Truth-Telling Dominating Strategy:
Impossibilities of Shill-Proofness*

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Abstract

A strategy is truth-telling dominating (TTD) if it weakly (strictly, resp.) dominates truth-telling for all (some, resp.) strategy profiles of others. A strategy is iteratively TTD (i-TTD) if any iterate is TTD and payoff improving. We show that any mechanism only with undominated equilibria is i-TTD non-manipulable. We also show that any TTD shill-bidding strategy is i-TTD. The Vickrey-Clarke-Groves (VCG) mechanism is not shill-proof, but neither the existence nor the nonexistence of TTD strategy had previously been known. We show both that VCG is TTD manipulable when externalities exist, but TTD non-manipulable in package auctions without externalities.

Keywords: obvious strategy-proofness, manipulability, VCG, shill bidding, package auction, externalities, referral, referrer's dilemma

JEL Classification: C78, D44, D62, D82

1 Introduction

Strategy-proofness is one of the most desirable properties in market design, but many non-strategy-proof mechanisms are widely used, e.g., the first-price auction, the generalized second-price auction (Edelman et al., 2007; Varian, 2007), core-selecting package auctions (Day and Milgrom, 2008; Day and Raghavan, 2007), the Anglo-Dutch auction (Binmore and

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Klemperer, 2002), the Deferred Acceptance algorithm (Gale and Shapley, 1962),[1] the Boston mechanism (Abdulkadiroğlu and Sönmez, 2003). However, it is often unclear whether there exists a strategy that weakly dominates truth-telling. If such a strategy exists and is known to the players, it is unlikely that the players will report truthfully. Otherwise, truth-telling may still be used for various reasons.

To the best of our knowledge, truth-telling dominance has not been formally studied. A strategy is (weakly) *truth-telling dominating* (TTD) if it weakly dominates truth-telling for all strategy profiles of others, and strictly dominates truth-telling for some strategy profile of others.[2] A mechanism is said to be *TTD manipulable* if there exists a TTD strategy for some agent.

One may think that a TTD strategy is too strong to exist, but it may trivially exist in some mechanisms. For instance, the first-price auction is TTD manipulable, since any underbidding weakly dominates truthful bidding. However, no one would think that the first-price auction is "easy" to manipulate in terms of the bidder's payoff maximization without any knowledge of other bidders. Note that by definition, TTD is "detail-free" as in the spirit of the Wilson doctrine (Wilson, 1987).

As a refinement, a strategy is *iteratively* TTD (i-TTD) if any *iterate*[3] of the strategy is also TTD and the $(k+1)$-th iterate weakly dominates the k-th iterate for all $k \in \mathbb{N}$.[4] As an i-TTD strategy iterates, the payoff is weakly increasing. A mechanism is said to be *i-TTD manipulable* if there exists an i-TTD strategy for some agent.

For instance, in the first-price auction, for a bidder with value θ, $\sigma(\theta) = \theta/2$ is TTD and $\sigma^2(\theta) = \theta/2^2$ is also TTD, but $\sigma^2(\theta)$ does not weakly dominate $\sigma(\theta)$; thus, $\sigma(\theta)$ is TTD but not i-TTD. Furthermore, one can easily see that any TTD bidding strategy should be a combination of underbidding and truth-telling, and for any two different TTD bidding strategies, neither weakly dominates the other; thus, there is no i-TTD bidding strategy. This may well reflect that the first-price auction is not easy to manipulate in practice, i.e., one may not always benefit from underbidding compared with less underbidding.

Is an iterative TTD strategy too strong to exist? Yes and no. If we only consider a direct mechanism in which the strategy (or message) space is the same as the type space (which we call a *canonical strategy space*), the answer is still yes and no. It may be yes for most nontrivial mechanisms that are used in practice, since otherwise the mechanism may not be

[1] The deferred acceptance algorithm is strategy-proof for the proposing side only.

[2] There is some technicality to avoid a trivial nonexistence, see Definition 1 and footnote 9. In addition, since weak TTD is already a strong requirement, a strict TTD strategy may not exist except for some trivial mechanisms. Thus, we will use TTD without "weak" to mean weak TTD throughout the paper.

[3] As in the iterate of a function, e.g., $\sigma^2 \equiv \sigma \circ \sigma$ and $\sigma^3 \equiv \sigma \circ \sigma \circ \sigma$.

[4] That is, $\sigma(\cdot)$ is i-TTD if for all $k \in \mathbb{N}$ and all type θ, $\sigma^k(\theta)$ is TTD and $\sigma^{k+1}(\theta)$ weakly dominates $\sigma^k(\theta)$. As in TTD, there is some technicality to avoid a trivial nonexistence, see Definition 2.

used (unless an i-TTD strategy is unknown yet). However, there are not only trivial but also nontrivial direct mechanisms that are i-TTD manipulable. Thus, characterizing i-TTD manipulable direct mechanisms may be of interest theoretically as well as practically.

In contrast, for indirect mechanisms (or more unambiguously, mechanisms with a non-canonical strategy space), the answer is no even for some widely used mechanisms. It is well-known that the Vickrey-Clarke-Groves (VCG) mechanism (Vickrey, 1961; Clarke, 1971; Groves, 1973) is strategy-proof under quite general conditions, which in turn implies that the VCG mechanism is not TTD manipulable with a canonical strategy space. Perhaps surprisingly, however, we show that if we allow a non-canonical strategy space, the VCG mechanism is i-TTD manipulable, e.g., i-TTD shill-bidding strategies exist in the VCG auction with externalities.

More generally, when we allow non-canonical strategy spaces for an arbitrary mechanism, checking TTD (i.e., not even i-TTD) manipulability may not be an easy question. In the case of the first-price auction, the nonexistence of an i-TTD *bidding* (i.e., canonical) strategy is easy to see. However, for an arbitrary mechanism with a non-canonical strategy space, neither the existence nor the nonexistence of a TTD strategy is obvious, since there can be nontrivial manipulation strategies, e.g., shill-bidding, merger, split, and collusion in auctions; and pre-arrangement in matching (Sönmez, 1999). Shill bidding—bidders may use additional fake identities and submit multiple bids—is a good example to see this non-obviousness. In Section 3, for the VCG mechanism, we show the existence of an i-TTD (i.e., not just TTD) shill-bidding strategy in one environment, but we also show the nonexistence of a TTD (i.e., not just i-TTD) shill-bidding strategy in another environment.

That is, it may be possible to show the nonexistence of (i-)TTD strategies for a given (or certain class of) mechanism and message space, but it may be quite difficult or even impossible to show the nonexistence for all message spaces even for one mechanism, since there may still be some undiscovered forms of manipulation beyond our imagination.

By definition, in practice, i-TTD manipulability may be more important than just TTD manipulability. We show that any mechanism only with undominated non-degenerate equilibria is i-TTD non-manipulable with a canonical strategy space. This simple result is useful to exclude the i-TTD manipulability of many usual mechanisms with a canonical strategy space, e.g., the first-price auction. One important class of TTD strategies with a non-canonical strategy space is shill bidding in auctions. We show that (under mild conditions) any TTD shill-bidding strategy must be i-TTD. Thus, when there exists a TTD shill-bidding strategy, the mechanism may be undesirable for the seller.

We also provide some necessary conditions for i-TTD strategies: no i-TTD strategy is *periodic*, i.e., if some (non-zero) iterate of the strategy becomes truth-telling, the strategy is not i-TTD; and no i-TTD strategy is *bijective*. Interestingly, i-TTD does not necessarily require some "rich" strategy space, e.g., there exists an i-TTD manipulable mechanism with a canonical space. However, any i-TTD strategy cannot be bijective at least. We also show some conditions for the supports of the iterates of an i-TTD strategy to form certain mathematical structure such as a chain, a tree, and a lattice.

Impossibilities of Shill-Proofness For the main application of TTD, we show impossibility results on shill-proofness, which originally motivated this paper and has its own importance in theory and practice. It is well-known that the VCG mechanism is not shill-proof in package auctions, i.e., a bidder may be better off using an additional false-name bid (Ausubel and Milgrom, 2002, 2006; Yokoo et al., 2004; Lehmann et al., 2006; Sher, 2012).[5] For instance, even a bidder who would otherwise lose may win by shill bidding and pay nothing in some cases. Especially on the Internet where a fake identity can easily be created, shill bidding can be a serious problem. However, to the best of our knowledge, both the existence and nonexistence of a TTD shill-bidding strategy has been unknown. A shill-bidding strategy is TTD if it weakly dominates truthful bidding regardless of others' bidding, i.e., other bidders are also allowed to use shill bidding or misreporting. Some may think that a TTD shill-bidding strategy is too strong to exist, and others may not.

