{h_{yy}}{h_y}
$$

then $\frac{dy}{dx} < 0$ *and* $\frac{dp}{dx} < 0$*.*

Proof: See the appendix. ■

Thus, given the conditions in the Lemma, we obtain a similar result as in the case when $u(q) = q^\alpha$ and $h(y) = 1+y$. The following proposition establishes that inflation falls when trade increases.

Proposition 2 *In an interior equilibrium, if*

$$
\frac{3\theta\sigma(q)}{1-\theta}+1 > -\frac{qu'''(q)}{u''(q)} > \sigma(q)(1-\theta)+(\theta g(y,x,\theta)+(1-\theta))
$$

then $\frac{d\pi}{dx} < 0$*.*

Proof: See the appendix. ■

Thus, if the condition in the proposition holds, then inflation decreases when trade increases, as in the case with $u(q) = q^{\alpha}$ and $h(y) = 1 + y$. Note that the interval is nonempty whenever

$$\sigma(q) > \frac{\theta(g(y,x,\theta) - 1)}{\frac{3\theta}{1-\theta} - (1-\theta)}.$$

The following proposition establishes conditions for the existence of an interior equilibrium.

Proposition 3 *If*

$$\frac{u(q)}{qu'(q)} > x + 1 \tag{12}$$

then $y > 0$ *and if*

$$\frac{\frac{u(q)}{qh(1-x)} - 1}{1 - (1-\theta)\frac{qu''(q)}{u'(q)}\left(1 - \theta + \theta\frac{u(q)}{qu'(q)}\right)} < \frac{h_y(1-x)}{h(1-x)}$$

then $y < 1$.

Proof: See the appendix. ■

Thus given some restrictions on the payoff functions u and h, an interior equilibrium exists. In particular, the condition is related to the concavity of the payoff function u, which is related to the parameter α when $u(q) = q^{\alpha}$. This parameter is important for the existence of an interior equilibrium in the case where $u(q) = q^{\alpha}$ and $h(y) = 1 + y$, see Lemma 1.

3.2 Endogenous supply

This section introduces entry of sellers into the model, since an increase in trade through demand, as it is modelled above normally leads to a supply side response. To model entry, we assume that there is a fixed cost c_s of entry for a potential seller s. The cost is different for potential sellers, and follows the distribution F. A seller enters if the cost of entry is lower than the expected benefit of entry;

$$c_s \leq \max_y \beta p\left[-qh(y) + W^{\mathcal{S}}\left(m^{\mathcal{B}}\right) - W^{\mathcal{S}}(0)\right] + \beta W^{\mathcal{S}}(0) - \beta W^{\mathcal{S}}(0)$$

The seller that is indifferent between entering or not satisfies, when $\beta \to 1$ and using $\Delta = \theta^B qh(y) + \theta^S u(q)$,

$$c_s = p\theta^S [u(q) - qh(y)]$$

In the general case, we have the following result.

Proposition 4 *If*

$$\left(\frac{s - u'(q)}{qh_y} \frac{h_y}{s-h} + f(c)(u'(q) - h(y)) \right) \frac{\theta u'(q) + (1-\theta) h(y)}{(1-\theta) h(y)} < -\frac{u''(q)}{u'(q)}$$

and

$$-\frac{h_y}{s-h} - \frac{F(c)}{p} + \frac{(1-\theta) h(y)}{\theta u'(q) + (1-\theta) h(y)} < \frac{h_{yy}}{h_y}$$

then

$$\frac{d\pi}{dx} < 0$$

3.3 The Ramsey problem

We now describe how a planner choosing the Ramsey optimal policy would choose the inflation rate. To solve for the Ramsey optimal policy, the policymaker solves

$$\max_{\{q,y,\pi\}} s^B V^B + \left(1 - s^B\right) V^S$$

where V^B and V^S are given by (3) and (5), subject to expressions (4) and (6). Using the definitions of the values, we can write the objective as $p(y)(u(q) - c(q))$. To get an idea for the solution, we can consider the planner problem, where the objective is maximized with respect to q and y without the constraints (4) and (6). The first-order condition with respect to q is then $u'(q) = c'(q)$ and the first-order condition with respect to y is $u(q) - c(q) - p(y) c_y(q) = 0$. This allocation can be achieved in the Ramsey problem. To see this, consider the first-order condition with respect to q, expression (4), which is a constraint to the planner problem. By setting

$$\frac{1 + \pi - \beta + \beta p}{\beta p - \theta^S (1 + \pi - \beta + \beta p)} \theta^B = 1,$$

the planner first-order condition with respect to q is satisfied. To do this, simply set $1 + \pi - \beta = 0$. Note also that the ratio is increasing in π, implying that the solution is unique. The planner thus chooses the Friedman rule. Then $u'(q) = h(y)$ and the specialization first-order condition (6) is

$$\frac{u(q)}{q} - h(y) = p(y) h_y$$

which coincides with the planner solution.

4 Empirical results (Preliminary)

We now test the model above using data from different currency areas in Europe during the late middle ages and early modern period, starting after the Black Death and ending in the year 1600. To do this we need data for prices, as well as proxies for specialization and demand. As a proxy for specialization, we use the number of guilds in the main city of the currency area, and as a proxy for demand, we use the population of the city. For prices, we use the database from Allen and Unger (2019) and focus on grain prices, since such prices are available for long periods of time in that database. Since agricultural prices are sensitive to shocks, e.g., bad harvest, we look at 20 year averages of inflation rates. For guilds, we use the Italian Guilds Database, Dutch Craft Guilds and Craft Guilds Flanders. We also compile data for several cities within the boundaries of what was then the holy Roman empire (Frankfurt am Main, Cologne, Vienna, Lübeck, Hamburg, Strasbourg) as well as Krakow in Poland. Note that the cities Hamburg, Lübeck and Strasbourg are used only for the instrument, discussed below. For Frankfurt am Main, we use the compilation of guild documents in Bücher and Schmidt (1914) and use the first date a guild is mentioned in this source as a foundation date. We proceed similarly for the other cities and for Cologne we use Stein (1893), Stein (1895) and Tuckermann (1911), for Würzburg Hoffman (1955) and Götz (1986), for Vienna Thiel (1911) and Gneiss (2017), for Krakow Bucher (1889), for Lübeck Mehrmann (1864), for Hamburg Rüdiger (1874) and for Strasbourg Heitz (1856).[3] Regarding population, we use the database "European urban population, 700–2000" created by Buringh (2021). When combining these data we have data in all variables available for the following cities; Firenze, Milano, Napoli, Antwerpen, Brugge, Leuven, Amsterdam, Leiden, Utrecht, Cologne, Frankfurt am Main,

[3] The latter three cities are used for the instrument.

Würzburg, Krakow and Vienna.

We then estimate the following regression

$$\log \pi_{it} = \alpha_i + \beta_G \log \hat{G}_{it} + \beta_{pop} \log \hat{x}_{it} + \beta_G \log \hat{G}_{it} * \log \hat{x}_{it} + \beta_S \log D\hat{S}_t + \varepsilon_{it}$$

where α_i is a city fixed effect, $\hat{G}_{it}$ the number of guilds in city i in period t, $\hat{x}_{it}$ the population in city i in period t, $D\hat{S}_t$ the London wheat price inflation in terms of silver and ε_{it} a disturbance term. Since the degree of specialization depends on the inflation rate, we need to find an instrument for this. To do this, we rely on the fact that members of a guild in a specific city, as part of their education, travelled to other cities to better learn their craft. Inspired by this, we use the number of guilds in the neighboring cities as an instrument, since more guilds in neighboring cities tend to reduce the costs of such travels and hence encourage guild formation in the city itself. During the latter part of this period, the European discovery of the Americas lead to a large influx of silver and gold into Europe in particular during the second half of the 16^{th} century. This lead to a substantial increase in the general price level. To take into account general changes in the price level in terms of silver, which was the main metal used in currency for this period, we use the change in the silver price of grain in London as a control variable.

As can be seen in the first column of Table 1, IV estimation yields a statistically significant negative estimate of the effect of specialization on inflation; β_G is equal to -1.698 (city-level clustered s.e. 0.609), in line with the prediction of the model. Moreover, note also that the change in the London grain price enters with the expected sign and we have β_S (-0.115, s.e. 0.062).

This analysis will be extended to take into account fiscal shocks which reasonably affect inflation, using data from the Brecke data base. This database covers all European conflicts from the year 900 to 2000 (and non-European conflicts from 1400 to 2000).

