Experiments in the Linear Convex Order *

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Abstract

This paper proposes two rankings of statistical experiments using the linear convex order. These rankings provide simpler and more tractable characterizations than Blackwell order, which relies on the convex order. We apply these rankings to compare statistical experiments in binary-action decision problems and in decision problems that aggregate payoffs over a collection of binary-action decision problems. Furthermore, these rankings enable comparisons of statistical experiments in moral hazard problems without requiring the validity of the first-order approach, thereby complementing the results in Holmström (1979) and Kim (1995).

Keywords: Information, Comparison of Experiments, Decision Theory, Blackwell Order, Principal-Agent Problems.

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1 Introduction

Statistical experiments formalize information and uncertainty in mathematical models: Given a set of unknown states of the world Θ, an experiment F generates a signal $x \in X$ with the conditional distribution F_θ when the true state is $\theta \in \Theta$. A decision maker (DM) with a prior belief $\boldsymbol{q}$ over Θ updates her posterior belief using Bayes' rule after observing the signal. For discrete state spaces $\Theta = \{\theta_0, \ldots, \theta_n\}$, the experiment has two equivalent representations: an n-dimensional random vector of likelihood ratios of $\theta_1, \ldots, \theta_n$ relative to θ_0, or an n-dimensional random vector of posterior beliefs about $\theta_1, \ldots, \theta_n$.

Blackwell's celebrated theorem (Blackwell et al. (1951) and Blackwell (1953)) provides a ranking of experiments by their informativeness, which is fundamental to information economics. An experiment F dominates another experiment G in Blackwell order if the random vectors of likelihood ratios and posterior beliefs generated by F are more dispersed than those generated by G. That is, the random vectors[1] generated by F dominate those generated by G in the *convex order*. Specifically, an n-dimensional random vector $\boldsymbol{\mu}$ dominates another n-dimensional random vector $\boldsymbol{\nu}$ in the convex order if for all convex functions C with n variables,

$$E[C(\boldsymbol{\mu})] \geq E[C(\boldsymbol{\nu})].$$

For the binary-state case (i.e., $n = 1$ and $\Theta = \{\theta_0, \theta_1\}$), Blackwell order has been extensively studied. In this case, the convex order over random variables is equivalent to majorization, Lorenz order, and second-order stochastic dominance. The set of univariate convex functions can be characterized as a one-parameter family of extremal rays, and the convex order has a simple characterization based on the pointwise comparison of the integral of distribution functions.

However, Blackwell order is significantly more complex in the multi-state cases than in the binary-state case, as noted by Blackwell and Girshick (1954). Jewitt (2007) identifies two major limitations of Blackwell order: First, as noted by Lehmann (1988), it fails to apply in certain cases where it intuitively should. Second, verifying the ranking between two experiments is computationally intractable, particularly with infinite signal spaces. These limitations stem from fundamental differences between one-dimensional and multi-dimensional convex orders. The convex order over multi-dimensional random vectors imposes stronger conditions and lacks a tractable characterization. Indeed, The set of multivariable convex

[1] Comparing the dispersion of likelihood-ratio vectors is equivalent to comparing the dispersion of posterior-belief vectors.

functions is too intricate to provide a practical characterization, as pointed out by Johansen (1972) and Johansen (1974).

There is a growing statistical literature (e.g., Marshall (1979), Bhandari (1988), Joe and Verducci (1992), Scarsini (1998), and Arnold (2012)) examining a weaker order over random vectors for comparing their dispersion: the *linear convex order*. An n-dimensional random vector $\boldsymbol{\mu}$ dominates another n-dimensional random vector $\boldsymbol{\nu}$ in the linear convex order if, for each vector $\boldsymbol{b} \in \mathbb{R}^n$, the random variable $\boldsymbol{b}'\boldsymbol{\mu}$ dominates the random variable $\boldsymbol{b}'\boldsymbol{\nu}$ in the convex order. In contrast to the convex order, the linear convex order compares all linear combinations of two random vectors, thereby simplifying the problem to a univariate comparison and enabling a straightforward characterization. Furthermore, Koshevoy and Moseler in a series of papers (Koshevoy (1995), Koshevoy and Mosler (1996), and Koshevoy (1997)) provide an elegant geometric interpretation of the linear convex order by generalizing the concept of Lorenz Curve to the multi-dimensional setting.

This paper proposes two rankings of experiments by their informativeness based on the linear convex order: *Posterior-Mean (PM) order* and *Linear-Blackwell (LB) order*. The PM and LB orders provide simpler and more tractable characterizations compared to Blackwell order.

Consider the case in which the states $\theta_0, \ldots, \theta_n \in \mathbb{R}$, with $\theta_0 < \ldots < \theta_n$. An experiment F dominates G in PM order if, for each prior $\boldsymbol{q}$, the posterior mean (conditional expectation of the state based on the realized signal) generated by F dominates that generated by G in the convex order. Note that the posterior mean is a random variable and can be derived as a linear combination of either the likelihood-ratio vector or the posterior-belief vector. PM order is founded on the principle that more information leads to greater dispersion in the posterior mean. There is a growing literature that models the concept of more information in this way (e.g., Ganuza and Penalva (2010), Gentzkow and Kamenica (2016), and Ravid et al. (2022)). These studies focus on the situation in which the prior is fixed. In contrast, PM order is defined for all possible priors. We show that PM order is preserved over all real values assigned to the state space satisfying $\theta_0 \leq \ldots \leq \theta_n$, i.e., only the ordinal ranking of the state space is relevant.

An experiment F dominates G in Linear-Blackwell (LB) order if the random vectors of the likelihood ratios and the posterior beliefs generated by F dominate those generated by G in the linear convex order.[2] We show that F dominates G in LB order if and only if F dominates G in PM order for each ordinal ranking

[2] Chencking the random vectors of likelihood ratios is equivalent to checking the random vectors of posterior beliefs.

imposed on the state space.

When the state space is binary, Blackwell order, Lehmann order,[3] LB order, and PM order are equivalent. In cases with more than two states, Blackwell order implies LB order, but not vice versa in general. We show that an experiment F dominates G in LB order if and only if, for each weighted dichotomy constructed on the state space,[4] the experiment F dominates G in Blackwell order. This finding enables us to generalize established results in the binary-state case regarding Blackwell order to the multi-state cases regarding LB order. Furthermore, we show that when both F and G satisfy the Monotonic Likelihood Ratio Property (MLRP), the experiment F dominates G in PM order if and only if F dominates G in Lehmann order. Hence, PM order generalizes Lehmann order without relying on MLRP. The MLRP assumption imposes a stringent requirement on the correlation between the signal and the state and lacks a solid microfoundation, as shown by Mensch (2021).

We apply PM order and LB order to compare experiments in decision problems. Consider a decision problem in which the DM chooses an action $a \in A$ and receives payoff $u(a, \theta)$ if the realized state is θ. The DM is uncertain about the state with prior $\boldsymbol{q}$ and can choose to observe a signal from either F or G, then updates her belief according to Bayes' Rule and chooses the action that maximizes her expected payoff. Blackwell's theorem states that F dominates G in Blackwell order if and only if in each decision problem, the DM receives a higher ex ante expected payoff under F than under G.

We show that F dominates G in LB order if and only if, in each decision problem with $|A| = 2$, i.e., in each binary-action decision problem, the DM receives a higher ex ante expected payoff under F than under G. Furthermore, given the ordinal ranking of the state space, the experiment F dominates G in PM order if and only if, in each binary decision problem satisfying the single-crossing property, the DM receives a higher ex ante expected payoff under F than under G.

The results above extend to the decision problems that aggregate payoffs over a collection of binary-action decision problems. Specifically, a decision problem is *binary-decomposable* if its value function[5] equals the sum of the value functions of a collection of binary-action decision problems.[6] We show that a sufficient condition

[3] Lehmann (1988) compares experiments satisfying the Monotonic Likelihood Ratio Property (MLRP). Lehmann's order is based on how strongly the signal is correlated with the state.

[4] As will be demonstrated, each weighted dichotomy corresponds to a binary-state space.

[5] The value function maps the DM's belief to her maximal expected payoff.

[6] de Oliveira et al. (2023) is the first paper to study this decomposition problem. They show that in the binary-state case, all decision problems are binary-decomposable. However, this result does not hold when there are more than two states.

for a decision problem to be binary-decomposable is that there exists an ordinal ranking of the action space A such that, for each belief, the DM's expected payoff is quasi-concave in a. Furthermore, under this quasi-concavity condition, there exists an ordinal ranking of Θ such that the decision problem can be decomposed into a collection of binary decision problems satisfying the single-crossing property. Finally, the quasi-concavity condition holds if the DM's optimal action is always continuous in her belief.

We also apply LB order to compare statistical experiments in moral hazard problems under the framework of Grossman and Hart (1983): An agent privately chooses a costly action (state) from $\Theta = \{\theta_0, \ldots, \theta_n\}$, and a principal observes one signal correlated with this action from a statistical experiment. The principal writes a contract with the agent that maps signals to payments for the agent. The agent is risk-averse and his utility is additively separable in the payment and the cost of the chosen action, while the principal is risk-neutral.

We show that a statistical experiment F dominates G in LB order if and only if, for each mixed action[7] $\delta \in \Delta(\Theta)$, whenever it is implementable under G with an incentive-compatible contract, it is also implementable under F with a lower expected payment. We prove this result by first establishing strong duality using the geometric illustration of the linear convex order from Koshevoy and Mosler (1996), then applying the conjugate duality approach introduced by Jewitt (2007). We further strengthen the informativeness principle in Holmström (1979) by showing that an additional signal is not valuable if and only if, given the existing signal, the additional signal is uninformative in the sense of LB order and also Blackwell order. This paper contributes to the literature on comparing statistical experiments in agency problems (e.g., Gjesdal (1982) and Kim (1995)). Most of the literature focuses on situations where the agent's action is a continuous, one-dimensional effort variable. These studies also assume that the statistical experiments satisfy MLRP and that the first-order approach is valid. However, this situation is restrictive since it relies exclusively on the binary-action (high- and low-effort) case, as noted by Hart and Holmstrom (1986), and the first-order approach is only valid in very special cases.