Perhaps surprisingly, this paper is the first to show both (in two different environments). When externalities exist, even in single-item auctions, there exist i-TTD shill-bidding strategies. In contrast, when externalities do not exist, even in package auctions, there exists no TTD shill-bidding strategy.

The first result—the existence of a TTD shill-bidding strategy when externalities exist—is quite strong in the following senses. Normally when there exists an impossibility result, the next step is to find a preference domain where some positive results survive. For instance, the aforementioned literature shows some conditions under which the VCG package auction (without externalities) is shill-proof. However, we show that in the VCG mechanism with externalities, there always exist a shill bid that results in zero revenue for the seller. Even when we restrict attention to the TTD shill-bidding strategies, while they do not always reduce the revenue to zero, they can still be "powerful" for shill bidders: First, for the winner, there always exists a profitable shill bidding unless there is a tie for the winner. Second, a losing bidder can always reduce his payment to zero by using shill bidding.[6] Third, it is "robust" to resale prohibition in the sense that the winner does not change, even if shills are

[5] See Rothkopf et al. (1990), Ausubel and Milgrom (2002), Ausubel and Milgrom (2006), and Rothkopf (2007) for other undesirable properties of the VCG mechanism.

[6] When externalities exists, a losing bidder may pay in the VCG mechanism or in any core outcomes in order to avoid undesirable allocations (see Jehiel et al. (1996); Jeong (2020a), for instance).

used. Thus, there is no need for the "real" bidder to worry about transferring the item from the shill bidder. Finally, it is also difficult to detect in the following sense: in addition to aforementioned no allocation change, shill bidding does not change other bidders' payoffs. Thus, unlike a shill bidding which can make some other bidder worse off in package auctions, even if other bidders may suspect shill bidding, they may care less.

In addition, the result holds more generally. What the literature usually means by the name "VCG" is the so-called "pivot" rule, i.e., one specific rule of the Groves' scheme. We show that there exists a TTD shill-bidding strategy in any Groves mechanisms. Furthermore, we also show that there is no efficient, individually rational, and shill-proof mechanism.

Related Literature In terms of quantifying the manipulability of a mechanism, Pathak and Sönmez (2013), Carroll (2013), and Troyan and Morrill (2020) are the most related ones. In particular, Troyan and Morrill (2020) introduce *obvious manipulation* inspired by Li (2017): a manipulation is *obvious* if it either makes the agent strictly better off in the worst case, or it makes the agent strictly better off in the best case. By definition, TTD manipulation is a refinement of obvious manipulation. For instance, Troyan and Morrill (2020) show that the Boston mechanism is obviously manipulable. However, from the student's viewpoint, manipulating it may not still be easy in the sense that by misreporting, a student may end up with a less preferred school than the school that can be matched with truth-telling. In contrast, TTD requires that no such case arises, and the Boston mechanism is not TTD manipulable (with a canonical space). Any manipulable (including obviously manipulable) mechanism can be further examined if it is (i-)TTD manipulable; and i-TTD manipulable mechanisms may need to be reconsidered.

Another strand of related literature is asymptotic strategy-proofness or large market: Immorlica and Mahdian (2005) and Kojima and Pathak (2009) show that manipulation incentives in the Deferred Acceptance algorithm vanish as the market size grows. Azevedo and Budish (2019) define a relevant concept, *strategy-proofness in the large.*

The remainder of the paper is organized as follows. Section 2 provides formal definitions and properties of TTD. Section 3 shows impossibilities of shill-proofness as an application of TTD. Section 4 shows more applications in market design. Section 5 concludes the paper.

2 Truth-Telling Dominating Strategy

The model has a set of agents $N = \{1, ..., n\}$ and a set of outcomes X. Agent i's *type* is $\theta_i \in \Theta_i$, and let $\Theta = \prod_{i \in N} \Theta_i$. A *mechanism* is a function $\phi : M \to X$, where $M = \prod_{i \in N} M_i$, and M_i is a *strategy* (or *message*) space of agent i. Mechanism ϕ is a *direct* mechanism if

$M_i = \Theta_i$, and an *indirect* mechanism otherwise. When a strategy space $M_i = \Theta_i$, it is called a *canonical* strategy space.[7] Agent i's utility is $u_i : X \times \Theta_i \to \mathbb{R}$ with a notation $u_i(x; \theta_i)$. Let $s_i : \Theta_i \to M_i$ denote a pure (type-)strategy of agent i, and $S_i(\theta_i)$ denote the set of pure strategies.[8] Let $\sigma_i : \Theta_i \to \Delta M_i$ denote a mixed (type-)strategy with finite support, where $\text{supp}(\sigma_i(\theta_i)) \subseteq S_i(\theta_i)$ denotes the support of $\sigma_i(\theta_i)$.

Definition 1. Agent i's strategy $\sigma_i(\cdot)$ under mechanism ϕ is (weakly) *truth-telling dominating* (TTD) if,

(i) for some $\theta_i \in \Theta_i$,

$$\forall m_i \in \text{supp}(\sigma_i(\theta_i)), \forall m_{-i} : u_i(\phi(m_i, m_{-i}); \theta_i) \geq u_i(\phi(m_i, m_{-i}); \theta_i); \tag{1}$$

$$\forall m_i \in \text{supp}(\sigma_i(\theta_i)), \exists m_{-i} : u_i(\phi(m_i, m_{-i}); \theta_i) > u_i(\phi(m_i, m_{-i}); \theta_i), \tag{2}$$

(ii) for all other $\theta_i \in \Theta_i$,[9]

$$\sigma_i(\theta_i) = \theta_i. \tag{3}$$

Mechanism ϕ is said to be *TTD manipulable* if there exists a TTD strategy for some agent.[10]

In words, a strategy is *truth-telling dominating* if (for some type) every realized strategy weakly dominates truth-telling for all strategy profiles of others, and strictly dominates truth-telling for some strategy profile of others (and for all other types, it is equivalent to truth-telling). With a canonical type space, by definition, a strategy-proof mechanism is not TTD manipulable. Also, a weakly dominant strategy which is not equivalent to truth-telling is a TTD strategy, but not vice versa. By definition, TTD manipulability implies *obvious manipulability* of Troyan and Morrill (2020), but not vice versa. Thus, TTD manipulability can be seen as a refinement of obvious manipulability.

One may think that for TTD manipulability, some non-canonical strategy space, e.g., including shill bidding, may be required, but this is not true.

[7] This is to avoid ambiguity such as whether we should see shill reporting (i.e., using additional fake identities) as a direct or a indirect mechanism. When the fact that shill reporting is not allowed should be emphasized, we will use an expression "a mechanism with a canonical strategy space" rather than just a "direct mechanism," since from each shill bidder's perspective, the mechanism is still a direct mechanism.

[8] A type-strategy is a function that maps a type to a strategy. In most cases, we will consider a type-strategy; thus, we may use strategy and type-strategy interchangeably.

[9] This is to avoid the nonexistence of manipulability due to a boundary case, e.g., in a first-price auction, without (3), clearly there is no TTD strategy for a bidder with zero value.

[10] In a symmetric environment, "for some agent" implies all agents. One may want to enforce "for all agents" in general, but in that case, there can be a trivial nonexistence, e.g., if bidder i's type (value) space is $\{0\}$ but all other bidders' are $[0, 1]$ in a first-price auction, then no TTD strategy exists for i, and just due to i, we cannot call a first-price auction with such type space is TTD manipulable, which may be undesired.

Example 1. The first-price auction is TTD manipulable with a canonical strategy space.

Example 1 follows straightforwardly from the definition, as explained in the introduction. Note that, however, as implied by Definition 1-(ii) and (3), some richness condition on the type space is needed in order to avoid a trivial nonexistence. For instance, in the first-price auction, if every bidder's type and strategy space is $\{0\}$, no one has a TTD strategy.

The richness condition depends on applications, e.g., Troyan and Morrill (2020) use a richness condition $|\Theta_i| \geq |\{x_i : x \in X\}|$, where x_i denotes i's allocation, for usual mechanisms in market design, which we also use for the finite type spaces. For infinite type spaces, we use $\Theta_i = [0, 1]$ for any auctions in this section, and we will specify the type space for other applications. That is, except for the case where we explain technicality, we assume a "usual" type space.