5 Conclusions

Usually, the amount and level of trade affect economic agents money holdings as well as that changes in trade can affect the degree of specialization of the economy. Both will affect the amount of goods that is traded, hence influencing money holdings. We

Table 1: Results from IV Regressions

	(1)
β_G	-1.698
	$(0.609)^{**}$
β_{pop}	-3.648
	$(1.888)^{*}$
$\beta_{G,pop}$	0.910
	$(0.380)^{**}$
β_S	0.115
	$(0.062)^{*}$
Dummies:	
City	YES

* (**) Denotes significance on the 10 (5) percent level from zero. Standard errors clustered on the firm level reported inside parenthesis.

introduce this into a model where a policymaker chooses the inflation tax from a fiscal perspective, taking the effects on trade and specialization into account. In the model, when specialization is fixed, an increase in trade leads to an increase in inflation, since the tax base increases. On the other hand, when specialization is endogenous, an increase in trade leads to more specialization, in turn affecting the incentives of the policymaker. In particular, in contrast to the case when specialization is fixed, an increase in trade leads to a fall in inflation. The reason is the following. If the specialization choice is endogenous, increases in inflation leads to a decrease in the quantity of goods traded. This, in turn, leads to leads to less specialization and a further reduction in the amount traded. Then, in equilibrium, in response to an increase in trade the policymaker reduces inflation since this leads to an increase in the quantity of goods traded, in turn increasing the tax base.

In an historical setting, currencies with low inflation were introduced in economies with a high level of trade, e.g., the Venetian ducato and the Florentine fiorino, see Spufford (1988). In other, in a trade perspective less developed areas of Europe, e.g., eastern Germany, Poland and Scandinavia, the monetary tax was higher, see Svensson and Westermark (2020). To study this empirical relationship more formally, we gather data to empirically analyze the model, with a historical perspective. Specifically, we look at several European currency areas for the period following the Black Death until the early modern period. Since we do not have direct data for specialization, we use the number of guilds as a proxy for specialization. We use existing databases for Italy and the Low countries, and collect information about the number of guilds for several cities located in the Holy Roman empire, as well as for the Polish city of Krakow. In the empirical analysis, we find a negative relationship between the degree of specialization and the inflation rate, in line with the predictions of the model.

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A Appendix

A.1 Data Appendix

For guilds, we use the following databases; Italian Guilds Database, Dutch Craft Guilds and Craft Guilds Flanders. For the population in the cities, we use the database compiled by Buringh (2021). These databases are available at:

Italian Guilds Database: https://dataverse.nl/dataset.xhtml?persistentId=hdl:10411/10110&studyL

Dutch Craft Guilds: https://dataverse.nl/dataset.xhtml?persistentId=hdl:10411/10101&studyListin

Craft Guilds Flanders: https://dataverse.nl/dataset.xhtml?persistentId=hdl:10411/10059&studyLis

European urban population, 700–2000: https://www.doi.org/10.17026/dansxzy-u62q.

A.2 Proofs

Proof of Lemma 1.

Step 1. Ruling out $y = 0$.

Note first that the marginal utility of buyers, as given by (4), is increasing in π. An increase in $u'(q)$ gives the following effect on the first-order condition (6) of sellers:

$$\left[\frac{1}{\alpha} - p\left(\frac{\theta^B (1+y)}{\left(\theta^B c'(q) + \theta^S u'(q)\right)^2}\right)\right] \theta^S du'(q) > 0.$$

Then, if the choice is not at the lower boundary ($y = 0$) at $\pi = 0$, it cannot be at the boundary for any $\pi > 0$. Consider (4) when $\pi = 0$ and $y = 0$. We have

$$u'(q) = \frac{(1-\beta) + \beta x}{\beta x \left(1 - \theta^S\right) - \theta^S (1-\beta)} \theta^B.$$

Thus, we require, for this value of $u'(q)$ that

$$\left[-1 + \frac{1}{\alpha} u'(q)\right] \left(\theta^B + \theta^S u'(q)\right) - x u'(q) > 0.$$

This can be simplified to, noting that $u'(q) = \frac{\theta^B}{1-\theta^S}$, when $\beta \to 1$,

$$\frac{1}{\alpha} \frac{\theta^B}{1-\theta^S} = \frac{1}{\alpha} > x + 1 \iff \alpha < \frac{1}{1+x}.$$

Step 2. Ruling out the upper bound $y = 1 - x$. For simplicity, we let $\beta \to 1$ in this case. From the specialization choice, we have, when the solution is interior,

$$-(2-x) + \frac{1}{\alpha} u'(q) - \frac{u'(q)}{\theta^B (2-x) + \theta^S u'(q)} < 0. \tag{A.1}$$

The slope of (4) is, when $\beta = 1$,

$$\frac{du'}{d\pi} = u' \frac{1}{(\pi + 1)\left(\left(1 - \theta^S\right) - \theta^S \pi\right)} \tag{A.2}$$

We now show that the inflation rate satisfying the seigniorage first-order condition (8) is finite. Consider a candidate solution for a solution at the boundary where $p = 1$. Since u' is positive in equilibrium, the denominator in (4) is positive. Hence, $\frac{du'}{d\pi}$ in (A.2) is positive. Also, the solution for (4) at $\pi = 0$ is $u' = 2 - x$ and when $\pi \to \frac{1-\theta^S}{\theta^S}$ from below, $u' \to \infty$. Regarding the seigniorage condition, we have, when $u' = 2 - x$, that $\pi = \left(\theta^B\right)(1-\alpha)\left(\theta^B + \frac{\theta^S}{\alpha}\right)$ and when $u' \to \infty$, that $\pi \to 0$.

To find an upper bound for marginal utility, note that inflation is bounded above by $\pi = \left(\theta^B\right)(1-\alpha)\left(\theta^B + \frac{\theta^S}{\alpha}\right)$. Then, from the quantity first-order condition, marginal utility is bounded above by

$$u'(q) = \frac{1+\pi}{\theta^B - \theta^S \pi} \theta^B (2-x) = \frac{1 + \left(\theta^B\right)(1-\alpha)\left(\theta^B + \frac{\theta^S}{\alpha}\right)}{1 - \theta^S (1-\alpha)\left(\theta^B + \frac{\theta^S}{\alpha}\right)} (2-x).$$

To ensure that there is no equilibrium at $y = 1 - x$, we then require that

$$-(2-x) + \frac{1}{\alpha} \frac{1 + \left(\theta^B\right)(1-\alpha)\left(\theta^B + \frac{1-\theta^B}{\alpha}\right)}{1 - \left(1-\theta^B\right)(1-\alpha)\left(\theta^B + \frac{1-\theta^B}{\alpha}\right)} (2-x) - \left(1 + \left(\theta^B\right)(1-\alpha)\left(\theta^B + \frac{1-\theta^B}{\alpha}\right)\right) < 0. \tag{A.3}$$

When $\theta^S \to 1$, the right-hand side is

$$\frac{2(1-\alpha)}{2\alpha - 1} < \frac{1}{(2-x)} \iff \frac{2\alpha - 1}{2(1-\alpha)} > (2-x) \iff x > 2 - \frac{2\alpha - 1}{2(1-\alpha)}.$$

Hence, we require

$$2 - \frac{2\alpha - 1}{2(1-\alpha)} < x \iff 5 - 6\alpha < x2(1-\alpha) \iff \frac{5-2x}{6-2x} < \alpha$$

The cutoff when (A.3) holds with equality is $\bar{\alpha} = 0.75$.

When $\theta^B \to 1$, the right-hand side is

$$-(2-x) + \frac{1}{\alpha}\frac{2-\alpha}{1}(2-x) - (2-\alpha) = \left(\frac{1}{\alpha}2(1-\alpha)\right)(2-x) - (2-\alpha),$$

and we require

$$2 - x < \alpha\frac{2-\alpha}{2(1-\alpha)} \iff x > 2 - \alpha\frac{2-\alpha}{2(1-\alpha)}.$$

Hence, we require

$$2 - \alpha\frac{2-\alpha}{2(1-\alpha)} < x \iff 4 - 2x - (6-2x)\alpha + \alpha^2 < 0$$