This paper proceeds as follows: Section 2 presents the preliminary notions regarding the statistical experiments, the convex order, and the linear convex order. Section 3 introduces Posterior-Mean order and Linear-Blackwell order, and compares them with Blackwell order and Lehmann order. Section 4 applies

[7] The mixed strategies model the situation in which the principal would like the agent to allocate time among multiple tasks.

Posterior-Mean order and Linear-Blackwell order to compare statistical experiments in decision problems. Section 5 applies Linear-Blackwell order to compare statistical experiments in moral hazard problems. Section 6 concludes.

2 Preliminaries

2.1 Statistical Experiments

Let $\Theta = \{\theta_0, \dots, \theta_n\} \subset \mathbb{R}$ be the set of states in which

$$\theta_0 < \dots < \theta_n.$$

The decision-maker (DM) holds a prior belief $\boldsymbol{q} = \{q_1, \dots, q_n\}$ with

$$q_i \in [0,1], \forall i \in \{1, \dots, n\},$$

$$\sum_{i=1}^{n} q_i \leq 1.$$

She believes that with probability q_i, the realized state is θ_i.[8]

A *statistical experiment* F consists of (i) a set of signals X, in which x denotes a generic signal, and (ii) a collection of probability measures $\{F(\cdot|\theta_0), \dots, F(\cdot|\theta_n)\}$ defined over X, in which $F(A|\theta)$ represents the probability of observing $A \subset X$ in state $\theta \in \Theta$. Similarly, for another statistical experiment G, let Y be the signal space and y be a generic signal.

We assume that the signal spaces X and Y are one-dimensional with

$$X = Y = [0,1],$$

under which $F(\cdot|\theta)$ and $G(\cdot|\theta)$ are cumulative distribution functions (c.d.f) over $[0,1]$ for each θ. This assumption is unnecessary but simplifies the exposition. We further assume that:

Assumption 1. *(Continuity) The c.d.fs $F(\cdot|\theta)$ and $G(\cdot|\theta)$ are continuous in $x \in [0,1]$ and $y \in [0,1]$ for each $\theta \in \Theta$. There exist density functions $f(\cdot|\theta)$ and $g(\cdot|\theta)$ such that*

$$F(x|\theta) = \int_0^x f(t|\theta)dt,$$

$$G(y|\theta) = \int_0^y g(t|\theta)dt.$$

[8] We omit the prior belief for θ_0, allowing the prior beliefs of other states to move freely.

Assumption 2. *(Bounded Density) There exists a constant $a > 0$ such that for each $x, y \in [0,1]$ and $\theta \in \Theta$,*

$$\frac{1}{a} < f(x|\theta) < a,$$

$$\frac{1}{a} < g(y|\theta) < a.$$

As shown in Lehmann (1988), Assumption 1 is without loss of generality. Assumption 2 ensures bounded likelihood ratios between any two states, though we can extend definitions and results to cases in which unbounded likelihood ratios are allowed.

Given a prior belief $\boldsymbol{q}$ and a statistical experiment F, let $F_{\boldsymbol{q}}(x)$ and $f_{\boldsymbol{q}}(x)$ denote the marginal distribution function and density of x, respectively. For a signal x is drawn from $F_{\boldsymbol{q}}(\cdot)$, the DM's posterior belief

$$\boldsymbol{p}^F(x;\boldsymbol{q}) = \{p_1^F(x;\boldsymbol{q}), \dots, p_n^F(x;\boldsymbol{q})\}$$

is defined as

$$p_i^F(x;\boldsymbol{q}) = q_i \cdot f(x|\theta_i)/f_q(x), \forall i \in \{1, \dots, n\}$$

by Bayes' Rule. Furthermore, the conditional expectation of the realized state θ given x (posterior mean) $m^F(x)$ is defined as

$$m^F(x;\boldsymbol{q}) = E(\theta|x; F, \boldsymbol{q}).$$

The random variable (signal) x drawn from $F_{\boldsymbol{q}}(\cdot)$ generates the random vector of posterior beliefs $\boldsymbol{p}^F(x;\boldsymbol{q})$ and the random variable of posterior mean $m^F(x;\boldsymbol{q})$. We omit x and denote them as $\boldsymbol{p}^F(\boldsymbol{q})$ and $m^F(\boldsymbol{q})$. Similarly, the random vector of posterior beliefs and random variable of posterior mean under G are denoted as $\boldsymbol{p}^G(\boldsymbol{q})$ and $m^G(\boldsymbol{q})$, respectively.

Conditional on state θ_0, for a signal x drawn from $F(\cdot|\theta_0)$, the (conditional) likelihood-ratio vector

$$\boldsymbol{\ell}^F(x) = \{l_1^F(x), \dots, l_n^F(x)\}$$

is defined as

$$l_i^F(x) = f(x|\theta_i)/f(x|\theta_0), \forall i \in \{1, \dots, n\}.$$

The random variable x drawn from $F(\cdot|\theta_0)$ generates the random vector of likelihood ratios $\boldsymbol{\ell}^F(x)$. We omit x and denote the random vector as $\boldsymbol{\ell}^F$. Similarly, the random vector of likelihood ratios under G is denoted as $\boldsymbol{\ell}^G$.

2.2 Convex Order and Linear Convex Order

Consider two real-valued random vectors $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ and $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n)$ with

$$E[\mu_i] = E[\nu_i] < \infty, \forall i \in \{1, \ldots, n\}.$$

The convex order is used to compare their dispersion:

Definition 1. *The random vector $\boldsymbol{\mu}$ dominates $\boldsymbol{\nu}$ in the **convex order**, i.e.,*

$$\boldsymbol{\mu} \overset{cx}{\succ} \boldsymbol{\nu},$$

if for each convex function $C : \mathbb{R}^n \to \mathbb{R}$,

$$E[C(\boldsymbol{\mu})] \geq E[C(\boldsymbol{\nu})].$$

When $n = 1$, the convex order has a simple characterization since the set of univariate convex functions can be represented as a one-parameter family of extremal rays. That is, the random variable μ dominates ν in the convex order if and only if[9]

$$E\left[(\mu - t)_+\right] \geq E\left[(\nu - t)_+\right] \quad \forall t \in \mathbb{R}. \tag{1}$$

The convex order is equivalent to majorization, Lorenz order, and second-order stochastic dominance. The condition (1) is equivalent to the pointwise comparison of the indefinite integral of distribution functions.[10]

The convex order over multi-dimensional random vectors is more stringent and lacks a simple characterization. The set of multivariable convex functions is too intricate for a practical characterization. There is a growing statistical literature examining a weaker order over random vectors for comparing their dispersion:

Definition 2. *The random vector $\boldsymbol{\mu}$ dominates $\boldsymbol{\nu}$ in the **linear convex order**, i.e.,*

$$\boldsymbol{\mu} \overset{lcx}{\succ} \boldsymbol{\nu},$$

if for all $\boldsymbol{b} \in \mathbb{R}^n$,

$$\boldsymbol{b}'\boldsymbol{\mu} \overset{cx}{\succ} \boldsymbol{b}'\boldsymbol{\nu}.$$

In contrast to the convex order, the linear convex order compares all linear combinations of two random vectors, thereby simplifying the problem to a uni-

[9] $(\mu - t)_+ = max(\mu - t, 0)$

[10] That is, compare $\int_{-\infty}^{x} F_\mu(t)dt$ and $\int_{-\infty}^{x} F_\nu(t)dt$ for each x where $F_\mu(\cdot)$ and $F_\nu(\cdot)$ are c.d.fs of μ and ν, respectively.

variate comparison and enabling a more straightforward characterization. The convex order implies the linear convex order, but not vice versa in general.

3 Ranking Experiments

3.1 Posterior-Mean Order

We first compare experiments by the dispersion of their posterior means:

Definition 3. *The experiment F dominates the experiment G in* ***Posterior-Mean (PM) order****, i.e.,*

$$F \overset{PM}{\succ} G,$$

if

$$m^F(\boldsymbol{q}) \overset{cx}{\succ} m^G(\boldsymbol{q}) \ \forall \boldsymbol{q}.$$

Note that the posterior means $m^F(\boldsymbol{q})$ and $m^G(\boldsymbol{q})$ are random variables that can be expressed as linear combinations of the likelihood-ratio vector (or, equivalently, the posterior-belief vector). Consider a vector $\boldsymbol{b} = (b_1, \ldots, b_n)$ satisfying the *single-crossing* property, i.e., for each pair of $i < j$, if $b_i > 0$, then $b_j \geq 0$. Construct two random variables $\boldsymbol{b'l^F}$ and $\boldsymbol{b'l^G}$ with c.d.fs $F_{\boldsymbol{b}}$ and $G_{\boldsymbol{b}}$ respectively.

Lemma 1. *$F \overset{PM}{\succ} G$ if and only if for each $\boldsymbol{b} \in \mathbb{R}^n$ satisfying the single crossing property,*

$$\int_{-\infty}^{k} F_{\boldsymbol{b}}(t)dt \geq \int_{-\infty}^{k} G_{\boldsymbol{b}}(t)dt \ \forall k > 0.$$

Hence, a sufficient condition for F to dominate G in PM order is that, for each $\boldsymbol{b} \in \mathbb{R}^n$ satisfying the single crossing property, the random variable $\boldsymbol{b'l^F}$ dominates $\boldsymbol{b'l^G}$ in the convex order.

Furthermore, Lemma 1 implies that PM order is preserved over all real values assigned to the state space satisfying

$$\theta_0 \leq \ldots \leq \theta_n,$$

i.e., only the ordinal ranking of the state space is relevant since it determines the single-crossing property. Specifically, for each increasing function ψ, let $m^F(\boldsymbol{q}, \psi)$ and $m^G(\boldsymbol{q}, \psi)$ denote the the conditional expectation of $\psi(\theta)$ given the signal realizations under the marginal distributions $F_{\boldsymbol{q}}$ and $G_{\boldsymbol{q}}$, respectively.