We provide an example, which we will repeatedly use it due to simplicity. Mechanisms widely used in practice are considered in Sections 3 and 4.

Example 2. The mechanism allocates the item free of charge to a bidder who reports the highest value. Bidder i's value is $\theta_i \in \Theta_i = \{1, 2, ..., 10\}$ for all $i \in N$. Obviously, reporting 10 is not only dominant but also TTD. However, even when a dominant strategy exists, the dominant strategy may not be the unique TTD strategy. For instance, reporting $\min\{10, \theta_i + k\}$ for any $k \in \mathbb{N}$ is a TTD strategy. A mixed TTD strategy can easily be made of any combination of pure TTD strategies. For instance, a mixed strategy $\sigma(\theta_i)$ that reports $\min\{10, \theta_i + 1\}$, $\min\{10, \theta_i + 2\}$, or $\min\{10, \theta_i + 3\}$ randomly is a TTD strategy.

In Example 2, a TTD strategy $\min\{10, \theta_i + 2\}$ can be seen as iterating a TTD strategy $\min\{10, \theta_i + 1\}$ twice. To formalize this idea, for any two mixed strategies σ_i' and σ_i'', $\sigma_i'' \circ \sigma_i'$ denotes a mixed strategy σ_i such that $\sigma_i(s_i''(s_i'(\theta_i))|\theta_i) = \sigma_i'(s_i'|\theta_i) \cdot \sigma_i''(s_i''|s_i'(\theta_i))$ for all $s_i' \in \text{supp}(\sigma_i'(\theta_i))$ and all $s_i'' \in \text{supp}(\sigma_i''(s_i'(\theta_i)))$, and 0 otherwise. Let σ_i^k denote the k-th *iterate* of σ_i, e.g., $\sigma_i^2 \equiv \sigma_i \circ \sigma_i$, and σ_i^0 is the identity map, i.e., $\sigma_i^0(\theta_i) = \theta_i$.

Definition 2. Agent i's strategy $\sigma_i(\cdot)$ under mechanism ϕ is (weakly) *iterative* truth-telling dominating (i-TTD) if,

(i) $\sigma_i^k(\cdot)$ is TTD and $\sigma^k(\cdot)$ weakly dominates $\sigma^{k-1}(\cdot)$ for all $k \geq 1$, or

(ii) $\exists k' \geq 2$ such that $\sigma_i^k(\cdot)$ is TTD and $\sigma^k(\cdot)$ weakly dominates $\sigma^{k-1}(\cdot)$ for all $1 \leq k \leq k'$ and $\sigma_i^k(\cdot) = \sigma_i^{k-1}(\cdot)$ for all $k > k'$.

In words, a strategy is i-TTD if any iterate is also TTD and weakly dominates the previous iterate; or is already optimal among all iterates. Part (ii) is to prevent the nonexistence of i-TTD due to reaching an optimum among all iterates of an i-TTD strategy. For instance, in Example 2, if $\sigma(\theta_i) = \min\{10, \theta_i + 1\}$, then for any θ_i, σ^k eventually becomes $\sigma^k(\cdot) = 10$ for some k. Then, $\sigma^{k+1}(\cdot) = \sigma^k(\cdot)$, but we say that $\sigma(\cdot)$ is i-TTD.

Note that, however, not only a TTD strategy but also any iterate of an i-TTD strategy may not be *optimal* among all strategies $m_i \in M_i$. For instance, consider a strategy space $M_i = \Theta_i \cup \{11\}$ in Example 2, then reporting 11 is a weakly dominant strategy, but $\sigma(\theta_i) = \min\{10, \theta_i + 1\}$ is still i-TTD, which reaches its constrained optimum 10.

Since any iterate of an i-TTD strategy weakly dominates the previous iterate until they become equivalent, the more iteration of an i-TTD strategy, the higher payoff the agent can get until the optimum of the i-TTD strategy is reached, which is summarized as Lemma 1 below. To formalize this, for a fixed realization of $\sigma_i^k(\theta_i)$, denoted by $\vartheta(\sigma_i^k, \theta_i)$, let $\vartheta^{m,k}(\sigma_i, \theta_i)$ denote the sub-realization up to the m-th iterate of σ_i^k for any $m \leq k$, where $\vartheta^{0,k}(\sigma_i, \theta_i) \equiv \theta_i$ for all $k \geq 0$. For instance, if $\vartheta^{4,4}(\sigma_i, \theta_i) = s_i'''(s_i'(s_i''(s_i'(\theta_i))))$, then $\vartheta^{2,4}(\sigma_i, \theta_i) = s_i''(s_i'(\theta_i))$.

Lemma 1. *For any i-TTD strategy $\sigma_i(\cdot)$ and its fixed realization $\vartheta(\sigma_i^\infty, \theta_i)$,*

$$(u_i(\phi(\vartheta^{k,\infty}(\sigma_i, \theta_i), m_{-i}); \theta_i))_{k \geq 0}$$

is a payoff-improving sequence, i.e.,

$$u_i(\phi(\vartheta^{0,\infty}(\sigma_i, \theta_i), m_{-i}); \theta_i) \leq u_i(\phi(\vartheta^{1,\infty}(\sigma_i, \theta_i), m_{-i}); \theta_i) \leq u_i(\phi(\vartheta^{2,\infty}(\sigma_i, \theta_i), m_{-i}); \theta_i) \leq, \ldots.$$

Characterization The nature of iteration offers one simple necessary condition for i-TTD. A strategy σ_i is *periodic* if $\sigma_i^{k+l} = \sigma_i^k$ for some $l \geq 1$ and all $k \geq 0$. Proposition 1 below shows that an i-TTD strategy cannot be periodic.

Proposition 1. *Any i-TTD strategy cannot be periodic.*

Proof. Suppose $\sigma_i(\cdot)$ is periodic. Then, there exists $l \geq 1$ such that $\sigma_i^{k+l}(\cdot) = \sigma_i^k(\cdot)$ for all $k \geq 0$. Since $\sigma_i^k(\theta_i) \in \Theta_i$ for all $k \geq 0$, the fact that $\sigma_i^{k+l}(\cdot) = \sigma_i^k(\cdot)$ implies that $\sigma_i^l(\cdot)$ is an identity map, i.e., $\sigma_i^l(\theta_i) = \theta_i$. However, truth-telling cannot be TTD. □

Example 2 shows that a non-canonical strategy space may not be necessary for the existence of i-TTD strategies. However, with a canonical space, the range of an i-TTD strategy should be at least a strict subset of a type space. For instance, In Example 2 with an i-TTD strategy $\sigma_i(\theta_i) = \min\{10, \theta_i + 1\}$, the range is $\sigma_i(\Theta_i) = \{2, ..., 10\} \subset \{1, 2, ..., 10\} = \Theta_i$. Proposition 2 below shows that any i-TTD strategy cannot be bijective.

Proposition 2. *Any i-TTD strategy cannot be bijective.*

Proof. Suppose an i-TTD strategy is bijective. First, assume that $\sigma_i(\cdot)$ is a pure strategy. Then, for each $\theta_i \in \Theta_i$, there exists k such that $\sigma_i^k(\theta_i) = \theta_i$, which implies that θ_i, i.e., truth-telling, is TTD, a contradiction. When, $\sigma_i(\cdot)$ is a mixed strategy, let $s_{i_1}(\cdot), s_{i_2}(\cdot), ..., s_{i_l}(\cdot) \in \text{supp}(\sigma_i(\cdot))$ be pure strategies in its support. Then, there exists $(k_j)_{1\leq j\leq l}$ such that $s_{i_j}^{k_j}(\theta_i) = \theta_i$ for all $1 \leq j \leq l$. Thus, $\sigma_i^{\text{lcm}((k_j)_{1\leq j\leq l})}(\cdot) = \theta_i$, where $\text{lcm}((a_j))$ denotes the least common multiple of (a_j). But this implies that a truth-telling is TTD, a contradiction. □

Interestingly, however, a bijective TTD (not i-TTD) strategy is possible, as shown in the example below, which is also useful to understand the nature of TTD.

Example 3. Each agent reports an ordered preference over items A and B, e.g., agent i reports (A, B) if i prefers A to B. A mechanism allocates less-preferred item to each agent. Then, reporting in the opposite way, denoted by σ_i, is not only TTD but also dominant; yet it is not i-TTD, since $\sigma_i^2(\theta_i) = \theta_i$.