The cutoff when (A.3) holds with equality is then $\bar{\alpha} = 2 - \sqrt{2} \approx 0.586$ and the condition holds for α at least as large as this cutoff. The effect of an increase in θ^B on the cutoff, determined by expression (A.3) holding with equality, is

$$\frac{d\bar{\alpha}}{d\theta^B} = -\frac{(1-\alpha)A}{B},$$

where

$$\begin{aligned} A \;=\; & \frac{(2-x)}{\alpha}\left[\frac{\left(\theta^B + \frac{1-\theta^B}{\alpha}\right) + \left(\theta^B\right)\left(1-\frac{1}{\alpha}\right)}{1-\left(1-\theta^B\right)(1-\alpha)\left(\theta^B+\frac{1-\theta^B}{\alpha}\right)}\right. \\ & \left. + \frac{1+\theta^B(1-\alpha)\left(\theta^B+\frac{1-\theta^B}{\alpha}\right)}{\left(1-\left(1-\theta^B\right)(1-\alpha)\left(\theta^B+\frac{1-\theta^B}{\alpha}\right)\right)^2}\left(-\left(\theta^B+\frac{1-\theta^B}{\alpha}\right)+\left(1-\theta^B\right)\left(1-\frac{1}{\alpha}\right)\right)\right] \\ & -\left(\left(\theta^B+\frac{1-\theta^B}{\alpha}\right)+\theta^B\left(1-\frac{1}{\alpha}\right)\right), \end{aligned}$$

and

$$B = \frac{(2-x)}{\alpha}\left[-\frac{1}{\alpha}\frac{1+\left(\theta^B\right)(1-\alpha)\left(\theta^B+\frac{1-\theta^B}{\alpha}\right)}{1-\left(1-\theta^B\right)(1-\alpha)\left(\theta^B+\frac{1-\theta^B}{\alpha}\right)}+\frac{-\left(\theta^B\right)\left(\theta^B+\frac{1-\theta^B}{\alpha}\right)-\left(\theta^B\right)(1-\alpha)\left(\frac{1-\theta^B}{\alpha^2}\right)}{1-\left(1-\theta^B\right)(1-\alpha)\left(\theta^B+\frac{1-\theta^B}{\alpha}\right)}\right.$$
$$\left.+\frac{1+\left(\theta^B\right)(1-\alpha)\left(\theta^B+\frac{1-\theta^B}{\alpha}\right)}{\left(1-\left(1-\theta^B\right)(1-\alpha)\left(\theta^B+\frac{1-\theta^B}{\alpha}\right)\right)^2}\left(1-\theta^B\right)\left(-\left(\theta^B+\frac{1-\theta^B}{\alpha}\right)-(1-\alpha)\left(\frac{1-\theta^B}{\alpha^2}\right)\right)\right]$$
$$+\left(\theta^B\right)\left(\left(\theta^B+\frac{1-\theta^B}{\alpha}\right)+(1-\alpha)\left(\frac{1-\theta^B}{\alpha^2}\right)\right).$$

Note that B is always negative, since the second term in the square brackets dominates the last term.

Now consider A. The term multiplying $2-x$ is

$$\frac{1}{\left(1-\left(1-\theta^B\right)(1-\alpha)\left(\theta^B+\frac{1-\theta^B}{\alpha}\right)\right)^2}\frac{1}{\alpha}\left(1-\frac{1}{\alpha}-(1-\alpha)\left(\theta^B+\frac{1-\theta^B}{\alpha}\right)^2\right).$$

The terms in front of $d\theta^B$ is then

$$\frac{2-x}{\left(1-\left(1-\theta^B\right)(1-\alpha)\left(\theta^B+\frac{1-\theta^B}{\alpha}\right)\right)^2}\frac{1}{\alpha}\left(1-\frac{1}{\alpha}-(1-\alpha)\left(\theta^B+\frac{1-\theta^B}{\alpha}\right)^2\right)$$
$$-\left(\left(\theta^B+\frac{1-\theta^B}{\alpha}\right)+\left(\theta^B\right)\left(1-\frac{1}{\alpha}\right)\right).$$

Note that the two ratios in the front of the expression is larger than one and that the term multiplying the ratios is negative. Hence, the expression is smaller than

$$\left(1-\frac{1}{\alpha}\right)(1-\theta)-(1-\alpha)\left(\theta+\frac{1-\theta}{\alpha}\right)^2-\left(\theta+\frac{1-\theta}{\alpha}\right).$$

The effect on the expression above of an increase in θ is

$$-2\left(1+(1-\alpha)\left(\theta+\frac{1-\theta}{\alpha}\right)\right)\left(1-\frac{1}{\alpha}\right)>0.$$

Hence, the expression is largest when $\theta=1$ when it is $-(1-\alpha)-1<0$. Hence, the term multiplying $d\theta^B$ is negative, implying that

$$\frac{d\bar{\alpha}}{d\theta^B} < 0$$

Thus, there is a cutoff $\bar{\alpha}\left(\theta^B\right)$ which is decreasing in θ^B with $\bar{\alpha}(0) = 0.75$ and $\bar{\alpha}(1) = 2 - \sqrt{2}$.■

Proof of Lemma 2.

Preliminaries

Using the definitions of s and t in the quantity first-order condition (4) gives

$$u'(q) = \frac{(1-\theta)\frac{\theta s(q)+(1-\theta)h(y)}{(\theta g(y,x,\theta)h(y)+(1-\theta)h(y))^3}h(y)\,t(q)+1}{1-\theta\left(\frac{\theta s(q)+(1-\theta)h(y)}{(\theta g(y,x,\theta)h(y)+(1-\theta)h(y))^3}h(y)\,t(q)\right)}h(y)$$

Thus

$$g(y,x,\theta) = \frac{\theta^B\frac{\theta^S s(q)+\theta^B h(y)}{\left(\theta^S g(y,x,\theta)h(y)+\theta^B h(y)\right)^3}h(y)\,t(q)+1}{1-\theta^S\left(\frac{\theta^S s(q)+\theta^B h(y)}{\left(\theta^S g(y,x,\theta)h(y)+\theta^B h(y)\right)^3}h(y)\,t(q)\right)}$$

Rearranging

$$\begin{aligned} & g(y,x,\theta)\left(1-\theta\left(\frac{\theta s(q)+(1-\theta)h(y)}{\left(\theta^S g(y,x,\theta)h(y)+\theta^B h(y)\right)^3}h(y)\,t(q)\right)\right) \\ = \; & (1-\theta)\frac{\theta s(q)+(1-\theta)h(y)}{(\theta g(y,x,\theta)h(y)+(1-\theta)h(y))^3}h(y)\,t+1 \end{aligned}$$

Letting

$$\begin{aligned} f(y,x,\theta) &= \frac{\theta s(q)+(1-\theta)h(y)}{(\theta g(y,x,\theta)h(y)+(1-\theta)h(y))^3}h(y)\,t(q) \\ &= \frac{\theta\frac{s(q)}{h(y)}+(1-\theta)}{(\theta g(y,x,\theta)+(1-\theta))^3}\frac{t(q)}{h(y)} \end{aligned}$$

which is positive and hence the expression above is

$$g(y,x,\theta)\left(1-\theta^S f(y,x,\theta)\right) = \theta^B f(y,x,\theta)+1. \tag{A.4}$$

Note that we must have $f(y,x,\theta) < \frac{1}{\theta}$.

Step 1. Computing g_y

Differentiating (A.4)

$$g_y(y,x,\theta)(1-\theta f(y,x,\theta)) - g(y,x,\theta)\theta f_y(y,x,\theta) = (1-\theta) f_y(y,x,\theta)$$

and hence

$$g_y(y,x,\theta) = \frac{((1-\theta)+g(y,x,\theta)\theta)}{(1-\theta f(y,x,\theta))} f_y(y,x,\theta)$$

where

$$\begin{aligned} f_y(y,x,\theta) \quad = \quad & \frac{\theta \frac{\partial \frac{s\left(u'^{-1}(g(y,x,\theta)h(y))\right)}{h(y)}}{\partial y}}{(\theta g(y,x,\theta)+(1-\theta))^3} \frac{t\left(u'^{-1}(g(y,x,\theta)h(y))\right)}{h(y)} - 3\frac{\theta f(y,x,\theta)}{\theta g(y,x,\theta)+(1-\theta)} g_y(y,x,\theta) \\ & + \frac{\theta \frac{s\left(u'^{-1}(g(y,x,\theta)h(y))\right)}{h(y)} + (1-\theta)}{(\theta g(y,x,\theta)+(1-\theta))^3} \frac{\partial \frac{t\left(u'^{-1}(g(y,x,\theta)h(y))\right)}{h(y)}}{\partial y} \end{aligned}$$

The derivatives of s and t are

$$\begin{aligned} \frac{\partial \frac{s\left(u'^{-1}(g(y,x,\theta)h(y))\right)}{h(y)}}{\partial y} &= \frac{s'}{h(y)} \frac{\partial u'^{-1}(g(y,x,\theta)h(y))}{\partial y} - \frac{s}{(h(y))^2} h_y \\ &= \frac{s'}{h(y)} \frac{\partial u'^{-1}(v)}{\partial v} (g_y(y,x,\theta)h(y) + g(y,x,\theta)h_y) - \frac{s}{(h(y))^2} h_y \end{aligned}$$

and

$$\frac{\partial \frac{t\left(u'^{-1}(g(y,x,\theta)h(y))\right)}{h(y)}}{\partial y} = \frac{t'}{h(y)} \frac{\partial u'^{-1}(v)}{\partial v} (g_y(y,x,\theta)h(y) + g(y,x,\theta)h_y) - \frac{t}{(h(y))^2} h_y$$