Proposition 1. *$F \overset{PM}{\succ} G$ if and only if $m^F(\boldsymbol{q}, \psi) \overset{cx}{\succ} m^G(\boldsymbol{q}, \psi)$ for each prior $\boldsymbol{q}$ and each increasing function ψ.*

Lemma 2 demonstrates the cases regarding combinations of experiments.

Lemma 2. *Consider experiments F_1, F_2, G_1, and G_2 such that*

$$F_1 \overset{PM}{\succ} G_1 \text{ and } F_2 \overset{PM}{\succ} G_2.$$

1. *The product experiment satisfies*

$$F_1 \otimes F_2 \overset{PM}{\succ} G_1 \otimes G_2,$$

which corresponds to the case in which the DM receives two conditionally independent signals from each experiment, respectively.

2. *The mixture experiment satisfies*

$$tF_1 + (1-t)F_2 \overset{PM}{\succ} tG_1 + (1-t)G_2 \;\; \forall t \in [0,1],$$

which corresponds to the case in which the DM receives a signal from the first experiment with probability t and from the second experiment with probability $1-t$.

3.2 Linear-Blackwell Order

Definition 4. *The experiment F dominates the experiment G in **Linear-Blackwell (LB) order** , i.e.,*

$$F \overset{LB}{\succ} G,$$

if

$$\boldsymbol{l}^F \overset{lcx}{\succ} \boldsymbol{l}^G.$$

A prior $\boldsymbol{q}$ is *interior* if

$$q_i \in (0,1), \forall i \in \{1,\ldots,n\},$$

$$\sum_{i=1}^{n} q_i < 1.$$

We can also characterize LB order using the random vectors of posterior beliefs.

Lemma 3. *The following statements are equivalent:*

1. $F \overset{LB}{\succ} G$*;*

2. $\boldsymbol{p}^F(\boldsymbol{q}) \overset{lcx}{\succ} \boldsymbol{p}^G(\boldsymbol{q})$, $\forall \boldsymbol{q}$*;*

3. *There exists an interior $\boldsymbol{q}$ such that $\mathbf{p}^F(\boldsymbol{q}) \overset{lcx}{\succ} \boldsymbol{p}^G(\boldsymbol{q})$.*

Lemma 1 implies that if $F \overset{LB}{\succ} G$, then $F \overset{PM}{\succ} G$ holds for each ordinal ranking imposed on the state space. Conversely, the reverse implication is also true.

Proposition 2. *$F \overset{LB}{\succ} G$ if and only if $F \overset{PM}{\succ} G$ for each permutation $\hat{\Theta}$ of the state space Θ.*

Analogous to Lemma 2,

Lemma 4. *Consider experiments F_1, F_2, G_1, and G_2 such that*

$$F_1 \overset{LB}{\succ} G_1 \text{ and } F_2 \overset{LB}{\succ} G_2.$$

1. *The product experiment satisfies*

$$F_1 \otimes F_2 \overset{LB}{\succ} G_1 \otimes G_2.$$

2. *The mixture experiment satisfies*

$$tF_1 + (1-t)F_2 \overset{LB}{\succ} tG_1 + (1-t)G_2 \ \forall t \in [0,1].$$

Lipnowski et al. (2020) and Wu (2023) quantify informativeness through the convex hull of the set of posterior beliefs. Let $\mathcal{P}^{F,\boldsymbol{q}}$ be the set of possible posterior beliefs generated by F given prior $\boldsymbol{q}$, and $conv(\mathcal{P}^{F,\boldsymbol{q}})$ be its convex hull. Based on Wu (2023),

Lemma 5. *$F \overset{LB}{\succ} G \implies \mathcal{P}^{G,\boldsymbol{q}} \subset conv(\mathcal{P}^{F,\boldsymbol{q}})$, $\forall \boldsymbol{q}$.*

Koshevoy and Moseler in a series of papers (Koshevoy (1995), Koshevoy and Mosler (1996), and Koshevoy (1997)) generalize Lorenz Curve to multi-dimensional settings and construct Lorenz Zonotope to illustrate the linear convex order. We follow their approach and provide a geometric illustration of LB order. Consider the set of functions

$$\mathcal{H} = \{h | h : [0,1] \to [0,1]\}.$$

Under experiment F, each $h \in \mathcal{H}$ maps to a point in $\mathbb{R}^{n+1}$:

$$z^F(h) = \left(\int_0^1 h(x) dF(x|\theta_0), \ldots, \int_0^1 h(x) dF(x|\theta_n) \right).$$

The *Lorenz Zonotope* of experiment F is the set

$$Z(F) = \{z^F(h) | h \in \mathcal{H}\}.$$

Consider the DM faces a decision problem with a binary action set $A = \{a_0, a_1\}$. The Lorenz Zonotope $Z(F)$ is the feasible set of F. It specifies the set of action distributions conditional on each state, that can be obtained by some choice of strategy. Specifically, each $h \in \mathcal{H}$ corresponds to a strategy

$$\sigma_h : [0,1] \to \Delta(A),$$

in which $h(x)$ is the probability of choosing a_0 upon observing x. Using Theorem 3.1 in Koshevoy (1997), we demonstrate that increased information under LB order expands the feasible set.

Proposition 3. $F \overset{LB}{\succ} G \Longleftrightarrow Z(G) \subset Z(F)$.

3.3 Relation to Blackwell Order and Lehmann Order

Blackwell et al. (1951) compares experiments in terms of their informativeness by examining the dispersion of the likelihood-ratio vectors and the posterior-belief vectors.

Definition 5. *The experiment F dominates the experiment G in* ***Blackwell order****, i.e.,*

$$F \overset{B}{\succ} G,$$

if the following conditions hold:

1. $\boldsymbol{l}^F \overset{cx}{\succ} \boldsymbol{l}^G$,
2. $\mathbf{p}^F(\boldsymbol{q}) \overset{cx}{\succ} \mathbf{p}^G(\boldsymbol{q})\ \forall \boldsymbol{q}$;
3. *There exists an interior* $\boldsymbol{q}$ *such that* $\mathbf{p}^F(\boldsymbol{q}) \overset{cx}{\succ} \mathbf{p}^G(\boldsymbol{q})$.

These three conditions are equivalent.

Since the convex order implies the linear convex order, it follows that

$$F \overset{B}{\succ} G \implies F \overset{LB}{\succ} G \implies F \overset{PM}{\succ} G$$

In the binary-state case,

$$F \overset{B}{\succ} G \Longleftrightarrow F \overset{LB}{\succ} G \Longleftrightarrow F \overset{PM}{\succ} G.$$

Furthermore, two experiments F and G are *Blackwell-equivalent*, i.e.,

$$F \overset{B}{\sim} G,$$

if they mutually dominate each other in Blackwell order. Similarly, two experiments F and G are *LB-equivalent*, i.e.,

$$F \overset{LB}{\sim} G,$$

if they mutually dominate each other in LB order. Using Theorem 4.1 in Koshevoy and Mosler (1996),

Lemma 6. *$F \overset{B}{\sim} G \Longleftrightarrow F \overset{LB}{\sim} G$.*

Consider a *weighted dichotomy*

$$\mathcal{W} = \{W_0, W_1, P_0(\cdot), P_1(\cdot), \omega_0, \omega_1\}$$

constructed on the state space Θ. The sets W_0 and W_1 form a partition of Θ satisfying

$$W_0 \cup W_1 = \Theta,$$

$$W_0 \cap W_1 = \emptyset.$$

The weighting functions P_0 and P_1 assign non-negative weights to each state in W_0 and W_1, respectively, such that

$$\sum_{\theta \in W_0} P_0(\theta) = 1,$$

$$\sum_{\theta \in W_1} P_1(\theta) = 1.$$

The partition and the weighting functions generate a dichotomy, or equivalently, a binary-state space $\{\omega_0, \omega_1\}$ in which an experiment F on Θ maps to an experiment $F^{\mathcal{W}}$ on $\{\omega_0, \omega_1\}$ defined by

$$F^{\mathcal{W}}(x|\omega_0) = \sum_{\theta \in W_0} P_0(\theta) F(x|\theta),$$

$$F^{\mathcal{W}}(x|\omega_1) = \sum_{\theta \in W_1} P_1(\theta) F(x|\theta).$$

Proposition 4. *$F \overset{LB}{\succ} G$ if and only if $F^{\mathcal{W}} \overset{B}{\succ} G^{\mathcal{W}}$ for each weighted dichotomy $\mathcal{W}$.*

A weighted dichotomy

$$\mathcal{W} = \{W_0, W_1, P_0(\cdot), P_1(\cdot), \omega_0, \omega_1\}$$

is *monotone* if the partitions $\{W_0, W_1\}$ are monotone on Θ.

Proposition 5. *$F \overset{PM}{\succ} G$ if and only if $F^{\mathcal{W}} \overset{B}{\succ} G^{\mathcal{W}}$ for each monotone weighted dichotomy $\mathcal{W}$.*

Proposition 4 and Proposition 5 offer another tractable way to compare experiments in LB order and PM order by leveraging the simple characterization of Blackwell order in the binary-state case (see Appendix A for details). Furthermore, we can use Proposition 4 and Proposition 5 to address the complexity in the multi-dimensional setting and to generalize known results in the binary case regarding Blackwell order to the multi-state case regarding LB order and PM order. In Appendix B, we extend the results in Börgers et al. (2013) on complementarity and substitutability of experiments.

In Appendix C, we consider the discrete-signal case and show that Blackwell order is equivalent to LB order when the probability matrix is full rank, i.e., there is no redundant signal.

Lehmann (1988) compares experiments satisfying the *Monotonic Likelihood Ratio Property (MLRP)*. Specifically, an experiment F satisfying MLRP if for each $\theta < \theta'$, the likelihood ratio $\frac{f(x|\theta')}{f(x|\theta)}$ is increasing in x. Lehmann's order is based on how strongly the signal is correlated with the state.