Example 1 (i.e., the first-price auction is TTD manipulable) and Examples 2 and 3 may imply that TTD (but not i-TTD) may not be too "undesirable." In fact, the first-price auction is one of the most widely used auction mechanisms. As implied by Lemma 1, the more undesirable one in practice may be i-TTD, especially if an agent's better-off (using an i-TTD strategy) implies someone else's worse-off. For instance, Section 3.1 will show an i-TTD shill-bidding strategy in the VCG mechanism with externalities that can decrease the seller's revenue. In addition, Section 4.2 will show that the VCG mechanism with referrals has an i-TTD shill-bidding strategy which generically leads to a negative revenue. Thus, characterizing i-TTD is of importance.

The proposition below offers an important sufficient condition for i-TTD non-manipulability. For instance, the fact that the first-price auction is not i-TTD also follows as a corollary.

Proposition 3. *Any undominated equilibrium strategy $\sigma_i(\theta_i) \neq \theta_i$ is not i-TTD. Thus, any mechanism only with undominated non-degenerate*[11] *equilibria is not i-TTD manipulable with a canonical strategy space.*

Proof. Suppose an undominated equilibrium strategy $\sigma_i(\theta_i) \neq \theta_i$ is i-TTD. By the definition of i-TTD, $\sigma_i^2(\theta_i)$ must also be TTD. Then, $\sigma_i^2(\theta_i)$ weakly dominates $\sigma_i(\theta_i)$, which contradicts that $\sigma_i(\theta_i)$ is an undominated equilibrium strategy. □

Corollary 1. *The first-price auction is not i-TTD manipulable with a canonical strategy space.*

[11]A strategy $\sigma_i(\cdot)$ is *degenerate* if there exists $\theta_i' \in \Theta_i$ such that $\sigma_i(\theta_i) = \theta_i'$ for all $\theta_i \in \Theta_i$. For instance, reporting 10 in Example 2 is degenerate.

In Example 3, σ_i is a unique equilibrium strategy (in dominant strategies) that is not truth-telling. Thus, σ_i is TTD, but by Proposition 3, σ_i is not i-TTD.

Another important result on i-TTD is characterizing i-TTD *shill-reporting* strategy, where shill reporting means that an agent may use additional fake identities, i.e., submitting two or more types. One natural question may be is there a TTD but not i-TTD shill-reporting strategy? For ease of exposition, we prove the result for shill bidding in single-item auctions without externalities, but the result can be ge generalized (e.g., package auctions with externalities in Theorem 4 and related lemmas). An auction mechanism is said to be *standard* if the winner is a bidder with the highest bid. An auction mechanism is said to be *pairwise stable* if the winner's payment should be at least the maximum bid of losers. We assume that all ties are broken by some tie-breaking rule.

Theorem 1. *For any auction mechanism that is individually rational and pairwise stable, any TTD shill-bidding strategy is i-TTD. In addition, any TTD shill-bidding strategy should preserve the allocation, i.e., the winner should not change due to shill bidding.*

Proof. We first prove the second part, i.e., the winner should not change. By individual rationality, a loser should not pay. Losing implies that there exists another bidder whose bid is higher. Thus, introducing a shill bid lower than his bid without shill bidding cannot change the winner. On the other hand, by pairwise stability and individual rationality, the loser is worse off winning by shill bidding which requires overbidding. Thus, for a TTD shill-bidding strategy for a loser, the winner should not change with shill bidding. Now, for the winner without shill bidding to be better off with shill bidding, he must still be the winner, since by no loser's payment, losing gives him a zero payoff, which concludes the second part.

We now prove the first part. For the winner to be better off by shill bidding, the payment must decrease (weakly for all bid profiles of others and strictly for some bid profiles of others). But a TTD strategy is a type-strategy; thus, the strategy must preserve the winner and reduce the payment for all types (except for some types for which it should be truth-telling, i.e., part (ii) of Definition 1). Thus, the TTD shill-bidding strategy should be i-TTD. □

For instance, in Section 3.1, we show i-TTD shill-bidding strategies, called the *valuation-split* strategy (Theorem 2) and the *valuation-convex-split* strategy (Corollary 5). Roughly speaking, iterating a valuation-split strategy results in a valuation-convex-split strategy.

Theorem 1 *per se* does not provide any implication about the existence of a TTD strategy However, it is still useful, for instance, to prove the nonexistence of a TTD shill-bidding strategy. For most simple cases, e.g., in single-item first-price and second-price auctions, it is obvious that there exists no TTD shill-bidding strategy, but it may not be obvious for more complicated cases. Perhaps surprisingly, for the same mechanism (but, of course, in

$$\theta_i = 7 \longrightarrow 8 \longrightarrow 9 \longrightarrow 10 = \theta_i^*$$

Figure 1: The lattice induced from $\sigma_i(\theta_i)$ for $\theta_i = 7$ in Example 2.

different environments), Section 3.1 will show that there exists i-TTD shill-bidding strategies in the VCG auction with externalities, and Section 3.2 will show that there exists no TTD shill-bidding strategy in the VCG package auction without externalities.

Mathematical structure of i-TTD strategies Due to the nature of iteration, i-TTD strategies form certain mathematical structures. Let $\Omega(\sigma_i, \theta_i) \equiv \cup_{k\geq 0} \operatorname{supp}(\sigma_i^k(\theta_i))$ denote the support of the iterates of a TTD strategy $\sigma_i(\theta_i)$. Then, $\Omega(\sigma_i, \theta_i)$ forms a semi-lattice or lattice, under certain conditions. Whether the payoff-improving sequence in Lemma 1 converges or not can also be determined from the structure.

A *lattice* is a tuple $(L, \leq)$ such that $\leq$ is a *partial order*[12] on L in a way that for any x and y of L, there exists a unique greatest lower bound (or *meet*) $x \wedge y$ and a unique lowest upper bound (or *join*) $x \vee y$. One can easily see that $\leq$ induces $\wedge$ and $\vee$, and vice versa. A *meet-semilattice* is a tuple $(L, \leq)$ such that $\leq$ is a partial order on L in a way that for any x and y of L, there exists a unique greatest lower bound (or meet) $x \wedge y$. A *complete lattice* $(L, \leq)$ is a lattice such that any subset $L' \subseteq L$ has both the unique greatest lower bound, denoted by $\wedge L'$, and the unique lowest upper bound, denoted by $\vee L'$, in $(L, \leq)$.

Example 4 (Example 2 continued)**.** Figure 1 shows the lattice induced from the mixed TTD strategy $\sigma(\theta_i)$ that reports $\min\{10, \theta_i + 1\}$, $\min\{10, \theta_i + 2\}$, or $\min\{10, \theta_i + 3\}$ randomly for $\theta_i = 7$ in Example 2, where $\leq$ is the usual total order. One can easily see that the lattice induced from a pure strategy $\min\{10, \theta_i + 1\}$ only is the same as the lattice induced from the mixed strategy. That is, the lattice is a chain.

Note that, however, not all supports of the iterates of an i-TTD strategy form a semilattice.

Example 5. Suppose $\operatorname{supp}(\sigma_i(\theta_i)) = \{s', s''\}$. If $s'^2 = s' \circ s''$ and $s''^2 = s'' \circ s' \neq s'^2$, and $\Omega(\sigma_i, \theta_i) = \{\theta_i, s', s'', s'^2, s''^2\}$, then $(\Omega, \leq)$ is not a semilattice, as shown in Figure 2. For instance, s' and s'' do not have a unique lowest upper bound, and s'^2 and s''^2 do not have a unique greatest lower bound.

Thus, for the support of the iterates of a TTD strategy to form a semilattice or a lattice, some conditions are needed.[13] We provide some sufficient conditions where $(\Omega(\sigma_i, \theta_i), \leq)$ is a

[12]The partial orders will be provided or clear from the context.

[13]Characterizing the conditions for the support of the iterates of an i-TTD strategy to be a (semi-)lattice or any other mathematical structure is in progress.

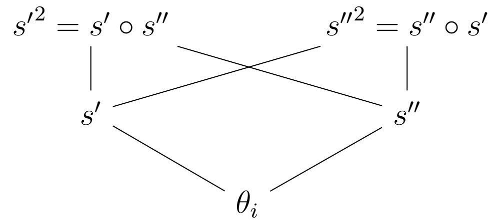


Figure 2: A non-lattice $(\{\theta_i, s', s'', s'^2, s''^2\}, \leq)$ in Example 5.

meet-semilattice or a special case of a meet-semilattice. First, the following condition makes $(\Omega(\sigma_i, \theta_i), \leq)$ a chain, as shown in Example 4 and Figure 1.