Then

$$\begin{aligned} & K g_y(y,x,\theta) \\ = \quad & \frac{\theta \frac{t}{h(y)}}{(\theta g(y,x,\theta)+(1-\theta))^2} \left(\frac{s'}{h(y)} \frac{\partial u'^{-1}(v)}{\partial v} g(y,x,\theta) - \frac{s}{(h(y))^2} \right) h_y \qquad \text{(A.5)} \\ & + \frac{\theta \frac{s}{h(y)} + (1-\theta)}{(\theta g(y,x,\theta)+(1-\theta))^2} \left(\frac{t'}{h(y)} \frac{\partial u'^{-1}(v)}{\partial v} g(y,x,\theta) - \frac{t}{(h(y))^2} \right) h_y \end{aligned}$$

where

$$K = (1-\theta f(y,x,\theta)) + 3\theta f(y,x,\theta) - \frac{\partial u'^{-1}(v)}{\partial v} \frac{\theta \frac{t}{h(y)} s' + \left(\theta \frac{s}{h(y)} + (1-\theta)\right) t'}{(\theta g(y,x,\theta)+(1-\theta))^2}$$

with

$$
\begin{aligned}
s' &= -\frac{s}{q} + \frac{1}{q} u'(q) \\
t' &= \frac{t}{q} - q u'''(q)
\end{aligned}
$$

implying

$$
\begin{aligned}
K g_y(y,x,\theta) &= \frac{\theta \frac{t}{h(y)}}{(\theta g(y,x,\theta) + (1-\theta))^2} \frac{1}{(h(y))^2} \left(s \left(\frac{1}{\sigma(q)} - 1 \right) - u'(q) \frac{1}{\sigma(q)} \right) h_y \\
&+ \frac{\theta \frac{s}{h(y)} + (1-\theta)}{(\theta g(y,x,\theta) + (1-\theta))^2} \frac{1}{(h(y))^2} \left(\left(-t \left(\frac{1}{\sigma(q)} + 1 \right) - \frac{q u'''(q)}{u''(q)} u'(q) \right) \right) h_y
\end{aligned}
$$

Combining gives

$$
\min \left(1 - \frac{q u'(q)}{u(q)}, -\frac{q u'''(q)}{u''(q)} - 1 \right) > \sigma(q)
$$

For $g_y > 0$ when $K < 0$ we require

$$
\begin{aligned}
s(1 - \sigma(q)) &< u'(q) \iff \frac{q u'(q)}{u(q)} (1 - \sigma(q)) < 1 \\
-\frac{q u'''(q)}{u''(q)} &< 1 + \sigma(q)
\end{aligned}
$$

Step 2. Condition for positive K

We have

$$
\begin{aligned}
K &= (1 + 2\theta f(y,x,\theta)) \\
&+ \frac{(\theta g(y,x,\theta) + (1-\theta)) + \left(\theta \frac{s}{h(y)} + (1-\theta) \right) \frac{q u'''(q)}{u''(q)}}{(\theta g(y,x,\theta) + (1-\theta))^2}
\end{aligned}
$$

where

$$
f(y,x,\theta) = \frac{\theta \frac{s}{h(y)} + (1-\theta)}{(\theta g(y,x,\theta) + (1-\theta))^3} g(y,x,\theta) \sigma(q).
$$

and hence

$$
K = 1 + 2\theta \frac{\theta \frac{s}{h(y)} + (1-\theta)}{(\theta g(y,x,\theta) + (1-\theta))^3} g(y,x,\theta) \sigma(q) + \frac{(\theta g(y,x,\theta) + (1-\theta)) + \left(\theta \frac{s}{h(y)} + (1-\theta) \right) \frac{q u'''(q)}{u''(q)}}{(\theta g(y,x,\theta) + (1-\theta))^2} \tag{A.6}
$$

Thus, we require

$$\left(1+2\theta\frac{\theta\frac{s}{h(y)}+(1-\theta)}{\left(\theta g\left(y,x,\theta\right)+(1-\theta)\right)^{3}}g\left(y,x,\theta\right)\sigma\left(q\right)\right)\left(\theta g\left(y,x,\theta\right)+(1-\theta)\right)^{2}$$
$$+\left(\theta g\left(y,x,\theta\right)+(1-\theta)\right)>-\left(\theta\frac{s}{h\left(y\right)}+(1-\theta)\right)\frac{qu'''\left(q\right)}{u''\left(q\right)}$$

Using the first-order condition with respect to specialization, restricting $\frac{h_y}{h} \leq 1$ which holds for $h = 1 + y$,

$$\theta\frac{s}{h\left(y\right)}+(1-\theta)=1+p\frac{\theta g}{\theta g+(1-\theta)}\frac{h_{y}}{h}<2$$

gives, using that $g\left(y,x,\theta\right) > 1$,

$$1+2\theta\frac{g\left(y,x,\theta\right)}{\theta g\left(y,x,\theta\right)+(1-\theta)}\sigma\left(q\right)>1+2\theta\sigma\left(q\right)>-\frac{qu'''\left(q\right)}{u''\left(q\right)} \tag{A.7}$$

When $u = q^{\alpha}$ we have $\sigma\left(q\right) = 1 - \alpha$ and $-\frac{qu'''(q)}{u''(q)} = 2 - \alpha$ and hence the condition (A.7) is

$$2\theta\frac{g\left(y,x,\theta\right)}{\theta g\left(y,x,\theta\right)+(1-\theta)}>1\iff\theta g\left(y,x,\theta\right)>(1-\theta)\Rightarrow\theta\geq\frac{1}{2}$$

since

$$\theta>\frac{1}{g\left(y,x,\theta\right)+1}<\frac{1}{2}$$

Step 3. Effect on the probability of trade.

The specialization first-order condition is

$$p=\frac{s\left(u'^{-1}\left(g\left(y,x,\theta\right)h\left(y\right)\right)\right)-h\left(y\right)}{h_{y}}\frac{\theta g\left(y,x,\theta\right)+(1-\theta)}{g\left(y,x,\theta\right)}$$

Differentiating gives

$$\begin{aligned}dx+dy &= \frac{s'}{h_y}\frac{\partial u'^{-1}\left(v\right)}{\partial v}\left(h\left(y\right)g_{y}+g\left(y,x,\theta\right)h_{y}\right)\frac{\theta g\left(y,x,\theta\right)+(1-\theta)}{g\left(y,x,\theta\right)}dy\\ &\quad -\frac{h_y}{h_y}\frac{\theta g\left(y,x,\theta\right)+(1-\theta)}{g\left(y,x,\theta\right)}dy-\frac{s-h\left(y\right)}{\left(h_{y}\right)^{2}}\frac{\theta g\left(y,x,\theta\right)+(1-\theta)}{g\left(y,x,\theta\right)}h_{yy}dy\\ &\quad +\frac{s-h\left(y\right)}{h_{y}}\left(-\frac{(1-\theta)}{\left(g\left(y,x,\theta\right)\right)^{2}}\right)g_{y}dy\end{aligned}$$

Using the first-order condition, we can write

$$
\begin{aligned}
dx+dy &= \frac{s'}{h_y}\frac{1}{u''(q)}\left(h(y)\,g_y+g(y,x,\theta)\,h_y\right)\frac{\theta g(y,x,\theta)+(1-\theta)}{g(y,x,\theta)}dy \\
&\quad -\frac{\theta g(y,x,\theta)+(1-\theta)}{g(y,x,\theta)}dy-p\frac{h_{yy}}{h_y}dy-\frac{p}{\theta g(y,x,\theta)+(1-\theta)}\frac{(1-\theta)}{g(y,x,\theta)}g_y dy
\end{aligned}
$$

Using that $s'=-\frac{s}{q}+\frac{u'(q)}{q}$ gives, defining $\sigma(q)=-\frac{qu''(q)}{u'(q)}$

$$
\begin{aligned}
dx+dy &= \frac{1-\sigma(q)-\frac{g(y,x,\theta)h(y)}{h_y}\frac{h_y}{h(y)}}{\sigma(q)}\frac{\theta^S g(y,x,\theta)+\theta^B}{g(y,x,\theta)}dy \\
&\quad +p\left(\frac{h_y}{h(y)}\frac{1}{\sigma(q)}-\frac{h_{yy}}{h_y}\right)dy \\
&\quad +\frac{\frac{s-g(y,x,\theta)h(y)}{h_y}\frac{1}{\sigma(q)}\frac{\theta^S g(y,x,\theta)+\theta^B}{g(y,x,\theta)}-\frac{p\theta^B}{\theta^S g(y,x,\theta)+\theta^B}}{g(y,x,\theta)}g_y dy
\end{aligned}
$$