Definition 6. *The experiment F dominates the experiment G in **Lehmann order** , i.e.,*

$$F \overset{L}{\succ} G,$$

if

$$F^{-1}\left(G(y|\theta)|\theta\right) \leq F^{-1}\left(G(y|\theta')|\theta'\right) \;\; \forall \theta < \theta' \text{ and } y \in [0,1].$$

Proposition 6. *If the experiments F and G satisfy MLRP,*

$$F \overset{PM}{\succ} G \Longleftrightarrow F \overset{L}{\succ} G.$$

Therefore, PM order generalizes Lehmann order without relying on MLRP. Although mathematically convenient, the MLRP assumption imposes a stringent requirement on the correlation between the signal and the state. For example, consider a local experiment in which the signal $x = \theta + \epsilon$. The MLRP assumption requires the noise ϵ to be one-dimensional and unimodal, which is violated in many situations (e.g., Rauh and Seccia (2005) and Cheynel and Levine (2020)). Furthermore, Mensch (2021) demonstrates that the MLRP assumption lacks a solid micro-foundation by showing that a DM acquires information satisfying MLRP only if she has attention costs proportional to entropy reduction. Jewitt (2007), Di Tillio et al. (2021), and Kim (2023) relax the MLRP assumption and generalize Lehmann order. In Appendix D, we examine how PM order relates to these alternative rankings and establish additional conditions for PM order.

Figure 1 illustrates the relations among Lehmann order, PM order, LB order, and Blackwell order.

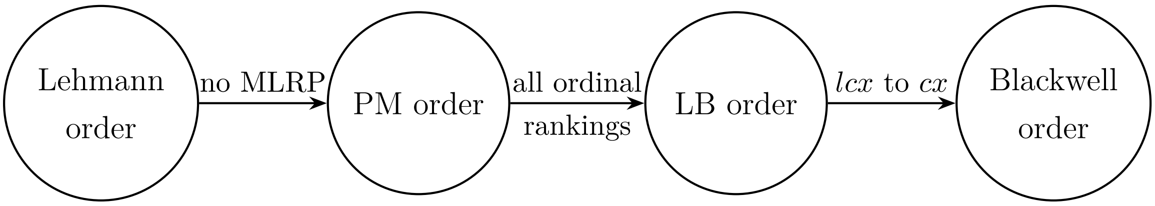

Figure 1: Relations among Lehmann order, PM order, LB order, and Blackwell order. PM order extends Lehmann order by relaxing the MLRP assumption. LB order is equivalent to PM order with respect to all ordinal rankings of the state space. Blackwell order is more stringent than LB order since it is based on the convex order instead of the linear convex order. LB order is equivalent to Blackwell order on each weighted dichotomy.

4 Decision Problem

4.1 Basic Setting

A *decision problem* $\{A, u\}$ consists of (i) a set of actions A with a denoting a generic action, and (ii) a payoff function

$$u : A \times \Theta \to \mathbb{R}$$

such that the DM receives payoff $u(a, \theta)$ when choosing a in state θ.

Given a belief $\boldsymbol{p}$ over the state space Θ, denote the expected payoff of choosing a as

$$u(a, \boldsymbol{p}) = E_{\boldsymbol{p}}[u(a, \theta)].$$

Denote the set of optimal actions under $\boldsymbol{p}$ as

$$A(\boldsymbol{p}) = argmax_{a \in A} u(a, \boldsymbol{p}).$$

Denote the *value function*, i.e., the maximal payoff the DM can receive under $\boldsymbol{p}$ as

$$V(\boldsymbol{p}) = \max_{a \in A} u(a, \boldsymbol{p}).$$

An experiment F and a prior $\boldsymbol{q}$ induce a distribution over posterior beliefs. Given this distribution, denote the ex ante expected payoff as

$$V^F(\boldsymbol{q}) = E[V(\boldsymbol{p}^F(\boldsymbol{q}))].$$

Under experiment F, the DM's strategy is characterized by a mapping

$$\sigma^F : [0,1] \to \Delta(A).$$

Let $\sigma^F(x,a)$ denote the probability (or density) that the DM chooses action a when receiving signal x. Denote the expected payoff generated by strategy σ^F in state θ as

$$u(\sigma^F, \theta) = \int_0^1 \int_A \sigma^F(x,a) u(a,\theta) f(x|\theta) da dx.$$

Given an ordinal ranking of A, i.e., $A \subset \mathbb{R}$, a decision problem satisfies the *single-crossing property* if for each pair of $a < a'$ and each pair of $\theta < \theta'$,

$$u(a',\theta) > u(a,\theta) \implies u(a',\theta') \geq u(a,\theta').$$

4.2 Binary-Action Case

We show that F dominates G in LB order if and only if for each decision problem with $|A| = 2$, i.e., for each binary-action decision problem, the DM receives a higher ex ante expected payoff under F than under G.

Theorem 1. *$F \overset{LB}{\succ} G$ if and only if for each decision problem $\{A,u\}$ with $|A| = 2$,*

$$V^F(\boldsymbol{q}) \geq V^G(\boldsymbol{q}) \ \forall \boldsymbol{q}.$$

PM order applies to binary-action decision problems satisfying the single-crossing property.

Theorem 2. *$F \overset{PM}{\succ} G$ if and only if for each decision problem $\{A,u\}$ with $|A| = 2$ satisfying the single-crossing property,*

$$V^F(\boldsymbol{q}) \geq V^G(\boldsymbol{q}) \ \forall \boldsymbol{q}.$$

In a binary decision problem, the ex ante expected payoff $V^F(\boldsymbol{q})$ is given by the sum of

$$\int_0^1 \left[\sum_{i=0}^n \Delta u_i \cdot q_i \cdot f(x|\theta_i) \right]_+ dx \tag{2}$$

and a constant K, in which Δu_i is the payoff difference between two actions in state θ_i. We rewrite (2) in the same form as (1) and compare linear combinations of random vectors of likelihood ratios generated by F and G, thereby allowing us to prove Theorem 1 and Theorem 2.

Furthermore, we strengthen Theorem 1 by Proposition 3 that increased information under LB order expands the feasible set in each binary-action decision problem.

Theorem 3. *$F \overset{LB}{\succ} G$ if and only if, in each decision problem $\{A, u\}$ with $|A| = 2$, for each strategy σ^G under G, there exists a strategy σ^F under F such that*

$$u(\sigma^F, \theta) \geq u(\sigma^G, \theta) \ \forall \theta.$$

Theorem 3 demonstrates that the DM with the *maxmin expected utility (MEU)* preferences (Gilboa and Schmeidler (1989)) prefers F to G in each binary-action decision problem, i.e.,

$$\max_{\sigma^F} \min_{\theta} u(\sigma^F, \theta) \geq \max_{\sigma^G} \min_{\theta} u(\sigma^G, \theta).$$

This result can be further extended to a broader family of ambiguity preferences characterized by Cerreia-Vioglio et al. (2011) using the proof in Li and Zhou (2020).

4.3 Binary-Decomposablility

We extend the results in Section 4.2 to decision problems that aggregate payoffs over a collection of binary-action decision problems.

Definition 7. *A decision problem $\{A, u\}$ is **binary-decomposable (BD)** if for its value function $V(\cdot)$, there exists a collection of binary-action decision problems $\{\{A_t, u_t\}\}_{t\in[0,1]}$ with value functions $\{V_t(\cdot)\}_{t\in[0,1]}$ such that*

$$V(\boldsymbol{p}) = \int_0^1 V_t(\boldsymbol{p})dt \ \forall \boldsymbol{p}.$$

Definition 8. *A decision problem $\{A, u\}$ is **monotonic-binary-decomposable (MBD)** if for its value function $V(\cdot)$, there exists a collection of binary-action decision problems $\{\{A_t, u_t\}\}_{t\in[0,1]}$ satisfying the single-crossing property with value* *functions* $\{V_t(\cdot)\}_{t\in[0,1]}$ *such that*

$$V(\boldsymbol{p}) = \int_0^1 V_t(\boldsymbol{p})dt \ \forall \boldsymbol{p}.$$

Corollary 1. $F \overset{LB}{\succ} G$ *if and only if for each BD decision problem,*

$$V^F(\boldsymbol{q}) \geq V^G(\boldsymbol{q}) \ \forall \boldsymbol{q}.$$

Corollary 2. $F \overset{PM}{\succ} G$ *if and only if for each MBD decision problem,*

$$V^F(\boldsymbol{q}) \geq V^G(\boldsymbol{q}) \ \forall \boldsymbol{q}.$$

Corollary 3. $F \overset{LB}{\succ} G$ *if and only if in each BD decision problem* $\{A, u\}$ *with* $|A| < \infty$*, for each strategy* σ^G *under* G*, there exists a strategy* σ^F *under* F *such that*[11]

$$u(\sigma^F, \theta) \geq u(\sigma^G, \theta) \ \forall \theta.$$

de Oliveira et al. (2023) demonstrates that if the state space is binary, i.e., $|\Theta| = 2$, all decision problems are *BD*. However, not all decision problems are *BD* if $|\Theta| > 2$. Figure 2 provides examples of both a BD decision problem and a non-BD decision problem.

The left panel of Figure 2 shows a BD decision problem $\{A, u\}$ with $A = \{a_0, a_1, a_2\}$. Note that the indifference line between a_0 and a_1 does not intersect the one between a_1 and a_2. For each belief, if a_0 is preferred to a_1, it is also preferred to a_2. Conversely, if a_2 is preferred to a_1, it is also preferred to a_0. Consider two binary-action decision problems $\{u_1, A_1\}$ and $\{u_2, A_2\}$ with $A_1 = \{a_0, a_1\}$ and $A_2 = \{a_1, a_2\}$ such that

$$u_1(a, \theta) = u(a, \theta) \, \forall a \in A_1, \theta \in \Theta,$$
$$u_2(a_1, \theta) = 0, \ u_2(a_2, \theta) = u(a_2, \theta) - u(a_1, \theta) \ \forall \theta \in \Theta.$$

For each belief, the value function of the original problem equals the sum of the value functions of these two binary-action decision problems.