Lemma 2 (Chain)**.** *For a TTD strategy $\sigma_i(\theta_i)$, if all pure strategies in* $\mathrm{supp}(\sigma_i(\theta_i))$ *can be iteratively induced from one of the pure strategies, $(\Omega(\sigma_i, \theta_i), \leq)$ is a chain.*

Proof. Let $s_i^*(\theta_i) \in \mathrm{supp}(\sigma_i(\theta_i))$ be the pure TTD strategy from which the all pure TTD strategies in $\mathrm{supp}(\sigma_i(\theta_i))$ can be iteratively induced. Then, any $s_i(\theta_i) \in \Omega$ can also be iteratively induced from $s_i^*(\theta_i)$. Thus, $(\Omega, \leq)$ is a lattice that is totally ordered. $\square$

Another simple case is when any $k \geq 1$ composition of pure TTD strategies lead to either a distinct strategy or the identical strategy of $k-1$ composition. In this case, $(\Omega(\sigma_i, \theta_i), \leq)$ is not only a meet-semilattice but also a tree. A *tree* is a partially ordered set where each element x (except for the least element, which is also called a *root*) has only one element y such that $y < x$ (y is called a parent of x). Note that any tree is a meet-semilattice but not vice versa.

Lemma 3 (Tree)**.** *For a TTD strategy $\sigma_i(\theta_i)$, if any $k \geq 1$ composition of pure TTD strategies lead to either a distinct strategy or the identical strategy of $k-1$ composition, then $(\Omega(\sigma_i, \theta_i), \leq)$ is a tree, where the least element is truth-telling θ_i.*

Proof. Any element $s_i \in \cup_{k\geq 0} \mathrm{supp}(\sigma_i^k(\theta_i))$ is a realization $\vartheta(\sigma_i^{k'}, \theta_i) = \vartheta^{k',k'}(\sigma_i, \theta_i)$, where $k' \geq 1$. Thus, $\vartheta^{k'-1,k'}(\sigma_i, \theta_i)$ is either the same as $\vartheta^{k',k'}(\sigma_i, \theta_i)$ or not. When $\vartheta^{k'-1,k'}(\sigma_i, \theta_i) \neq \vartheta^{k',k'}(\sigma_i, \theta_i)$, $\vartheta^{k'-1,k'}(\sigma_i, \theta_i)$ is the parent of $\vartheta^{k',k'}(\sigma_i, \theta_i)$. $\square$

Figure 5 shows the lattice of the i-TTD *fake referral* strategy (in Example 13 in Section 4.2), which is a complete lattice. By Tarski's fixed point theorem, we get the following result.

Corollary 2. *If $(\Omega(\sigma_i, \theta_i), \leq)$ is a complete lattice, optimal strategies among $\Omega(\sigma_i, \theta_i)$ form a complete lattice.*

Regardless of whether $(\Omega(\sigma_i; \theta_i), \leq)$ is a meet-semilattice or a lattice, the set of all realized payoffs $\{u_i(\phi(s_i, \theta_{-i}); \theta_i) : s_i \in \Omega(\sigma_i; \theta_i)\} \subseteq \mathbb{R}$ has its usual total order $\leq$ and therefore forms a lattice.

Corollary 3. *For any i-TTD strategy σ_i, $(\{u_i(\phi(s_i, \theta_{-i}); \theta_i) : s_i \in \Omega(\sigma_i; \theta_i)\}, \leq)$ is a lattice.*

Corollary 4. *For any i-TTD strategy σ_i, if $\{u_i(\phi(s_i, \theta_{-i}); \theta_i) : s_i \in \Omega(\sigma_i; \theta_i)\}$ is compact, an optimal i-TTD strategy exists.*

Note that, however, an optimal i-TTD strategy is not necessarily optimal (among all strategies), since for a fixed strategy profile of others, there can be a better strategy which is not TTD. The full-fake-referral strategy in Section 4.2 is an optimal TTD strategy that is optimal. The optimal valuation-split strategy in Proposition 2 may not be optimal depending on type profiles.

3 Application: Impossibilities of Shill-Proofness

As the main application of truth-telling dominance (TTD), we show impossibility results on shill-proofness, which has its own importance in practice. While the literature has studied shill bidding in the Vickrey-Clarke-Groves (VCG) auction, TTD shill-bidding strategies have not been studied yet. Section 3.1 shows that when externalities exist, even for single-item auctions, the VCG mechanism is TTD manipulable. In contrast, Section 3.2 shows that without externalities, even for package auctions, the VCG mechanism is not TTD manipulable.

3.1 Impossibility for auctions with externalities

This section shows impossibility results on the Vickrey-Clarke-Groves (VCG) auction with externalities and generalizes them. For expositional simplicity, we first define the model without shill bidding and extend it to the model with shill bidding. The model has one seller, denoted by player 0, with one indivisible item to be auctioned and a set $N = \{1, 2, ..., n\}$ of bidders with $n \geq 2$. The set of all players is denoted by $N^0 = N \cup \{0\}$. Player j's type is denoted by a column vector $t_j = (t_{ij})_{i \in N^0}$, where $t_{ij} \in \mathbb{R}$ is the externality imposed on player j when player $i \neq j$ wins the item, and $t_{jj} \in \mathbb{R}_+$ is player j's valuation of the item. We assume that $t_{j0} = 0$ and $t_{0j} = 0$ for all $j \in N^0$, i.e., the seller neither imposes nor suffers any externalities. Each bidder's type is independent and private information. The type profile of all players is denoted by $T = (t_{ij})_{i,j \in N^0} \in \mathcal{T} \subset \mathbb{R}^{(n+1)\times(n+1)}$, where $\mathcal{T}$ is the type space.

A direct auction mechanism $\varphi = (x, \rho)$ is a pair that consists of an allocation rule $x : \mathcal{T} \to N^0$ (with a tie-breaking rule), and a payment rule $\rho : \mathcal{T} \to \mathbb{R}^n$. Any *bid* (i.e., *reported type*) of bidder j is denoted by b_j and a bid profile is denoted by $B = (b_{ij})_{i,j \in N^0} \in \mathcal{T}$. An auction outcome is denoted by (w, p), where $w = x(B)$ is the winner, and $p = \rho(B)$ is the payment. An auction outcome is *efficient* if $w \in \arg\max_{i\in N^0} \sum_{j\in N^0} t_{ij}$. An allocation rule that chooses an efficient outcome with respect to any reported bid profile B is denoted by x^*, i.e., $x(B) \in \arg\max_{i\in N^0} \sum_{j\in N^0} b_{ij}$.

The payoff of $j \in N$ is $\pi_j = t_{wj} - p_j$ when her payment is p_j and $w \in N^0$ wins. If bidder j does not participate, $\pi_j = t_{ij}$ if i wins. The seller's payoff or revenue is $\pi_0 = \sum_{j\in N} p_j$. As usual, $T_S = (t_{ij})_{i,j\in S} \in \mathcal{T}_S \subseteq \mathbb{R}^{|S|\times|S|}$ for $S \subseteq N^0$ and $T_{-S} = T_{S^c}$, where $S^c = N^0 \backslash S$.

We now define the Groves mechanism and the Vickrey-Clarke-Groves (VCG) mechanism. Note that the VCG mechanism is one particular kind of the Groves mechanism.

Definition 3. The Groves mechanism is (x^*, ρ^G), where $\rho_j^G(B) = h_j(B_{-j}) - \sum_{k\in N^0\backslash\{j\}} b_{x(B),k}$ with some $h_j : \mathcal{T}_{-j} \to N^0\backslash\{j\}$ for all $j \in N$.[14]

Note that h_j is any function that is independent of b_j, but we impose the following technical assumption for Proposition 2, which implies that if a bidder has a zero bid and all other bidders do not suffer externalities from the bidder, adding such bidder does not change the value of $h(\cdot)$.

Assumption 1. *For any Groves mechanism in Definition 3,* $h_j(B_{-j}) = h_j\left(\begin{bmatrix} B_{-j} & 0 \\ 0 & 0 \end{bmatrix}\right)$ *for all* $j \in N$ *and* $B_{-j} \in \mathcal{T}_{-j}$.