Combining the first and last rows gives, using $s=h+p\frac{g}{\theta^S g+\theta^B}h_y$,

$$
\begin{aligned}
\frac{dp}{dx} &= -\left[\left(1+\frac{h}{h_y}\frac{g_y}{g}\right)\frac{(g(y,x,\theta)-1)}{\sigma(q)}+1\right]\frac{\theta g(y,x,\theta)+(1-\theta)}{g(y,x,\theta)}\frac{dy}{dx} \\
&\quad +p\left(\frac{h_y}{h}\frac{1}{\sigma(q)}-\frac{h_{yy}}{h_y}\right)\frac{dy}{dx}+p\left(\frac{1}{\sigma(q)}-\frac{(1-\theta)}{\theta g(y,x,\theta)+(1-\theta)}\right)\frac{g_y}{g}\frac{dy}{dx}
\end{aligned}
$$

Using the specialization first-order condition gives

$$
\begin{aligned}
\frac{dp}{dx} &= -\left[\frac{1}{\sigma(q)}\left(\frac{h_y}{h}+\frac{g_y}{g}\right)\frac{u'(q)-h}{s-h}p+\frac{\theta g(y,x,\theta)+(1-\theta)}{g(y,x,\theta)}\right]\frac{dy}{dx} \\
&\quad +p\left(\frac{h_y}{h}\frac{1}{\sigma(q)}-\frac{h_{yy}}{h_y}\right)\frac{dy}{dx}+p\left(\frac{1}{\sigma(q)}-\frac{(1-\theta)}{\theta g(y,x,\theta)+(1-\theta)}\right)\frac{g_y}{g}\frac{dy}{dx}
\end{aligned}
$$

We can write the above expression as, using the specialization first-order condition,

$$
\begin{aligned}
\frac{dp}{dx} &= -p\left[\frac{1}{\sigma(q)}\left(\frac{h_y}{h}+\frac{g_y}{g}\right)\frac{u'(q)-h}{s-h}+\frac{h_y}{h}\frac{h}{s-h}\right]\frac{dy}{dx} \\
&\quad +p\left(\frac{h_y}{h}\frac{1}{\sigma(q)}-\frac{h_{yy}}{h_y}\right)\frac{dy}{dx}+p\left(\frac{1}{\sigma(q)}-\frac{(1-\theta)}{\theta g(y,x,\theta)+(1-\theta)^B}\right)\frac{g_y}{g}\frac{dy}{dx}
\end{aligned}
$$

or

$$\begin{aligned}\frac{dp}{dx} &= p\left(\frac{h_y}{h}\left(\frac{1}{\sigma(q)}-\frac{1}{\sigma(q)}\frac{u'(q)-h}{s-h}-\frac{h}{s-h}\right)-\frac{h_{yy}}{h_y}\right)\frac{dy}{dx}\\ &+p\left(\frac{1}{\sigma(q)}\left(1-\frac{u'(q)-h}{s-h}\right)-\frac{(1-\theta)}{\theta g(y,x,\theta)+(1-\theta)}\right)\frac{g_y}{g}\frac{dy}{dx}\end{aligned}$$

We require

$$\frac{h_y}{h}\frac{1}{s-h}\left(\frac{s-u'(q)}{\sigma(q)}-h\right)>\frac{h_{yy}}{h_y} \tag{A.8}$$

and, for the coefficient in front of $\frac{g_y}{g}$, when $g_y<0$

$$\frac{1}{\sigma(q)}\frac{s-u'(q)}{s-h}=\frac{1}{\sigma(q)}\frac{\frac{u(q)}{qu'(q)}-1}{\frac{u(q)}{qu'(q)}-\frac{h}{u'(q)}}<\frac{\theta^B}{\theta^S g\left(y,\theta^S,\theta^B\right)+\theta^B} \tag{A.9}$$

and, when $g_y>0$

$$\frac{1}{\sigma(q)}\frac{s-u'(q)}{s-h}>\frac{(1-\theta)}{\theta g(y,x,\theta)+(1-\theta)}$$

Thus, if expression (A.8) and (A.9) holds then $dx/dy<0$. Condition (A.9) can be rearranged as

$$\frac{u(q)}{qu'(q)}\left(1-\sigma(q)\frac{(1-\theta)}{\theta g\left(y,\theta^S,\theta^B\right)+(1-\theta)}\right)<1-\sigma(q)\frac{(1-\theta)}{\theta g\left(y,\theta^S,\theta^B\right)+(1-\theta)}\frac{h}{u'(q)}$$

Step 4. Checking dy/dx

Noting that

$$\begin{aligned}1+\frac{dy}{dx} &= -\left(\frac{s-u'(q)}{qh_y}\frac{dq}{dy}+1\right)\frac{h_y}{s-h}p\frac{dy}{dx}\\ &-p\frac{h_{yy}}{h_y}\frac{dy}{dx}-p\frac{(1-\theta)h(y)}{\theta u'(q)+(1-\theta)h(y)}\frac{u''(q)}{u'(q)}\frac{dq}{dy}\frac{dy}{dx}+p\frac{(1-\theta)h(y)}{\theta u'(q)+(1-\theta)h(y)}\frac{dy}{dx}\end{aligned}$$

and hence

$$\frac{dy}{dx}=\frac{1}{-\left(\frac{s-u'(q)}{qh_y}\frac{dq}{dy}+1\right)\frac{h_y}{s-h}p-1-p\frac{h_{yy}}{h_y}-p\frac{(1-\theta)h(y)}{\theta u'(q)+(1-\theta)h(y)}\frac{u''(q)}{u'(q)}\frac{dq}{dy}+p\frac{(1-\theta)h(y)}{\theta u'(q)+(1-\theta)h(y)}}$$

Thus, for $dy/dx<0$, we require

$$-\frac{h_y}{s-h}p-1-p\frac{h_{yy}}{h_y}-p\left(\frac{(1-\theta)h(y)}{\theta u'(q)+(1-\theta)h(y)}\frac{qu''(q)}{u'(q)}+\frac{s-u'(q)}{s-h}\right)\frac{1}{q}\frac{dq}{dy}+p\frac{(1-\theta)h(y)}{\theta u'(q)+(1-\theta)h(y)}<0$$

We require

$$\frac{(1-\theta)\,h(y)}{\theta u'(q)+(1-\theta)\,h(y)}\frac{qu''(q)}{u'(q)}+\frac{s-u'(q)}{s-h}<0 \iff \frac{s-u'(q)}{s-h}\frac{\theta u'(q)+(1-\theta)\,h(y)}{(1-\theta)\,h(y)}<\sigma(q)$$

An alternative condition is

$$-\frac{yh_{yy}}{h_y}-\frac{s-u'(q)}{s-h}\frac{y}{q}\frac{dq}{dy}<0 \iff \frac{yh_{yy}}{h_y}>-\frac{s-u'(q)}{s-h}\frac{y}{q}\frac{dq}{dy}$$

Rearranging, using $x+y<1 \iff y<1-x$,

$$\begin{aligned}
&\left(\frac{s-h}{h}\right)\frac{1}{s-h}p-1+p\left(-\frac{(1-\theta)\,h(y)}{\theta u'(q)+(1-\theta)\,h(y)}\right)\frac{1}{h}+p\frac{(1-\theta)\,h(y)}{\theta u'(q)+(1-\theta)\,h(y)}\\
=&\ \frac{1}{h}p-1-p\frac{(1-\theta)\,h(y)}{\theta u'(q)+(1-\theta)\,h(y)}\left(\frac{1}{h}-1\right)\\
=&\ \frac{1}{1+y}\left(x-1+p\frac{(1-\theta)\,h(y)}{\theta u'(q)+(1-\theta)\,h(y)}y\right)\\
<&\ \frac{1}{1+y}\left(1-p\frac{(1-\theta)\,h(y)}{\theta u'(q)+(1-\theta)\,h(y)}\right)(x-1)<0
\end{aligned}$$

which holds.

Proof of Proposition 2.