The right panel of Figure 2 shows a non-BD decision problem since the indifference line between a_0 and a_1 intersects at P^* with the indifference line between

[11] Such σ^F may not exist because we are considering a decision problem that, in terms of value functions, is payoff-equivalent to a collection of binary-action decision problems rather than one that explicitly comprises a collection of binary-action decision problems. By Theorem 2 in Cheng and Borgers (2023), the existence of such σ^F is guaranteed when the action space is finite.

a_1 and a_2. In this case, if a_0 is preferred to a_1, it does not necessarily follow that a_0 is preferred to a_2.

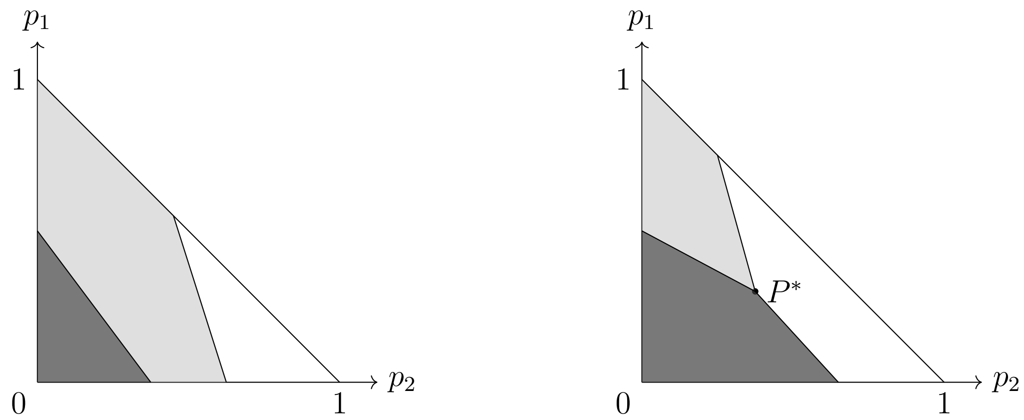

Figure 2: BD and non-BD decision Problems. Let $\Theta = \{\theta_0, \theta_1, \theta_2\}$. Consider a decision problem with $A = \{a_0, a_1, a_2\}$. It is characterized by a partition of the belief simplex $\{(p_1, p_2)|p_1 + p_2 \leq 1, p_1 \geq 0, p_2 \geq 0\}$ into three regions corresponding to different preferred actions. The dark grey area is the set of beliefs under which a_0 is preferred by the DM. The light grey area is the set of beliefs under which a_1 is preferred. The white area is the set of beliefs under which a_2 is preferred. The left panel illustrates a BD decision problem, while the right panel illustrates a non-BD decision problem.

We generalize the example above and establish a sufficient condition for a decision problem $\{A, u\}$ to be BD: there exists an ordinal ranking of the action space, i.e., $A \subset \mathbb{R}$, such that for each belief, every local optimum is also a global optimum. That is, when A is discrete, an action is optimal under a belief if it is preferred to its adjacent actions; when A is an interval of $\mathbb{R}$, an action is optimal under a belief if it satisfies the first-order condition. This condition is equivalent to requiring that the decision problem be quasi-concave.

Definition 9. *A decision problem $\{A, u\}$ with $A \subset \mathbb{R}$ is **quasi-concave (QCC)** if for each belief $\boldsymbol{p}$, the expected payoff $u(a, \boldsymbol{p})$ is quasi-concave in a.*

Kolotilin et al. (Forthcoming) considers a similar condition named aggregate single-crossing, and applies it to analyze the Bayesian persuasion problems.

Proposition 7. *A QCC decision problem is BD.*

Furthermore, for each QCC decision problem, we can find an ordinal order of the state space and establish the single-crossing property.

Lemma 7. *If a decision problem is QCC, there exists a permutation $\hat{\Theta}$ of Θ under which the decision problem is MBD.*

Finally, consider a *continuous* decision problem $\{A, u\}$ in which (i) A is an interval of $\mathbb{R}$, and (ii) $u(a, \theta)$ is continuous and differentiable in a for each θ.

Lemma 8. *A continuous decision problem $\{A, u\}$ is QCC if the set of optimal actions $A(\boldsymbol{p})$ is an interval or a singleton for each belief $\boldsymbol{p}$.*

In the left panel of Figure 2, the optimal action changes incrementally as the belief moves continuously from one point to another, while in the right panel, the optimal action may jump directly from a_1 to a_3. Lemma 8 generalizes this observation by using Berge's maximum theorem to ensure the continuity of $A(\boldsymbol{p})$.

Persico (2000) studies the continuous decision problems and derives the marginal value of information using the envelope theorem. In Appendix E, we generalize the approach in Persico (2000) and strengthen the results in Di Tillio et al. (2021) concerning the informativeness of order statistics.

When MLRP holds, PM order applies to all decision problems satisfying the single-crossing property since it is equivalent to Lehmann order by Proposition 6. Without MLRP, PM order further requires decision problems to be MBD. This is because, when MLRP holds, the posterior belief evolves along a single trajectory as the one-dimensional signal increases. We can eliminate actions that are dominated under each belief on the trajectory and construct a QCC decision problem that is payoff-equivalent to the original one (See Jewitt (2007), Quah and Strulovici (2009), and Di Tillio et al. (2021)). However, without MLRP, this approach fails since the beliefs are not restricted to a single trajectory.

5 Moral Hazard Problem

5.1 Discrete Action Space

Consider a moral hazard problem (i.e., a hidden-action agency model) between a principal and an agent, in the state-space formulation introduced by Wilson (1967), Spence and Zeckhauser (1971), and Ross (1973). The agent privately chooses an action (state) $\theta \in \Theta$ with $\Theta = \{\theta_0, \ldots, \theta_n\}$, incurring a cost $c(\theta)$. Conditional on the chosen action, a public signal $x \in [0, 1]$ is drawn according to a statistical experiment F. The principal's problem is to construct a contract (reward scheme) $s(x)$ that maps signals to payments for the agent. We follow Grossman and Hart (1983) and assume that the agent is risk-averse and his utility is additively separable in payment and cost. Specifically, when the agent chooses action θ and the realized signal is x, his total utility is

$$u(s(x)) - c(\theta),$$

in which u is a continuous concave function.

We assume that the principal can only offer bounded contracts, i.e., there exists $\underline{u}, \bar{u} \in \mathbb{R}$ such that for each contract $s(x)$ offered,[12]

$$\underline{u} \leq u(s(x)) \leq \bar{u} \ \ \forall x \in [0,1].$$

The principal cannot offer contracts with unbounded punishments or rewards.

A contract $s(\cdot)$ *implements* an action $\theta \in \Theta$ under F if it satisfies the incentive compatibility (IC) constraints,

$$E[u(s(x))|\theta; F] - c(\theta) \geq E[u(s(x))|\theta'; F] - c(\theta') \ \ \forall \theta' \in \Theta,$$

and the individual rationality (IR) constraint,

$$E[u(s(x))|\theta; F] - c(\theta) \geq 0.$$

Let $S^F(\theta)$ denote the set of contracts implementing θ under F. An action θ is *implementable* under F if $S^F(\theta)$ is not empty.

Consider the cost-minimization problem of a risk-neutral principal who seeks to implement θ. Let $W^F(\theta)$ denote the infimum of the expected payments generated by the contracts implementing θ under F,

$$W^F(\theta) = \inf_{s(\cdot) \in S^F(\theta)} E[s(x)|\theta; F].$$

If $S^F(\theta)$ is empty, i.e., the principal's minimization problem is not feasible, let

$$W^F(\theta) = \infty.$$

We apply LB order to compare statistical experiments in the moral hazard problem described above.

Theorem 4. *If the statistical experiment F dominates another statistical experiment G in LB order, then $W^F(\theta) \leq W^G(\theta)$ for each θ, each utility function $u(\cdot)$, and each cost function $c(\cdot)$.*

[12] In the discrete-signal case, it suffices to assume that $u(s(x)) \geq \underline{u}$ for each $x \in [0,1]$.

Theorem 4 is based on Proposition 13 in Jewitt (2007). We complete the proof by showing that if the principal's minimization problem is feasible under G, then it is feasible under F, thereby validating the strong duality using Slater's condition.[13]

We now provide a sketch of the proof. Instead of choosing a contract $s(x)$ mapping from the realized signal to the payment for the agent, let the principal choose a contract $v(x)$ mapping from the realized signal into the agent's utility, in which

$$v(x) = u(s(x)).$$

Rewrite the cost-minimizing problem of the principal as

$$\inf_{v(\cdot)} E[u^{-1}(v(x))|\theta; F],$$

with the IC constraints,

$$E[v(x)|\theta; F] - c(\theta) \geq E[v(x)|\theta'; F] - c(\theta') \ \forall \theta' \in \Theta,$$

and the IR constraint,

$$E[v(x)|\theta; F] - c(\theta) \geq 0.$$

Lemma 9. *Let $F \overset{LB}{\succ} G$. For each utility function $u(\cdot)$, cost function $c(\cdot)$, and action θ, if θ is implementable under G, then it is implementable under F.*

Proof. Consider a contract $v(\cdot)$ implementing θ under G. Let

$$\underline{v} = \inf_{y \in [0,1]} v(y),$$

$$\bar{v} = \sup_{y \in [0,1]} v(y).$$

By Proposition 3, there exists a contract $v'(\cdot)$ under F whose range is a subset of $[\underline{v}, \bar{v}]$ such that

$$E[v'(x)|\theta; F] = E[v(y)|\theta; G] \ \forall \theta.$$

Therefore, the contract $v'(\cdot)$ under F satisfies the IC and IR constraints and implements θ. $\square$

Let $L^F(v, \boldsymbol{\lambda}, \gamma; \theta)$ be the Lagrangian of the principal's minimization problem for implementing θ under F, in which $\boldsymbol{\lambda} = \{\lambda_0, \ldots, \lambda_n\}$ are the Lagrange multipliers

[13] See the discussion in Chi and Choi (2023).

for the IC constraints such that λ_i corresponds to the IC constraint between θ and θ_i, and γ is the Lagrange multiplier for the IR constraint. It follows that

$$W^F(\theta) = \inf_{v(\cdot)} \sup_{\lambda_i,\gamma\geq 0} L^F(v,\boldsymbol{\lambda},\gamma;\theta).$$