Definition 4. The Vickrey-Clarke-Groves (VCG) mechanism is (x^*, ρ^V), where

$$\rho_j^V(B) = \sum_{k\in N^0\backslash\{j\}} b_{x(B_{-j}),k} - \sum_{k\in N^0\backslash\{j\}} b_{x(B),k}, \forall j \in N.$$

That is, the VCG mechanism is a Groves mechanism with $h(B_{-j}) = \sum_{k\in N^0\backslash\{j\}} b_{x(B_{-j}),k}$.

Shill bidding There is a finite set of identities $I = \{id_1, id_2, ..., id_M\}$, and each bidder $j \in N$ can use any identities in $I_j \in 2^I$ such that $\{I_j : j \in N\}$ is a partition of I, i.e., each I_j is mutually exclusive and $I = \cup_{j\in N} I_j$. We call any bidder $j \in N$ a *real* bidder, and a *shill* bidder otherwise; however, any $j \in I$ may be simply called a bidder when j is whether real or shill is apparent from context. A bid of a shill bidder is called a *shill bid*. Without loss of generality, we assume that $id_1 = 1$, $id_2 = 2$, ..., $id_n = n$. Any bidder $j \in N$ may submit shill bids. Each bidder j chooses a set of ids $\iota_j \subseteq I_j$ with $j \in \iota_j$ and submits a bid b_{j_k} for each $j_k \in \iota_j$, and an auction is run for the reported bids denoted by $\tilde{B} \in \tilde{\mathcal{B}} \subset \mathbb{R}^{(\sum_{j=1}^n |\iota_j|+1)\times(\sum_{j=1}^n |\iota_j|+1)}$, where $\tilde{\mathcal{B}}$ is the reported profile space including shill

[14] Abusing the notation, we use $t_{i,j}$ instead of t_{ij} when it helps exposition.

bidding.[15] Based on the motivation of the shill bidding problem, the existence of shill bidding of j is only known to bidder j himself, and each bidder only knows the real bidders' identities. The *extended type* due to shill bids is denoted by $\tilde{T} = (\tilde{t}_{ij})$, where $\tilde{t}_{ij} = 0$ for all $i \notin N$, and $\tilde{t}_{ij} = t_{ij}$ otherwise.

The shill-bidding strategy of bidder $j \in N$ is denoted by β_j which maps t_j to a list of bids. For instance, $\beta_j(t_j) = \left(\left((1-\alpha)\,\tilde{t}_{jj}, \tilde{t}_{-j,j}\right), \left(\alpha\tilde{t}_{jj}, \mathbf{0}\right)\right)$ implies that j submits two bids: $((1-\alpha)\,t_{jj}, t_{-j,j}) \equiv (t_{0j}, t_{1j}, ..., t_{j-1,j}, (1-\alpha)\,t_{jj}, t_{j+1,j}, ..., t_{n+1,j})$ as bidder j, and $(\alpha t_{jj}, \mathbf{0}) \equiv (0, 0, ..., 0, \alpha t_{jj}, 0, ..., 0)$ as a shill bidder. Let $\beta_j^0(\cdot)$ denote the truthful bidding strategy, i.e., $\beta_j^0(t_j) = \tilde{t}_j$. We may use β_j to denote a realized shill-bidding strategy, i.e., $\beta_j = \beta_j(t_j)$.

The *effective* (or *aggregated*) payoff of bidder $j \in N$ under a reported profile $\tilde{B}$ is denoted by $\tilde{\pi}_j(\tilde{b}_j) = \tilde{\pi}_j(\tilde{B}; t_j) \equiv t_{x(\tilde{B}),j} - \tilde{p}_j$, where $\tilde{p}_j \equiv \sum_{k \in \iota_j} \rho_k(\tilde{B})$ denotes the *effective* payment, i.e., the sum of the payments of j and j's shill bidders.

Definition 5. A mechanism is *shill-proof* if $\tilde{\pi}_j((\tilde{t}_j, \tilde{b}_{-j}); t_j) \geq \tilde{\pi}_j((\tilde{b}_j, \tilde{b}_{-j}); t_j)$ for all $\tilde{B} \in \mathcal{B}$, all $T \in \mathcal{T}$, and all $j \in N$.

TTD shill-bidding strategies We now introduce two kinds of TTD shill-bidding strategy: the *valuation-split* strategy and the *externality-split* strategy. For each strategy, we provide an example, a theorem (strategy) and a corollary (generalized strategy), and then explain the nontriviality of the strategy.

Example 6. Suppose

$$T_N = \begin{bmatrix} 5 & 0 & -2 \\ -4 & 5 & 0 \\ 0 & -3 & 5 \end{bmatrix}.$$

The VCG outcome is $w = 1$ and $p = (7, 2, 0)$. However, if bidder 1 uses shill bidding $\beta_1 = ((2, -4, 0, 0), (3, 0, 0, 0))$,

$$\tilde{B}_N = \begin{bmatrix} 2 & 0 & -2 & 3 \\ -4 & 5 & 0 & 0 \\ 0 & -3 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

then the outcome is $w' = 1$ and $p' = (\mathbf{4}, 2, 0, \mathbf{2})$. Thus, the effective payoff of bidder 1 is $\tilde{\pi}_1 = 5 - (4 + 2) = -1 > \pi_1 = 5 - 7 = -2$, i.e., bidder 1 is better off. □

[15] Abusing the notation, the definition of φ is extended for $\tilde{B}$ (instead of defining φ for varying population. For defining a mechanism for varying population, see Yokoo et al. (2004) and Jeong (2020a), for instance.

The above shill-bidding strategy divides the valuation into two bids. Proposition 2 shows that this strategy works more generally and is TTD; and Corollary 5 shows that the TTD strategy is iterative so that any convex combination of the value is also TTD. The main idea (which also applies to the externality-split strategy in Proposition 3) is that using this strategy does not change the row sum, i.e., $t_{ij} = \tilde{t}_{ij} + \tilde{t}_{i,n+1}$ for all $i \in N^0$. Therefore, the winner does not change (up to ties), and the payoffs of all other bidders do not change.

Theorem 2 (Valuation Split)**.** *In any Groves mechanism,*

$$\beta_j(t_j) = (((1-\alpha)\tilde{t}_{jj}, \tilde{t}_{-j,j}), (\alpha\tilde{t}_{jj}, \mathbf{0})) \tag{4}$$

for any $0 < \alpha < 1$ *is a TTD shill-bidding strategy for* $j \in N$*. In addition, the winner does not change (up to ties), and all other bidders' payoffs remain unchanged.*

Corollary 5 (Valuation Convex Split)**.** *In any Groves mechanism,*

$$\beta_j(t_j) = ((\alpha_1\tilde{t}_{jj}, \tilde{t}_{-j,j}), (\alpha_2\tilde{t}_{jj}, \mathbf{0}), (\alpha_3\tilde{t}_{jj}, \mathbf{0}), ..., (\alpha_m\tilde{t}_{jj}, \mathbf{0})) \tag{5}$$

for any $m \in \mathbb{N}$ *and any* $\alpha = (\alpha_k) > 0$ *(componentwise) such that* $\sum_{k=1}^{m} \alpha_k = 1$ *is a TTD shill-bidding strategy for* $j \in N$*. In addition, the winner does not change (up to ties), and all other bidders' payoffs remain unchanged.*

Proof of Proposition 2. Without loss of generality, suppose bidder 1 uses the shill-bidding strategy and the shill bidder is denoted by $s = n+1$. Since $\sum_{j=1}^{n} t_{ij} = \sum_{j=1}^{n+1} \tilde{b}_{ij}$ for all $i \in N$, the winner does not change (up to ties), i.e., $w' = w$. In addition, $w'_{-1} = w_{-1}$. Likewise, other bidders' payoffs do not change, i.e., $\pi'_j = \pi_j$ for all $2 \leq j \leq n$. Thus, it is sufficient to show that $\tilde{p}_1 = p'_1 + p'_s \leq p_1$ for all t_{-1} and $\tilde{p}_1 < p_1$ for some t_{-1}.

1. $w \neq 1$: $p'_1 = p_1$. Since a loser's payment is zero if the winner does not change with his absence, $p'_s = 0$. Thus, $\tilde{p}_1 = p'_1 + p'_s = p_1 + 0 \leq p_1$.

2. $w = 1$: $p'_1 = p_1 - \alpha t_{11}$. But this does not necessarily mean that $\tilde{p}_1 \leq p_1$ yet, because we also need to check p'_s.