We first define

$$\begin{aligned}
s(q) &\equiv \frac{u(q)}{q}=s\left(u'^{-1}\left(g(y,x,\theta)\,h(y)\right)\right)\\
t(q) &\equiv -qu''(q)=t\left(u'^{-1}\left(g(y,x,\theta)\,h(y)\right)\right).
\end{aligned}$$

Then the first-order condition with respect to y and π are

$$\begin{aligned}
s(q)-h(y) &= p\frac{u'(q)}{v'(q)}h_y\\
\pi &= (1-\theta)\frac{\theta s(q)+(1-\theta)\,h(y)}{\left(\theta g(y,x,\theta)\,h(y)+(1-\theta)\,h(y)\right)^3}h(y)\,t(q)\,p.
\end{aligned}$$

We have

$$
\begin{aligned}
\pi &= (1-\theta)\frac{\theta s\left(u'^{-1}\left(g\left(y,x,\theta\right)h\left(y\right)\right)\right)+(1-\theta)h\left(y\right)}{\left(\theta g\left(y,x,\theta\right)h\left(y\right)+(1-\theta)h\left(y\right)\right)^{3}}h\left(y\right)t\left(u'^{-1}\left(g\left(y,x,\theta\right)h\left(y\right)\right)\right)p \\
&= (1-\theta)\frac{\theta\frac{s}{h(y)}+(1-\theta)}{\left(\theta g\left(y,x,\theta\right)+(1-\theta)\right)^{3}}\frac{t}{h\left(y\right)}p \\
&= (1-\theta)\frac{\theta s+(1-\theta)h\left(y\right)}{\left(\theta g\left(y,x,\theta\right)h\left(y\right)+(1-\theta)h\left(y\right)\right)^{2}}t\frac{s-h\left(y\right)}{g\left(y,x,\theta\right)h_{y}}
\end{aligned}
$$

In general

$$
\begin{aligned}
\frac{d\pi}{dx} &= (1-\theta)\frac{\theta\frac{s}{h(y)}+(1-\theta)}{\left(\theta g\left(y,x,\theta\right)+(1-\theta)\right)^{3}}\frac{t}{h\left(y\right)}\frac{dp}{dx}+\frac{(1-\theta)\theta}{\left(\theta g\left(y,x,\theta\right)+(1-\theta)\right)^{3}}\frac{t}{h\left(y\right)}p\frac{d\left(\frac{s}{h(y)}\right)}{dx} \\
&\quad +(1-\theta)\frac{\theta\frac{s}{h(y)}+(1-\theta)}{\left(\theta g\left(y,x,\theta\right)+(1-\theta)\right)^{3}}p\frac{d\left(\frac{t}{h(y)}\right)}{dx}-\frac{3(1-\theta)\left(\theta\frac{s}{h(y)}+(1-\theta)\right)}{\left(\theta g\left(y,x,\theta\right)+(1-\theta)\right)^{4}}\frac{t}{h\left(y\right)}p\theta g_{y}\frac{dy}{dx}
\end{aligned}
$$

Here, using that we in general have $\frac{\partial u'^{-1}(v)}{\partial v}=\frac{1}{u''(q)}$, $s'=-\frac{s}{q}+\frac{u'(q)}{q}$ and $t'=\frac{t}{q}-qu'''(q)$ gives

$$
\begin{aligned}
\frac{d\left(\frac{s}{h(y)}\right)}{dx} &= \left(\frac{-\frac{s}{q}+\frac{u'(q)}{q}}{h\left(y\right)}\frac{1}{u''\left(q\right)}\left(h\left(y\right)g_{y}+g\left(y,x,\theta\right)h_{y}\right)-\frac{s}{\left(h\left(y\right)\right)^{2}}h_{y}\right)\frac{dy}{dx} \\
\frac{d\left(\frac{t}{h(y)}\right)}{dx} &= \left(\frac{\frac{t}{q}-qu'''\left(q\right)}{h\left(y\right)}\frac{1}{u''\left(q\right)}\left(h\left(y\right)g_{y}+g\left(y,x,\theta\right)h_{y}\right)-\frac{t}{\left(h\left(y\right)\right)^{2}}h_{y}\right)\frac{dy}{dx}
\end{aligned}
$$

Rearranging

$$
\begin{aligned}
\frac{d\left(\frac{s}{h(y)}\right)}{dx} &= \left(\frac{s-u'\left(q\right)}{h\left(y\right)}\frac{1}{\sigma\left(q\right)}\frac{g_{y}}{g}+\left(\frac{s-u'\left(q\right)}{h\left(y\right)}\frac{1}{\sigma\left(q\right)}-\frac{s}{h\left(y\right)}\right)\frac{h_{y}}{h}\right)\frac{dy}{dx} \\
\frac{d\left(\frac{t}{h(y)}\right)}{dx} &= \left(-\frac{t-q^{2}u'''\left(q\right)}{h\left(y\right)}\frac{1}{\sigma\left(q\right)}\frac{g_{y}}{g}-\left(\frac{t-q^{2}u'''\left(q\right)}{h\left(y\right)}\frac{1}{\sigma\left(q\right)}+\frac{t}{h\left(y\right)}\right)\frac{h_{y}}{h}\right)\frac{dy}{dx}
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{d\pi}{dx} &= (1-\theta)\frac{\theta\frac{s}{h(y)}+(1-\theta)}{(\theta g(y,x,\theta)+(1-\theta))^3}\frac{t}{h(y)}\frac{dp}{dx} \\
&+(1-\theta)\frac{\theta\left(\frac{g_y}{g}+\frac{h_y}{h}\right)\frac{1}{\sigma(q)}}{(\theta g(y,x,\theta)+(1-\theta))^3}\frac{p}{h(y)^2}\left(sq^2u'''(q)-tu'(q)\right)\frac{dy}{dx} \\
&+(1-\theta)\frac{(1-\theta)\left(-\frac{t-q^2u'''(q)}{h(y)}\frac{1}{\sigma(q)}\frac{g_y}{g}-\left(\frac{t-q^2u'''(q)}{h(y)}\frac{1}{\sigma(q)}+\frac{t}{h(y)}\right)\frac{h_y}{h}\right)}{(\theta g(y,x,\theta)+(1-\theta))^3}p\frac{dy}{dx} \\
&-3(1-\theta)\frac{\theta\frac{s}{h(y)}+(1-\theta)}{(\theta g(y,x,\theta)+(1-\theta))^4}\frac{t}{h(y)}p\theta g_y\frac{dy}{dx}
\end{aligned}
$$

Alternatively

$$
\begin{aligned}
\frac{d\pi}{dx} &= \frac{\frac{1}{(h(y))^2}\left(sq^2u'''(q)-tu'(q)\right)\frac{1}{\sigma(q)}\left(\frac{g_y}{g}+\frac{h_y}{h}\right)}{(\theta g(y,x,\theta)+(1-\theta))^3}(1-\theta)\theta p\frac{dy}{dx} \\
&-3\theta^B\frac{\theta\frac{s}{h(y)}+(1-\theta)}{(\theta g(y,x,\theta)+(1-\theta))^4}\frac{t}{h(y)}p\theta g_y\frac{dy}{dx} \\
&+\frac{-\frac{t-q^2u'''(q)}{h(y)}\frac{1}{\sigma(q)}\left(\frac{g_y}{g}+\frac{h_y}{h}\right)-\frac{t}{h(y)}\frac{h_y}{h}}{(\theta g(y,x,\theta)+(1-\theta))^3}(1-\theta)^2p\frac{dy}{dx} \\
&+(1-\theta)\frac{\theta\frac{s}{h(y)}+(1-\theta)}{(\theta g(y,x,\theta)+(1-\theta))^3}\frac{t}{h(y)}\frac{dp}{dx}
\end{aligned}
$$

The coefficients in front of $\frac{h_y}{h}$ is

$$
\begin{aligned}
&(1-\theta)\frac{\theta\frac{1}{\sigma(q)}\frac{u'(q)}{h}\left(\frac{s}{u'(q)}q^2u'''(q)-t\right)-(1-\theta)\left((t-q^2u'''(q))\frac{1}{\sigma(q)}+t\right)}{(\theta g(y,x,\theta)+(1-\theta))^3}\frac{p}{h} \\
=\ &(1-\theta)\frac{\frac{1}{\sigma(q)}\theta\frac{u'(q)}{h}\left(\frac{s}{u'(q)}q^2u'''(q)-t\right)+(1-\theta)\left((q^2u'''(q)-t)\frac{1}{\sigma(q)}-t\right)}{(\theta g(y,x,\theta)+(1-\theta))^3}\frac{p}{h}
\end{aligned}
$$

Hence, we require

$$
\frac{1}{\sigma(q)}\theta g\left(\frac{s}{u'(q)}q^2u'''(q)-t\right)+(1-\theta)\left(\left(q^2u'''(q)-t\right)\frac{1}{\sigma(q)}-t\right)>0
$$

Since $s > u'(q)$ the above expression is larger than

$$\begin{aligned} & \frac{1}{\sigma(q)}(\theta g + (1-\theta))\left(q^2 u'''(q) - t\right) - (1-\theta)t \\ = & -qu''(q)\left[\frac{1}{\sigma(q)}(\theta g + (1-\theta))\left(-\frac{qu'''(q)}{u''(q)} - 1\right) - (1-\theta)\right] \end{aligned}$$