When θ is implementable under F, i.e., the principal's problem is feasible, Slater's condition is satisfied since (i) $u^{-1}(\cdot)$ is convex, and (ii) the IC and IR constraints are linear in $v(\cdot)$. Hence, strong duality holds with

$$\inf_{v(\cdot)} \sup_{\lambda_i,\gamma\geq 0} L^F(v,\boldsymbol{\lambda},\gamma;\theta) = \sup_{\lambda_i,\gamma\geq 0} \inf_{v(\cdot)} L^F(v,\boldsymbol{\lambda},\gamma;\theta). \tag{3}$$

It follows that

$$\inf_{v(\cdot)} L^F(v,\boldsymbol{\lambda},\gamma;\theta) = \gamma c(\theta) + \sum_i \lambda_i[c(\theta)-c(\theta_i)] - \int_0^1 \pi\left(\sum_i \lambda_i \left(1-\frac{f(x|\theta_i)}{f(x|\theta)}\right)+\gamma\right) f(x|\theta)dx,$$

where $\pi(\cdot)$ is a convex function defined by

$$\pi(t) = \sup_{v\in[\underline{u},\bar{u}]} [tv - u^{-1}(v)].$$

The condition that $F \overset{LB}{\succ} G$ implies that the random vector of likelihood ratios under F is more dispersed in the sense of the linear convex order. Therefore, for each pair of $\boldsymbol{\lambda}$ and γ,

$$\int_0^1 \pi\left(\sum_i \lambda_i \left(1-\frac{f(x|\theta_i)}{f(x|\theta)}\right)+\gamma\right) f(x|\theta)dx \geq \int_0^1 \pi\left(\sum_i \lambda_i \left(1-\frac{g(y|\theta_i)}{g(y|\theta)}\right)+\gamma\right) g(y|\theta)dx.$$

Hence,

$$\sup_{\lambda_i,\gamma\geq 0} \inf_{v(\cdot)} L^F(v,\boldsymbol{\lambda},\gamma;\theta) \leq \sup_{\lambda_i,\gamma\geq 0} \inf_{v(\cdot)} L^G(v,\boldsymbol{\lambda},\gamma;\theta).$$

Consider the case in which θ is implementable under G. By Lemma 9, it must also be implementable under F. Thus, by (3),

$$W^F(\theta) = \inf_{v(\cdot)} \sup_{\lambda_i,\gamma\geq 0} L^F(v,\boldsymbol{\lambda},\gamma;\theta) \leq \inf_{v(\cdot)} \sup_{\lambda_i,\gamma\geq 0} L^G(v,\boldsymbol{\lambda},\gamma;\theta) = W^G(\theta),$$

which completes the proof.

5.2 General Case

Hart and Holmstrom (1986) proposes a more general framework than the state-space formulation in Section 5.1, in which the agent directly chooses the signal distribution from a set of feasible distributions $\mathbb{P}$.[14] Because the agent can randomize, the set $\mathbb{P}$ is convex, and so is his cost function.

Specifically, we consider a modification of the basic model in Section 5.1: The agent privately chooses an action

$$\boldsymbol{\delta} = (\delta_0, \dots, \delta_n) \in \Delta[0,1]^{n+1}.$$

He incurs a cost $c(\boldsymbol{\delta})$ that is convex in $\boldsymbol{\delta}$. Given the agent's action, a public signal $x \in [0,1]$ is drawn according to a statistical experiment F with

$$F(x|\boldsymbol{\delta}) = \sum_{i=0}^{n} \delta_i F(x|\theta_i).$$

The agent is risk-averse and his utility is additively separable in payment and cost.

The model above can also be interpreted as a situation in which the agent has multiple tasks and needs to decide how to allocate time among them, which is related to Holmstrom and Milgrom (1991).

Analogously to Section 5.1, we consider the cost-minimization problem of a risk-neutral principal who seeks to implement $\boldsymbol{\delta}$. Let $W^F(\boldsymbol{\delta})$ denote the infimum of the expected payments generated by the contracts implementing $\boldsymbol{\delta}$ under F. If $\boldsymbol{\delta}$ is not implementable, let

$$W^F(\boldsymbol{\delta}) = \infty.$$

We generalize Theorem 4 and show that LB order dominance is a necessary and sufficient condition for comparing statistical experiments in this case.

Theorem 5. *$F \overset{LB}{\succ} G$ if and only if $W^F(\boldsymbol{\delta}) \leq W^G(\boldsymbol{\delta})$ for each action $\boldsymbol{\delta}$, utility function $u(\cdot)$, and cost function $c(\cdot)$.*

Furthermore, consider a situation in which the principal observes a signal x from F, and can additionally receive a signal y from G, which is potentially correlated with x. When is the additional statistical experiment G valuable? Let $F \times G$ be the experiment combining F and G, which characterizes the joint distribution of the pair of signal (x, y). By Theorem 5, the additional statistical experiment G provides no value to the principal if and only if $F \times G \overset{LB}{\sim} F$. By Lemma 6,

Corollary 4. *Given the existing statistical experiment F, an additional statistical experiment G provides no value to the principal if and only if $F \times G \overset{B}{\sim} F$.*

[14] See also Georgiadis and Szentes (2020), Georgiadis (2022), and Georgiadis et al. (2024).

Corollary 4 strengthens the informativeness principle in Holmström (1979), which applies to the binary action case and to cases in which the first-order approach is valid. It also completes the results in Gjesdal (1982) and Grossman and Hart (1983) obtained using Blackwell order.

6 Concluding Remarks

This paper contributes to the literature on comparisons of statistical experiments pioneered by Blackwell et al. (1951). While we consider the situation in which the DM observes a single signal, Moscarini and Smith (2002) and Mu et al. (2021) examine the situation in which the DM observes a large number of signals through repeated sampling. Mu et al. (2021) considers the binary-state case and show that an experiment is more informative than another in large samples if and only if it has higher Rényi divergence. The multi-state case remains unresolved due to the complexity of convex order in multidimensions, suggesting a natural extension to analyze LB and PM orders in large samples and their connection to Rényi divergence.

Appendices

The appendices proceed as follows:

1. Appendix A presents a characterization of Blackwell order in the binary-state case.

2. Appendix B discusses complementarity and substitutability concerning two experiments.

3. Appendix C demonstrates a sufficient condition under which LB order is equivalent to Blackwell order when the signal space is discrete.

4. Appendix D provides relations between PM order and several variations of Lehmann order.

5. Appendix E discusses the marginal value of information and its application to informativeness of order statistics.

6. Appendix F presents omitted proofs.

Appendix A Blakcwell order in the Binary-State Case

Let $\Theta = \{\theta_0, \theta_1\}$. Consider two experiments F and G. Without loss of generality, assume that F and G satisfy MLRP by ordering signals based on the likelihood ratio.

For each $t \in [0, 1]$, select $x(t)$ such that

$$x(t) = \inf_{x \in [0,1]} \{x | F(x|\theta_0) = t\}.$$

Define the *Lorenz curve* $L^F : [0, 1] \to [0, 1]$ by

$$L^F(t) = F(x(t)|\theta_1).$$

Note that L^F is convex due to MLRP. Let L^G denote the Lorenz curve of G.

Based on Lehmann order—and noting that in the binary-state case, Lehmann order is equivalent to Blackwell order,[15]

Lemma 10. $F \overset{B}{\succ} G \iff L^F(t) \leq L^G(t), \forall t \in [0, 1]$.

Appendix B Complementarity and Substitutability of Experiments

B.1 Basic Setting

Börgers et al. (2013) introduces novel notions of complementarity and substitutability of two experiments in the spirit of Blackwell order. Consider two experiments F and G with generic signals, x and y respectively. Let $F \times G$ be the experiment combining F and G, which characterizes the joint distribution of the pair of signals (x, y). Based on the definition of the value function in Section 4.1,

Definition 10. *An experiment F* ***complements*** *another experiment G if for each decision problem,*

$$V^F(\boldsymbol{q}) - V^{\emptyset}(\boldsymbol{q}) \leq V^{F\times G}(\boldsymbol{q}) - V^G(\boldsymbol{q}),\ \forall \boldsymbol{q}.$$

Definition 11. *An experiment F* ***substitutes*** *another experiment G if for each*

[15] See Jewitt (2007).

decision problem,

$$V^F(\boldsymbol{q}) - V^{\emptyset}(\boldsymbol{q}) \geq V^{F\times G}(\boldsymbol{q}) - V^G(\boldsymbol{q}),\ \forall \boldsymbol{q}.$$

Note that $V^{\emptyset}(\boldsymbol{q})$ is the value function in the case where the principal has no information and chooses the action solely based on her prior belief.

Börgers et al. (2013) provides necessary and sufficient conditions for complementarity and substitutability of two experiments in the binary-binary case, where both the state space and the signal space are binary. We provide necessary and sufficient conditions in the case where the state space is binary, and extend the results to the case with more than two states in the spirit of LB order and PM order.

B.2 Binary-State Case

Consider Lorenz curves L^F and L^G of experiments F and G, respectively. Let ψ_F and ψ_G be the right derivatives of L^F and L^G such that

$$L^F(t) = \int_0^t \psi_F(k)dk,$$

$$L^G(t) = \int_0^t \psi_G(k)dk.$$

Since both L^F and L^G are increasing and convex, both ψ_F and ψ_G are non-negative and increasing. Define $\psi_{F\oplus G}$ by

$$\psi^{-1}_{F\oplus G}(k) = \frac{1}{2}\psi^{-1}_F(k) + \frac{1}{2}\psi^{-1}_G(k), \forall k \in [0,1].$$

where

$$\psi^{-1}(k) = inf\{t|\psi(t) = k\}.$$

Define $L^F(t) \oplus L^G(t)$ by

$$L^F(t) \oplus L^G(t) = \int_0^t \psi_{F\oplus G}(k)dk, \forall t.$$

Let $L^{\emptyset}(t) = t$ for each $t \in [0,1]$. Based on Lemma 10,

Proposition 8. *An experiment F complements another experiment G if and only if*

$$L^F(t) \oplus L^G(t) \leq L^{F\times G}(t) \oplus L^{\emptyset}(t), \forall t \in [0,1].$$

Proposition 9. *An experiment F substitutes another experiment G if and only if*

$$L^F(t) \oplus L^G(t) \geq L^{F\times G}(t) \oplus L^{\emptyset}(t), \forall t \in [0,1].$$

B.3 General Case

We provide a tractable framework to characterize the complementarity and substitutability of two experiments in the spirit of LB order and PM order.