   (a) $w'_{-s} = w$: $p'_s = 0$; thus, $\tilde{p}_1 = (p_1 - \alpha t_{11}) + 0 < p_1$.

   (b) $w'_{-s} \neq w$: $p'_s = \sum_{k=1}^{n} t_{w'_{-s},k} - (\sum_{k=1}^{n} t_{w,k} - \alpha t_{11}) = \alpha t_{11} + \sum_{k=1}^{n} t_{w'_{-s},k} - \sum_{k=1}^{n} t_{w,k} \leq \alpha t_{11}$. The last inequality holds because $\sum_{k=1}^{n} t_{w'_{-s},k} \leq \sum_{k=1}^{n} t_{w,k}$. Otherwise, w'_{-s} must be the winner for the original auction without shill bidding, i.e., $w'_{-s} = x(T) = w$, a contradiction. Thus, $\tilde{p}_1 \leq (p_1 - \alpha t_{11}) + \alpha t_{11} = p_1$.

The existence of t_{-1} such that $\tilde{p}_1 < p_1$ is obvious. $\square$

Note that the valuation-(convex)-split TTD shill-bidding strategy requires positive externalities. One might think that a shill bid that imposes a positive externality on himself would always benefit, but this is not true.

Example 7. With the same T in Example 6, if bidder 2 uses a shill bid of $(0,3,0,0)$, then $w' = 2 \neq w$ and $p' = (0, \mathbf{6}, 1, \mathbf{2})$. That is, $\tilde{\pi}_2 = 5 - (6+2) = -3 < \pi_2 = 0 - 2 = -2$, so bidder 2 is worse off.

In general, if positive externalities are allowed, a subsidy may occur. Thus, the seller may not want to allow positive externalities from the beginning. In addition, shill bidding by using positive externalities might also be undesirable to bidders because shill bidding might be easily recognized, i.e., a bid with a positive externality on "someone else" (which is in fact himself) may look suspicious. However, even if positive externalities are not allowed, there exists another TTD shill-bidding strategy that uses negative externalities.

Example 8. Suppose

$$T = \begin{bmatrix} 5 & 0 & -2 & 0 \\ 0 & 6 & 0 & 0 \\ -3 & 0 & 7 & 0 \\ -5 & 0 & 0 & 8 \end{bmatrix}.$$

The VCG outcome is $w = 2$, $p = (2, 4, 0, 0)$. However, if bidder 1 uses shill bidding $\tilde{b}_1 = ((5, 0, 0, -2, 0), (0, 0, -3, -3, 0))$,

$$\tilde{T}_N = \begin{bmatrix} 5 & 0 & -2 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 & -3 \\ -2 & 0 & 0 & 8 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

then the outcome is $w' = 2$, $p' = (\mathbf{0}, 4, 0, 0, \mathbf{1})$. Thus, the effective payment of bidder 1 is $\tilde{p}_1 = p'_1 + p'_5 = 0 + 1 < p_1 = 2$. Since the winner does not change, $\tilde{\pi}_1 > \pi_1$. In fact, bidder 1's payment can be even reduced to zero: if bidder 1 uses $\tilde{b}_1 = ((5, 0, -1, -3, 0), (0, 0, -2, -2, 0))$, then $w' = 2$, $p' = (\mathbf{0}, 4, 0, 0, \mathbf{0})$, and $\tilde{p}_1 = 0 + 0$. $\square$

The idea of splitting externalities works in general, as in the following theorem. Let $\mathbf{1}_{\tilde{t}_j < 0}$ denote a column vector where i-th element is 1 if $\tilde{t}_{ij} < 0$, and 0 otherwise. For instance, in Example 8, $\mathbf{1}_{\tilde{t}_1} = (0, 0, 1, 1, 0)$.

Theorem 3 (Externality Split)**.** *In the VCG mechanism,*

$$\beta_j(t_j) = (\tilde{t}_j + \alpha \mathbf{1}_{\tilde{t}_j<0}, -\alpha \mathbf{1}_{\tilde{t}_j<0}) \tag{6}$$

for some α such that $0 < \alpha \leq -\min_i \tilde{t}_{ij}$ is a TTD shill-bidding strategy for $j \in N$. In addition, the winner does not change (up to ties), and all other bidders' payoffs remain unchanged.

Corollary 6 (Externality Convex Split)**.** *In the VCG mechanism,*

$$\beta_j(t_j) = (\tilde{t}_j + \alpha \mathbf{1}_{\tilde{t}_j<0}, -\alpha_1 \mathbf{1}_{\tilde{t}_j<0}, -\alpha_2 \mathbf{1}_{\tilde{t}_j<0}, ..., -\alpha_m \mathbf{1}_{\tilde{t}_j<0}) \tag{7}$$

for any $m \in \mathbb{N}$ and any $\alpha = (\alpha_k) > 0$ such that $\sum_{k=1}^m \alpha_k \leq -\min_i \tilde{t}_{ij}$ is a TTD shill-bidding strategy for $j \in N$. In addition, the winner does not change (up to ties), and all other bidders' payoffs remain unchanged.

Proof of Proposition 3. As in Proposition 2, the winner does not change (up to ties), i.e., $w' = w$, and other bidders' payoffs do not change, i.e., $\pi'_j = \pi_j$ for all $2 \leq j \leq n$. Note that, however, both $w'_{-1} = w_{-1}$ and $w'_{-1} \neq w_{-1}$ are possible as opposed to the valuation-split strategy. Without loss of generality, suppose bidder 1 uses the shill-bidding strategy, and let $s = n+1$. Then, it is sufficient to show that $\tilde{p}_1 \equiv p'_1 + p'_s \leq p_1$ for all t_{-1} and $\tilde{p}_1 < p_1$ for some t_{-1}.

1. $w \neq 1$: $p'_1 = \left[\sum_{k=2}^n t_{w'_{-1},k} - \alpha \mathbf{1}_{t_{w'_{-1},1}<0}\right] - \left[\sum_{k=2}^n t_{w,k} - \alpha \mathbf{1}_{t_{w,1}<0}\right]$.
   (a) $w_{-1} = w$: $p_1 = 0$. Note that $w'_{-1} = w$; otherwise, $w'_{-1} = w_{-1}$, a contradiction to $w_{-1} = w$. Thus, $p'_1 = 0$. Likewise, $w'_{-s} = w$; thus, $p'_s = 0$. Therefore, $\tilde{p}_1 = 0 = p_1$.
   (b) $w_{-1} \neq w$:
      i. $w'_{-1} = w$: $p'_1 = 0$. For the shill bidder, if $w'_{-s} = w$, then $p'_s = 0$ and therefore $\tilde{p}_1 \leq p_1$. Even when $w'_{-s} \neq w$ (which requires $t_{w'_{-s},1} < 0$ and $t_{w,1} = 0$), $p'_s \leq p_1$ can be shown as follows. $p'_s = \sum_{k=1}^n t'_{w'_{-s},k} - \left[\sum_{k=2}^n t_{w,k} + t'_{w,1}\right] = \sum_{k=1}^n t'_{w'_{-s},k} - \sum_{k=2}^n t_{w,k}$. The last equality holds due to $t_{w,1} = 0$. Then, $p'_s \leq p_1$ is equivalent to $\sum_{k=2}^n t_{w_{-1},k} \geq \sum_{k=1}^n t'_{w'_{-s},k}$. However, $\sum_{k=2}^n t_{w_{-1},1} = \max_{i\neq 1, i\in N} \sum_{k=2}^n t_{ik} = \max_{i\neq 1, i \in N} \sum_{k=2}^n t'_{ik}$ and $\sum_{k=1}^n t'_{w'_{-s},k} = \max_{i\in N} \sum_{k=1}^n t'_{ik} = \max_{i\neq 1, i\in N} \sum_{k=1}^n t'_{ik}$. The last equality holds, otherwise bidder 1 should be the winner in the original auction, a contradiction. Now, $\sum_{k=2}^n t_{w_{-1},k} \geq \sum_{k=1}^n t'_{w'_{-s},k}$ because $t'_{i,1} \leq 0$. Thus, $\tilde{p}_1 \leq p_1$.

ii. $w'_{-1} \neq w$: $t_{w'_{-1},1} < 0$ and $t_{w,1} = 0$, otherwise w'_{-1} must be the winner for the original auction, a contradiction. In addition, $w'_{-1} = w_{-1}$. Thus, $p'_1 = p_1 - \alpha$. For the shill bidder, if $w'_{-s} = w$, then $p'_s = 0$ and therefore $\tilde{p}_1 \leq p_1$. Even when $w'_{-s} \neq w$ (which requires $t_{w'_{-s},1} < 0$), $p'_s \leq \alpha \mathbf{1}_{t_{w'_{-s},1} < 0} = \alpha$, otherwise w'_{-s} must be the winner for the original auction, i.e., $w'_{-s} = w$, a contradiction. Thus, $\tilde{p}_1 \leq p_1$. It can be similarly proven when bidder 1 is the winner.