Thus

$$\frac{1}{\sigma(q)}(\theta g + (1-\theta))\left(-\frac{qu'''(q)}{u''(q)} - 1\right) > (1-\theta)$$

or

$$-\frac{qu'''(q)}{u''(q)} > \sigma(q)(1-\theta) + (\theta g + (1-\theta))$$

The terms involving g_y are

$$\frac{\left(\theta g \frac{\frac{s}{u'(q)}q^2 u'''(q) - t}{h} + (1-\theta)\frac{q^2 u'''(q) - t}{h}\right)\frac{1}{\sigma(q)} - 3\left(\theta\frac{s}{h} + (1-\theta)\right)\frac{t}{h(y)}\frac{g}{\theta g\left(y,\theta^S,\theta^B\right) + (1-\theta)}\theta}{\left(\theta g\left(y,\theta^S,\theta^B\right) + (1-\theta)\right)^3}(1-\theta)p\frac{g_y}{g}\frac{dy}{dx}$$

We require, when $g_y > 0$,

$$\begin{aligned} & \left(\theta g \frac{\frac{s}{u'(q)}q^2 u'''(q) - t}{h} + (1-\theta)\frac{q^2 u'''(q) - t}{h}\right)\frac{1}{\sigma(q)} \\ > \; & 3\left(\theta\frac{s}{h} + (1-\theta)\right)\frac{t}{h(y)}\frac{g}{\theta g\left(y,\theta^S,\theta^B\right) + (1-\theta)}\theta \end{aligned}$$

Using the definition of t

$$\begin{aligned} & \left(\theta g\left(-\frac{s}{u'(q)}\frac{qu'''(q)}{u''(q)} - 1\right) + (1-\theta)\left(-\frac{qu'''(q)}{u''(q)} - 1\right)\right)\frac{1}{\sigma(q)} \\ > \; & 3\left(\theta\frac{s}{h} + (1-\theta)\right)\frac{g}{\theta g\left(y,\theta^S,\theta^B\right) + (1-\theta)}\theta \end{aligned}$$

Note that the left-hand side is larger than $(\theta g + (1-\theta))\left(-\frac{qu'''(q)}{u''(q)} - 1\right)\frac{1}{\sigma(q)}$. Hence, if

$$\begin{aligned} (\theta g + (1-\theta))\left(-\frac{qu'''(q)}{u''(q)} - 1\right)\frac{1}{\sigma(q)} & > 3\left(\theta\frac{s}{h} + (1-\theta)\right)\frac{g}{\theta g + (1-\theta)}\theta \\ & > 3(\theta g + (1-\theta))\frac{g}{\theta g + (1-\theta)}\theta \end{aligned}$$

and the condition is satisfied whenever

$$-\frac{qu'''(q)}{u''(q)}-1>3\theta\sigma(q)$$

If

$$sq^2u'''(q)>tu'(q)=-qu''(q)\,u'(q)\iff-\frac{qu'''(q)}{u''(q)}>\frac{u'(q)}{s}=\frac{qu'(q)}{u(q)}$$

and

$$\frac{dq}{dy}=-\frac{1}{\sigma(q)}\left(\frac{g_y}{g}+\frac{h_y}{h}\right)<0$$

we can restrict attention to the terms involving $\frac{h_y}{h}$ in the third row is

$$\begin{aligned}&\frac{-\frac{t-q^2u'''(q)}{h(y)}\frac{1}{\sigma(q)}-\frac{t}{h(y)}}{\left(\theta g\left(y,\theta^S,\theta^B\right)+(1-\theta)\right)^3}\frac{h_y}{h}(1-\theta)^2p\frac{dy}{dx}\\=\;&-qu''(q)\frac{\left(-1-\frac{qu'''(q)}{u''(q)}\right)\frac{1}{\sigma(q)}-1}{\left(\theta g\left(y,\theta^S,\theta^B\right)+(1-\theta)\right)^3}\frac{h_y}{h}\frac{1}{h(y)}(1-\theta)^2p\frac{dy}{dx}\end{aligned}$$

Hence, we require

$$\left(-1-\frac{qu'''(q)}{u''(q)}\right)\frac{1}{\sigma(q)}-1>0\iff-\frac{qu'''(q)}{u''(q)}>\sigma(q)+1$$

The term involving g_y on the third row and the second row is

$$\begin{aligned}&-3(1-\theta)\frac{\theta\frac{s}{h(y)}+(1-\theta)}{(\theta g(y,x,\theta)+(1-\theta))^3}\frac{g}{(\theta g(y,x,\theta)+(1-\theta))\,h(y)}\frac{t}{h(y)}p\theta\frac{g_y}{g}\frac{dy}{dx}\\&+\frac{-\frac{t-q^2u'''(q)}{h(y)}\frac{1}{\sigma(q)}\frac{g_y}{g}}{(\theta g(y,x,\theta)+(1-\theta))^3}(1-\theta)^2p\frac{dy}{dx}\\=\;&\left(-\left(t-q^2u'''(q)\right)\frac{(1-\theta)}{\sigma(q)}-3\left(\theta\frac{s}{h(y)}+(1-\theta)\right)\theta\frac{g}{(\theta g(y,x,\theta)+(1-\theta))}t\right)\\&\times\frac{(1-\theta)}{(\theta g(y,x,\theta)+(1-\theta))^3}\frac{1}{h(y)}p\frac{g_y}{g}\frac{dy}{dx}\end{aligned}$$

We require

$$-\left(t-q^2u'''(q)\right)\frac{(1-\theta)}{\sigma(q)}-3\left(\theta\frac{s}{h(y)}+(1-\theta)\right)\theta\frac{g}{(\theta g(y,x,\theta)+(1-\theta))}t<0$$

Using the definition of t gives

$$\left(-1-\frac{qu'''(q)}{u''(q)}\right)\frac{(1-\theta)}{\sigma(q)}-3\left(\theta\frac{s}{h(y)}+(1-\theta)\right)\frac{g}{(\theta g(y,x,\theta)+(1-\theta))}<0$$

If $\frac{s}{h(y)}>g$ then we have

$$\begin{aligned}&\left(-1-\frac{qu'''(q)}{u''(q)}\right)\frac{(1-\theta)}{\sigma(q)}-3\left(\theta\frac{s}{h(y)}+(1-\theta)\right)\theta\frac{g}{(\theta g(y,x,\theta)+(1-\theta))}\\<&\left(-1-\frac{qu'''(q)}{u''(q)}\right)\frac{(1-\theta)}{\sigma(q)}-3\theta\end{aligned}$$

Thus, if

$$-\frac{qu'''(q)}{u''(q)}<\frac{3\theta\sigma(q)+(1-\theta)}{1-\theta}=\frac{3\theta\sigma(q)}{1-\theta}+1$$

the condition is satisfied. ■

Proof of Proposition 3.

Step 1. Ruling out $y=0$.

Note first that the marginal utility of buyers, as given by (4), is increasing in π. An increase in $u'(q)$ gives the following effect on the first-order condition (6) of sellers:

$$\left[\frac{d\left(\frac{u(q)}{q}\right)}{du'(q)}-p\left(\frac{\theta^{B}h(y)}{\left(\theta^{B}h(y)+\theta^{S}u'(q)\right)^{2}}\right)\right]\theta^{S}du'(q)>0$$

where

$$\frac{d\left(\frac{u(q)}{q}\right)}{dq}\frac{dq}{du'(q)}=\frac{\frac{u'(q)}{q}-\frac{u(q)}{q^{2}}}{u''(q)}=\left(1-\frac{u(q)}{qu'(q)}\right)\frac{u'(q)}{qu''(q)}$$

Then, if the choice is not at the lower boundary ($y=0$) at $\pi=0$, it cannot be at the boundary for any $\pi>0$. Consider (4) when $\pi=0$ and $y=0$ and hence $h(0)=1$. We have

$$u'(q)=\frac{(1-\beta)+\beta x}{\beta x\left(1-\theta^{S}\right)-\theta^{S}(1-\beta)}\theta^{B}.$$

Thus, we require, for this value of $u'(q)$ that

$$\left[\frac{u(q)}{qu'(q)}-\frac{1}{u'(q)}\right]\left(\theta^{B}+\theta^{S}u'(q)\right)-x>0.$$

This can be simplified to, noting that $u'(q) = \frac{\theta^B}{1-\theta^S}$, when $\beta \to 1$,

$$\frac{u(q)}{qu'(q)} > x + 1.$$

Step 2. Ruling out the upper bound $y = 1 - x$. For simplicity, we let $\beta \to 1$ in this case. From the specialization choice, we have, when the solution is interior,

$$-h(1-x) + \frac{u(q)}{q} - h_y(1-x)\frac{u'(q)}{\theta^B h(1-x) + \theta^S u'(q)} < 0.$$

The slope of (4) is, when $\beta = 1$,

$$\frac{du'}{d\pi} = u' \frac{1}{(\pi+1)\left(\left(1-\theta^S\right) - \theta^S \pi\right)}$$

We now show that the inflation rate satisfying the seigniorage first-order condition (8) is finite. Consider a candidate solution for a solution at the boundary where $p = 1$. Since u' is positive in equilibrium, the denominator in (4) is positive. Hence, $\frac{du'}{d\pi}$ in (A.2) is positive. Also, the solution for (4) at $\pi = 0$ is $u' = 2 - x$ and when $\pi \to \frac{1-\theta^S}{\theta^S}$ from below, $u' \to \infty$. Regarding the seigniorage condition, we have, when $u' = 2 - x$, that $\pi = \left(\theta^B\right)(1-\alpha)\left(\theta^B + \frac{\theta^S}{\alpha}\right)$ and when $u' \to \infty$, that $\pi \to 0$.