Definition 12. *An experiment F* ***LB-complements*** *another experiment G if for each BD decision problem,*

$$V^F(\boldsymbol{q}) - V^{\emptyset}(\boldsymbol{q}) \leq V^{F\times G}(\boldsymbol{q}) - V^G(\boldsymbol{q}),\ \forall \boldsymbol{q}.$$

Conversely, an experiment F ***LB-substitutes*** *another experiment G if for each BD decision problem,*

$$V^F(\boldsymbol{q}) - V^{\emptyset}(\boldsymbol{q}) \geq V^{F\times G}(\boldsymbol{q}) - V^G(\boldsymbol{q}),\ \forall \boldsymbol{q}.$$

Definition 13. *An experiment F* ***PM-complements*** *another experiment G if for each MBD decision problem,*

$$V^F(\boldsymbol{q}) - V^{\emptyset}(\boldsymbol{q}) \leq V^{F\times G}(\boldsymbol{q}) - V^G(\boldsymbol{q}),\ \forall \boldsymbol{q}.$$

Conversely, an experiment F ***PM-substitutes*** *another experiment G if for each MBD decision problem,*

$$V^F(\boldsymbol{q}) - V^{\emptyset}(\boldsymbol{q}) \geq V^{F\times G}(\boldsymbol{q}) - V^G(\boldsymbol{q}),\ \forall \boldsymbol{q}.$$

Let $Z(\emptyset)$ denote set $\{(t,\dots,t)|t \in [0,1]\} \subset \mathbb{R}^{n+1}$. Based on Proposition 3,

Proposition 10. *An experiment F LB-complements another experiment G if and only if*

$$Z(F) + Z(G) \subset Z(F \times G) + Z(\emptyset),$$

where the operator "+" is the Minkowski sum.

Proposition 11. *An experiment F LB-substitutes another experiment G if and only if*

$$Z(F \times G) + Z(\emptyset) \subset Z(F) + Z(G).$$

Furthermore, based on Proposition 4 and Proposition 5,

Proposition 12. *An experiment F LB-complements another experiment G if and only if for each weighted dichotomy $\mathcal{W}$, the experiment $F^{\mathcal{W}}$ complements the experiment $G^{\mathcal{W}}$. Conversely, an experiment F LB-substitutes another experiment G if and only if for each weighted dichotomy $\mathcal{W}$, the experiment $F^{\mathcal{W}}$ substitutes the experiment $G^{\mathcal{W}}$.*

Proposition 13. *An experiment F PM-complements another experiment G if and only if for each monotone weighted dichotomy $\mathcal{W}$, the experiment $F^{\mathcal{W}}$ complements the experiment $G^{\mathcal{W}}$. Conversely, an experiment F PM-substitutes another experiment G if and only if for each monotone weighted dichotomy $\mathcal{W}$, the experiment $F^{\mathcal{W}}$ substitutes the experiment $G^{\mathcal{W}}$.*

Appendix C Equivalence between Blackwell Order and LB Order

Consider an experiment F with a discrete signal space

$$X = \{x_0, \ldots, x_m\}.$$

The experiment F is characterized by the probability matrix $M^F = (m_{ij}^F)$ such that

$$m_{ij}^F = Pr(x_j|\theta_i; F).$$

Without loss of generality, assume that every pair of columns of M^F is linearly independent.

Using Theorem 2 in Wu (2023) and Lemma 5,

Proposition 14. *Consider experiments F and G with discrete signal spaces. If $rank(M^F) = \min\{n+1, m\}$, then $F \overset{B}{\succ} G \Longleftrightarrow F \overset{LB}{\succ} G$.*

Appendix D Variations of Lehmann Order

Jewitt (2007) proposes a ranking, called *L-order*, based on comparing distributions of likelihood ratios across each dichotomy. Define $F \overset{L}{\succ} G$ if F dominates G in L-order.

Proposition 15. *(i) $F \overset{L}{\succ} G \implies F \overset{PM}{\succ} G$. (ii) If F satisfies MLRP, then $F \overset{L}{\succ} G \Longleftrightarrow F \overset{PM}{\succ} G$.*

Kim (2023) proposes a ranking, called *Monotone-Quasi-Gambling order*, based on adding reversely monotone noise. Define $F \overset{MQG}{\succ} G$ if F dominates G in Monotone-Quasi-Gambling order.

Proposition 16. *(i) If F satisfies MLRP, then $F \overset{PM}{\succ} G \implies F \overset{MQG}{\succ} G$. (ii) If G satisfies MLRP, then $F \overset{MQG}{\succ} G \implies F \overset{PM}{\succ} G$.*

Di Tillio et al. (2021) proposes a ranking, called *accuracy order*, by extending Lehmann order to experiments that generate multi-dimensional signals satisfying the generalized MLRP. Define $F \overset{A}{\succ} G$ if F dominates G in accuracy order.

Proposition 17. *If both F and G satisfy generalized MLRP, then $F \overset{A}{\succ} G \implies F \overset{PM}{\succ} G$.*

Appendix E Marginal Value of Information

E.1 Basic Setting

Consider a DM who faces a decision problem $\{A, u\}$ and holds a uniform prior[16] $\boldsymbol{q}$ over Θ. The DM observes a signal $x^\eta \in X^\eta$ from an experiment F^η chosen from a family of experiments $\{F^\eta\}_{\eta\in[0,1]}$. For brevity, denote the ex ante expected payoff $V^{F^\eta}(\boldsymbol{q})$, as defined in Section 4.1, by $V(\eta)$. Assume that $V(\eta)$ is differentiable in η.

The *marginal value of information* is defined as

$$MR(\eta) := \frac{\partial}{\partial \eta} V(\eta).$$

This section analyzes the marginal value of information in MBD decision problems. Without loss of generality, consider a binary-choice decision problem with

$$A = \{a_0, a_1\},$$

that satisfies the single-crossing property. Normalize the payoff such that

$$u(a_1, \theta_i) = 0 \ \forall\, i,$$

and, for notational simplicity, denote

$$u(\theta) = u(a_0, \theta).$$

[16]This is without loss of generality in this section since varying the prior is equivalent to varying the payoff intensity of the decision problem.

There exists a cut-off $\hat{\theta}$ such that

$$
\begin{aligned}
u(\theta) &\geq 0, \text{ if } \theta \leq \hat{\theta}, \\
u(\theta) &\leq 0, \text{ if } \theta > \hat{\theta}.
\end{aligned}
$$

Definition 14. *The family of experiments* $\{F^{\eta}\}_{\eta\in[0,1]}$ *is* ***welfare-increasing*** *at* $\hat{\eta} \in [0,1]$ *if for each MBD decision problems,*

$$MR(\hat{\eta}) \geq 0.$$

Note that if the family of experiments $\{F^{\eta}\}_{\eta\in[0,1]}$ is welfare-increasing at all $\hat{\eta} \in [0,1]$, then for any pair of $\eta' < \eta''$, the experiment $F^{\eta''}$ dominates $F^{\eta'}$ in the PM order. Moreover, if both experiments satisfy MLRP, this PM order dominance is equivalent to dominance in the Lehmann order (See Proposition 6).

E.2 Single Dimension with MLRP

Consider the univariate case with

$$X^{\eta} = [0,1] \ \forall \eta.$$

Assume that F^{η} satisfies MLRP for each η.

Under the experiment F^{η}, the DM's optimal strategy is characterized by a cut-off $\hat{x}^{\eta}$ such that

$$\sum_{i=0}^{n} u(\theta_i) f^{\eta}(\hat{x}^{\eta}|\theta_i) = 0. \tag{4}$$

The DM chooses a_0 when $x \leq \hat{x}^{\eta}$ and chooses a_1 otherwise. It follows that

$$V^{\eta} = \sum_{i=0}^{n} u(\theta_i) F^{\eta}(\hat{x}^{\eta}|\theta_i).$$

Using envelop theorem and (4),

$$MR(\eta) = \sum_{i=0}^{n} \left[u(\theta_i) \cdot \left. \frac{\partial F^{\eta}(x|\theta_i)}{\partial \eta} \right|_{x=\hat{x}^{\eta}} \right].$$

Rewrite it as

$$MR(\eta) = \sum_{i=0}^{n} \left[u(\theta_i) f^{\eta}(\hat{x}^{\eta}|\theta_i) \cdot K(\hat{x}^{\eta}, \theta_i, \eta) \right],$$

where

$$K(x, \theta, \eta) = \frac{1}{f^{\eta}(x|\theta)} \frac{\partial F^{\eta}(x|\theta)}{\partial \eta}.$$

Using Lemma 1 in Persico (2000),

Proposition 18. *The family of experiments $\{F^\eta\}_{\eta\in[0,1]}$ is welfare-increasing at $\hat{\eta}\in[0,1]$ if and only if, for each $x\in[0,1]$, the function $K(x,\theta,\hat{\eta})$ is decreasing in θ.*

E.3 Multi Dimensions with Generalized MLRP

Consider the case in which

$$X^\eta = [0,1]^N \ \ \forall \eta,$$

with $N > 1$. Assume that F^η satisfies generalized MLRP for every η, i.e., for each pair of vectors x' and x with $x' > x$ (componentwise), the likelihood ratio

$$\frac{f(x'|\theta)}{f(x|\theta)}$$

is increasing in θ.

For a binary-action decision problem under the experiment F^η, we can partition the set of signal X into three subsets D_1^η, D_2^η and ΔD^η such that

$$\begin{aligned} D_1^\eta &= \left\{ x \middle| \sum_{i=0}^{n} u(\theta_i) f^\eta(x|\theta_i) > 0 \right\}, \\ D_2^\eta &= \left\{ x \middle| \sum_{i=0}^{n} u(\theta_i) f^\eta(x|\theta_i) < 0 \right\}, \\ \Delta D^\eta &= \left\{ x \middle| \sum_{i=0}^{n} u(\theta_i) f^\eta(x|\theta_i) = 0 \right\}. \end{aligned}$$

The DM's optimal strategy—i.e., the cutoff rules—is characterized by ΔD^η. Furthermore, the subsets D_1^η and D_2^η display a monotonicity property,

$$\begin{aligned} x \in D_1^\eta &\implies x' \in D_1^\eta \ \forall x' < x, \\ x \in D_2^\eta &\implies x' \in D_2^\eta \ \forall x' > x. \end{aligned}$$

Similar to the univariate case,

$$MR(\eta) = \sum_{i=0}^{n} \left[u(\theta) \cdot \left. \frac{\partial F^\eta(D|\theta_i)}{\partial \eta} \right|_{D=D_1^\eta} \right].$$

Because the optimal strategy is no longer characterized by a single cut-off, incorporating the first-order condition into the marginal revenue becomes more challenging.

For a vector $x = (x_1, \dots, x_n)$, let $x_{1:k} = (x_1, \dots, x_k)$ denote the subvector consisting of the first k components. Denote the marginal density of $x_{1:k}$ by f_k.

Proposition 19. *The family of experiments $\{F^\eta\}_{\eta\in[0,1]}$ is welfare-increasing at $\hat{\eta} \in [0,1]$ if for each $k \in \{1, \dots, N\}$ and each $x \in [0,1]^N$, the ratio*

$$\frac{1}{f_k^\eta(x_{1:k}|\theta)} \frac{\partial}{\partial \eta} \left[\int_0^{x_k} f_k^\eta(x_{1:k-1}, t) dt \right]$$

is decreasing in θ.

E.4 General Case

We can interpret the posterior-belief vector $\boldsymbol{p} = \{p_0, p_1, \dots, p_n\}$ as the signal. The c.d.f and density remain denoted by F^η and f^η, respectively. Furthermore,

$$\sum_{i=0}^{n} p_i = 1,$$

$$\frac{f(p|\theta_i)}{\sum_i f(p|\theta_i)} = p_i,$$
$$\sum_i E(p_j|\theta_i) = 1 \ \forall j.$$

Partition the set of posterior beliefs into three subsets,

$$D_1^\eta = \left\{ p | \sum_i u(\theta_i) p_i > 0 \right\},$$
$$D_2^\eta = \left\{ p | \sum_i u(\theta_i) p_i < 0 \right\},$$
$$\Delta D^\eta = \left\{ p | \sum_i u(\theta_i) p_i = 0 \right\}.$$

It follows that

$$MR(\eta) = \sum_{i=0}^{n} \left[u(\theta) \cdot \left. \frac{\partial F^\eta(D|\theta_i)}{\partial \eta} \right|_{D=D_1^\eta} \right].$$

However, we cannot obtain the same result as in Proposition 19 because the analogous monotonic property does not hold for D_1^η and D_2^η.

E.5 Sample Selection

We apply Proposition 18 and Proposition 19 to study sample selection. Consider a signal $x \in [0,1]$ drawn from an experiment F satisfying MLRP. Draw the signal N times, with the draws being conditionally i.i.d. Let z_i be the i^{th} highest value among the N draws, and consider a *selected sample* with size k with $k < N$,

$$z^k = \{z_1, \ldots, z_k\}.$$

Denote the distribution of z^k as $F^{N,k}$. Does $F^{N,k}$ become more or less informative as N increases?

Di Tillio et al. (2021) has studied this problem by extending Lehmann order to the generalized MLRP case in a novel manner. They focus on effectiveness in statistical decision theory; specifically, an experiment F is more effective than an experiment G if, for each decision problem and each strategy under G, there exists a strategy under F that yields a higher expected payoff for any prior.

This section focuses on informativeness in Bayesian decision theory,[17]. Specifically, one experiment F is more informative than another experiment G if, for each decision problem and each prior, the optimal strategy under F yields a higher ex ante expected payoff than the optimal strategy under G. By focusing on the MBD decision problems, we can always identify the optimal strategy using the first-order condition, and leverage the envelope theorem to control for the effects of varying optimal strategies.

Proposition 20. *For each k, N, and N' such that $k \leq N < N'$, the experiment $F^{N',k}$ dominates $F^{N,k}$ in PM order if $-log[F(x|\theta)]$ is log-supermodular.*

Proposition 21. *For each N and N' such that $N < N'$, the experiment $F^{N,1}$ dominates $F^{N',1}$ in PM order if $-log[F(x|\theta)]$ is log-submodular.*

Appendix F Omitted Proofs

F.1 Proofs of Results in Section 3 and Section 4

From Definition 4 and (1),

Lemma 11. $F \overset{LB}{\succ} G$ *if and only if for each vector $\boldsymbol{b} = \{b_0, b_1, \ldots, b_n\} \in \mathbb{R}^{n+1}$,*

$$\int_0^1 \left[\sum_{i=0}^n b_i \cdot f(x|\theta_i)\right]_+ dx \geq \int_0^1 \left[\sum_{i=0}^n b_i \cdot g(y|\theta_i)\right]_+ dy.$$

Furthermore,

[17] Effectiveness in statistical decision theory is a stronger notion than Informativeness in Bayseion decision theory. Chi (2014) establishes the equivalence between these two terms for some special cases.

Lemma 12. *$F \overset{LB}{\succ} G$ if and only if for each vector $\boldsymbol{b} = \{b_0, b_1, \ldots, b_n\} \in \mathbb{R}^{n+1}$ satisfying the single-crossing property,*

$$\int_0^1 \left[\sum_{i=0}^{n} b_i \cdot f(x|\theta_i) \right]_+ dx \geq \int_0^1 \left[\sum_{i=0}^{n} b_i \cdot g(y|\theta_i) \right]_+ dy.$$

Proof. From Definition 3 and (1), the experiment $F \overset{PM}{\succ} G$ if and only for each $t \in \mathbb{R}$ and each prior belief $\boldsymbol{q} = \{q_0, \ldots, q_n\} \in \Delta[0,1]^{n+1}$,

$$\int_0^1 \left[\sum_{i=0}^{n} (\theta_i - t) \cdot q_i \cdot f(x|\theta_i) \right]_+ dx \geq \int_0^1 \left[\sum_{i=0}^{n} (\theta_i - t) \cdot q_i \cdot g(y|\theta_i) \right]_+ dy. \tag{5}$$

Consider a vector $\boldsymbol{b}(\boldsymbol{q}, t)$ with

$$b_i(q, t) = (\theta_i - t) \cdot q_i \;\; \forall i \in \{0, \ldots, n\}.$$

Note that for each t and each prior belief $\boldsymbol{q}$, the vector $\boldsymbol{b}(\boldsymbol{q}, t)$ satisfies the single-crossing property. Conversely, for each vector $\boldsymbol{b}$ satisfying the single-crossing property, we can find t, a prior belief $\boldsymbol{q}$, and $K \in \mathbb{R}^+$ such that

$$\boldsymbol{b} = K \cdot \boldsymbol{b}(\boldsymbol{q}, t)$$

□

Using Lemmas 11 and 12, we can prove the results in Sections 3 and 4 through simple algebra.

F.2 Proof of Theorem 5

Let the agent choose the action

$$\boldsymbol{\delta} = (\delta_1, \ldots, \delta_n)$$

with

$$\delta_i \in [0,1] \;\; \forall i \in \{1, \ldots, n\},$$
$$\sum_{i=1}^{n} \delta_i \leq 1.$$

The agent incurs a cost $c(\boldsymbol{\delta})$ that is convex in $\boldsymbol{\delta}$. Given the agent's action, a public signal $x \in [0,1]$ is drawn according to a statistical experiment F with

$$F(x|\boldsymbol{\delta}) = \sum_{i=1}^{n} \delta_i F(x|\theta_i) + \left(1 - \sum_{i=1}^{n} \delta_i\right) F(x|\theta_0).$$

Without loss of generality, consider when the principal wishes to implement an interior action $\hat{\boldsymbol{\delta}}$ with

$$\hat{\delta}_i \in (0,1), \forall i \in \{1, \ldots, n\},$$

$$\sum_{i=1}^{n} \hat{\delta}_i < 1.$$

Given a contract $v(\cdot)$, define the agents net-payoff as

$$U(\boldsymbol{\delta}; v) = \int_0^1 v(x) dF(x|\boldsymbol{\delta}) - c(\delta)$$

We can write the principal's problem under the experiment F as

$$\inf_{v(\cdot)} E[u^{-1}(v(x))|\hat{\boldsymbol{\delta}}; F],$$

with the IC constraints,

$$U(\hat{\boldsymbol{\delta}}; v) \geq U(\boldsymbol{\delta}'; v) \ \ \forall \boldsymbol{\delta}',$$

and the IR constraint,

$$U(\hat{\boldsymbol{\delta}}; v) \geq 0.$$

Note that the agency's payoff is linear in $\boldsymbol{\delta}$ while his cost is convex in $\boldsymbol{\delta}$. Therefore, his net-payoff $U(\boldsymbol{\delta}; v)$ is concave in $\boldsymbol{\delta}$. We can replace the global IC constraints by local IC constraints,

$$\frac{\partial}{\partial \delta_i} U(\boldsymbol{\delta}; v)\bigg|_{\boldsymbol{\delta}=\hat{\boldsymbol{\delta}}} = 0 \ \ \forall i \in \{1, \ldots, n\}.$$

That is

$$E[v(x)|\theta_i; F] - E[v(x)|\theta_0; F] = \frac{\partial}{\partial \delta_i} c(\boldsymbol{\delta})\bigg|_{\boldsymbol{\delta}=\hat{\boldsymbol{\delta}}}.$$

Therefore, we can prove the sufficiency by using the approach in Section 5.1. The necessity is proved by constructing supporting hyperplanes.

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