To find an upper bound for marginal utility, note that inflation is bounded above by $\pi = -\theta^B \frac{qu''(q)}{u'(q)}\left(\theta^B + \theta^S \frac{u(q)}{qu'(q)}\right)$. Then, from the quantity first-order condition, marginal utility is bounded above by

$$u'(q) = \frac{1+\pi}{\theta^B - \theta^S \pi}\theta^B h(1-x) = \frac{1 - \left(\theta^B\right)\frac{qu''(q)}{u'(q)}\left(\theta^B + \theta^S \frac{u(q)}{qu'(q)}\right)}{1 + \theta^S \frac{qu''(q)}{u'(q)}\left(\theta^B + \theta^S \frac{u(q)}{qu'(q)}\right)} h(1-x).$$

To ensure that there is no equilibrium at $y = 1 - x$, we then require that

$$-h(1-x) + \frac{u(q)}{q} - h_y(1-x)\frac{u'(q)}{\theta^B h(1-x) + \theta^S u'(q)} < 0.$$

When $\theta^S \to 1$, the right-hand side is, using $u'(q) = \frac{1}{1+\frac{qu''(q)}{u'(q)}\frac{u(q)}{qu'(q)}} h(1-x)$,

$$-h(1-x) + \frac{u(q)}{q} - h_y(1-x) = -h(1-x) + \frac{u(q)}{qu'(q)}\frac{1}{1+\frac{qu''(q)}{u'(q)}\frac{u(q)}{qu'(q)}}h(1-x) - h_y(1-x) < 0.$$

and hence

$$-\frac{1+\left(\frac{qu''(q)}{u'(q)}-1\right)\frac{u(q)}{qu'(q)}}{1+\frac{qu''(q)}{u'(q)}\frac{u(q)}{qu'(q)}}<\frac{h_y\left(1-x\right)}{h\left(1-x\right)}$$

When $\theta^B \to 1$, the right-hand side is

$$-h\left(1-x\right)+\frac{u\left(q\right)}{q}-h_y\left(1-x\right)\frac{u'\left(q\right)}{h\left(1-x\right)}<0.$$

and we require, using $u'\left(q\right)=\left(1-\frac{qu''(q)}{u'(q)}\right)h\left(1-x\right)$,

$$\frac{\frac{u(q)}{qu'(q)}\left(1-\frac{qu''(q)}{u'(q)}\right)}{h\left(1-x\right)}-\frac{1}{\left(1-\frac{qu''(q)}{u'(q)}\right)}<\frac{h_y\left(1-x\right)}{h\left(1-x\right)}.$$

In general, using the definition of $u'\left(q\right)$ from the quantity first-order condition,

$$\begin{aligned}&\frac{\left(\frac{u(q)}{qh(1-x)}-1\right)\left(1+\theta^B\theta^S\frac{qu''(q)}{u'(q)}\left(\theta^B+\theta^S\frac{u(q)}{qu'(q)}\right)-\theta^S\theta^B\frac{qu''(q)}{u'(q)}\left(\theta^B+\theta^S\frac{u(q)}{qu'(q)}\right)\right)}{1-\left(\theta^B\right)\frac{qu''(q)}{u'(q)}\left(\theta^B+\theta^S\frac{u(q)}{qu'(q)}\right)}\\ = &\frac{\frac{u(q)}{qh(1-x)}-1}{1-\left(\theta^B\right)\frac{qu''(q)}{u'(q)}\left(\theta^B+\theta^S\frac{u(q)}{qu'(q)}\right)}<\frac{h_y\left(1-x\right)}{h\left(1-x\right)}.\end{aligned}$$

■

Proof of Proposition 4.

Steps 1 and 2 are as in Lemma **2.**

Step 3. Effect on the probability of trade.

The specialization first-order condition is

$$p=\frac{s\left(u'^{-1}\left(g\left(y,x,\theta\right)h\left(y\right)\right)\right)-h\left(y\right)}{h_y}\frac{\theta g\left(y,x,\theta\right)+\left(1-\theta\right)}{g\left(y,x,\theta\right)}$$

The entry condition is

$$c=p\left(u\left(q\right)-qh\left(y\right)\right)$$

and

$$p=xG+yF\left(c\right)$$

where G is the population of buyers and F the seller cumulative distribution of entry costs. Then

$$\begin{aligned}
& xdG + F(c)\,dy + f(c)\,(p\,(u'(q) - h(y)))\,\frac{dq}{dy}dy \\
= \; & \frac{s'\left(u'^{-1}\left(g(y,x,\theta)\,h(y)\right)\right)}{h_y}\frac{\partial u'^{-1}(v)}{\partial v}\left(h(y)\,g_y + g(y,x,\theta)\,h_y\right)\frac{\theta g(y,x,\theta) + (1-\theta)}{g(y,x,\theta)}dy \\
& - \frac{h_y}{h_y}\frac{\theta g(y,x,\theta) + (1-\theta)}{g(y,x,\theta)}dy - \frac{s\left(u'^{-1}\left(g(y,x,\theta)\,h(y)\right)\right) - h(y)}{\left(h_y\right)^2}\frac{\theta g(y,x,\theta) + (1-\theta)}{g(y,x,\theta)}h_{yy}dy \\
& + \frac{s\left(u'^{-1}\left(g(y,x,\theta)\,h(y)\right)\right) - h(y)}{h_y}\left(-\frac{(1-\theta)}{\left(g(y,x,\theta)\right)^2}\right)g_y dy
\end{aligned}$$

Proceeding as above still gives

$$\begin{aligned}
\frac{dp}{dx} &= p\left(\frac{h_y}{h}\left(\frac{1}{\sigma(q)} - \frac{1}{\sigma(q)}\frac{u'(q) - h}{s-h} - \frac{h}{s-h}\right) - \frac{h_{yy}}{h_y}\right)\frac{dy}{dx} \\
& + p\left(\frac{1}{\sigma(q)}\left(1 - \frac{u'(q) - h}{s-h}\right) - \frac{(1-\theta)}{\theta g(y,x,\theta) + (1-\theta)}\right)\frac{g_y}{g}\frac{dy}{dx}
\end{aligned}$$

Step 4. Checking dy/dx

We now have

$$\frac{dy}{dx} = \frac{G}{-\left(\frac{s-u'(q)}{qh_y}\frac{dq}{dy} + 1\right)\frac{h_y}{s-h}p - F(c) - f(c)\,(p\,(u'(q) - h(y)))\,\frac{dq}{dy} - p\frac{h_{yy}}{h_y} + p\frac{(1-\theta)h(y)}{\theta u'(q) + (1-\theta)h(y)}\left(1 - \frac{u''(q)}{u'(q)}\frac{dq}{dy}\right)}$$

Hence, we require

$$-\left(\frac{s-u'(q)}{qh_y}\frac{dq}{dy} + 1\right)\frac{h_y}{s-h}p - F(c) - f(c)\,(p\,(u'(q) - h(y)))\,\frac{dq}{dy} - p\frac{h_{yy}}{h_y} + p\frac{(1-\theta)\,h(y)}{\theta u'(q) + (1-\theta)\,h(y)}\left(1 - \frac{u}{u}\right.$$

Rearranging the expression above gives

$$-\left(\frac{s-u'(q)}{qh_y}\frac{h_y}{s-h}p + f(c)\,(p\,(u'(q) - h(y))) + p\frac{(1-\theta)\,h(y)}{\theta u'(q) + (1-\theta)\,h(y)}\frac{u''(q)}{u'(q)}\right)\frac{dq}{dy} - \frac{h_y}{s-h}p - F(c) - p\frac{h}{h}$$

and hence, if

$$\left(\frac{s-u'(q)}{qh_y}\frac{h_y}{s-h} + f(c)\,((u'(q) - h(y)))\right)\frac{\theta u'(q) + (1-\theta)\,h(y)}{(1-\theta)\,h(y)} < -\frac{u''(q)}{u'(q)}$$

and

$$-\frac{h_y}{s-h} - \frac{F(c)}{p} + \frac{(1-\theta)\,h(y)}{\theta u'(q) + (1-\theta)\,h(y)} < \frac{h_{yy}}{h_y}$$

we have $\frac{dy}{dx} < 0$.

■