The existence of t_{-1} such that $\tilde{p}_1 < p_1$ is obvious. □

One might think that splitting one externality might be sufficient even when a bidder suffers negative externalities imposed by more than one bidders. However, this is not true, as illustrated in the following example.

Example 9. In Example 8, if bidder 1 uses $\tilde{b}_1 = ((5, 0, -3, 0, 0), (0, 0, 0, -5, 0))$, then $w' = 2$, $p' = (1, 4, 0, 0, 2)$, and $\tilde{p}_1 = 1 + 2 > p_1$. Likewise, if $\tilde{b}_1 = ((5, 0, 0, -5, 0), (0, 0, -3, 0, 0))$ is used, $p' = (\mathbf{2}, 4, 0, 0, \mathbf{1})$, and $\tilde{p}_1 = 2 + 1 > p_1$. In addition, similar to Example 7 (but in the opposite direction), putting a negative externality to other bidders does not always work.

Propositions 2 and 3 show that the VCG mechanism is not only non-shill-proof but also TTD manipulable. One may still hope that there might be other mechanism that is shill-proof (with some other desirable properties). However, there is no such mechanism.

Corollary 7. *In single-item auctions with externalities, there exists no shill-proof mechanism that is also efficient and individually rational.*

Proof. By Holmstrom (1979), the Groves mechanism is the unique mechanism that is efficient, individually rational, and strategy-proof. However, Proposition 2 shows that any Groves mechanism is not shill-proof. □

The VCG mechanism is not shill-proof in package auctions without externalities either. Thus, the literature shows some type domains in which the VCG mechanism is shill-proof. However, when externalities exist, due to Propositions 2 and 3, finding such a type domain is hopeless. Even worse, we will show that there always exists a shill bid that makes zero revenue. While both the valuation-split and externality-split strategies decrease revenue, they do not always make zero revenue. However, if we forgo TTD, there always exists a shill bid that makes zero revenue.

Proposition 4. *In single-item VCG auctions with externalities, for any bidder, there exists a shill bid that makes zero revenue.*[16]

[16] If a subsidy is allowed (i.e., the seller pays bidders), it is easy to see that even negative revenue is possible.

Proof. Without loss of generality, suppose bidder 1 is the winner. Then, together with $\tilde{t}_1$, a shill bid $b_s = (-\sum_{k\neq 1} t_{1k}, \min\{-\sum_{k\neq 1} t_{2k}, 0\}, ..., \min\{-\sum_{k\neq n+1} t_{n+1,k}, 0\})$ makes zero revenue as follows: $p'_1 = \sum_{k=2}^{n+1} t_{w_{-1},k} - \sum_{k=2}^{n+1} t_{1k} = 0 - 0 = 0$; and $p'_j = 0$ for all $j \neq 1$, since the winner does not change with j's absence. □

The main idea of Proposition 4 is that externalities in a shill bid should be large enough negative so that $w_{-k} = w$ (i.e., the winner does not change with k's absence, and therefore k's payment is zero) for any $k \in N\backslash\{1\}$, and put some positive externality on himself so that his payment could be zero. Then, one may wonder if this idea could be used for a TTD shill-bidding strategy. However, in that case, the VCG mechanism is not well-defined due to $\infty - \infty$.

3.2 Impossibility for package auctions without externalities

It is well-known that the VCG mechanism is not shill-proof in package auctions (even without externalities). However, neither the existence nor the non-existence of a TTD shill-bidding strategy has been known. This section shows that a TTD shill-bidding strategy is impossible in package auctions. That is, the VCG package auction is not TTD manipulable in terms of shill bidding.

The model has a finite collection Y of goods, where Y is a *multiset* to allow multiple units of homogeneous goods. A *package* Z is a subset of Y. Each bidder $i \in N$ has a valuation $v_i : 2^Y \to \mathbb{R}_+$ on each package. Let $Z_i \subseteq Y$ denote the package that bidder i receives, and the set of all feasible allocations is denoted by $\mathcal{Z} = \{(Z_i)_{(i\in N)} : Z_i \cap Z_j = \emptyset, \forall i, j \in N\}$. Let $\mathcal{Z}_{-i}$ denote the set of all feasible allocations excluding i, i.e., $Z_i = \emptyset$. The efficient allocation is denoted by $(Z_i^*)_{i\in N} = \arg\max_{(Z_i)_{i\in N}\in\mathcal{Z}} \sum_{i\in N} v_i(Z_i)$. The VCG package auction (i.e., the VCG mechanism in package auctions) is an efficient mechanism in which each bidder i pays

$$p_i = \max\left\{\sum_{j\neq i} v_j(Z_i) : (Z_j)_{j\neq i} \in \mathcal{Z}_{-i}\right\} - \sum_{j\neq i} v_j(Z_j^*). \tag{8}$$

We first show an example where the VCG mechanism is not shill-proof.

Example 10. There are two goods A and B, i.e., $Y = \{A, B\}$. Bidder j's values are represented by a triplet $(v_j(\{A\}), v_j(\{B\}), v_j(\{A, B\}))$. The VCG mechanism is used in which each bidder bids truthfully.

(i) There are two bidders:

- bidder 1: $(7, 7, 16)$,

- bidder 2: $(0, 0, 12)$.

The bidder 1 receives both A and B and pays 12; thus, the payoff is $\pi_1 = 16 - 12 = 4$.

(ii) There are three bidders:

- bidder 1: $(7, 0, 8)$,
- bidder 2: $(0, 0, 12)$,
- bidder 3: $(0, 7, 8)$.

Then, bidder 1 receives A and pays $12 - 7 = 5$, and bidder 3 receives B and pays $12 - 7 = 5$. That is, bidder 1 in (i) can be better off using a shill bid of bidder 3 in (ii), since bidder 1's effective payoff is $\tilde{\pi}_1 = 16 - (5 + 5) = 6 > 4 = \pi_1$.

This shill-bidding strategy may look similar to the valuation-split strategy in Proposition 2, so one may think this may always work, but this is not true. Bidder 1 can also be worse off using the same shill bid depending on bidder 2's bid as follows.

(i-1) There are two bidders:

- bidder 1: $(7, 7, 16)$,
- bidder 2: $(0, 8, 12)$.

As in (i), Bidder 1 receives both A and B and pays 12, and $\pi_1 = 16 - 12 = 4$.

(ii-2) There are three bidders:

- bidder 1: $(7, 0, 8)$,
- bidder 2: $(0, 8, 12)$,
- bidder 3: $(0, 7, 8)$.

Bidder 1 now receives A and pays $12 - 8 = 4$, and bidder 2 receives B and pays $14 - 7 = 7$, and bidder 3 (the shill bidder of bidder 1) loses. Thus, bidder 1's effective payoff is now $\tilde{\pi}_1 = 7 - 4 = 3 < 4 = \pi_1$, i.e., lower than that of (i-1). $\square$

We also provide an example of homogeneous goods, which should be carefully considered in Proposition 4.

Example 11. There are two identical goods, i.e. $Y = \{A, A\}$. The value will be represented by $(v_j(\{A\}), v_j(\{A, A\}))$. The VCG mechanism is used in which bidders bid truthfully.

(i) There are two bidders:

- bidder 1: $(0, 3)$,
- bidder 2: $(1, 2)$.

Bidder 2 loses.

(ii) There are three bidders:

- bidder 1: $(0, 3)$,
- bidder 2: $(3, 3)$,
- bidder 3: $(3, 3)$.

Each of bidders 2 and 3 receives one unit of A but pays nothing. That is, if bidder 2 in (i) uses a shill bid of bidder 3 in (ii), even a loser can win and pay nothing by shill bidding. Note that this is an interesting case where bidder 2 can win an item at zero even though there is another bid of 3 by bidder 3. However, with the same shill bid, bidder 2 can still be worse off depending on others' bids as follows.

(ii-1) There are four bidders:

- bidder 1: $(0, 3)$,
- bidder 2: $(3, 3)$,
- bidder 3: $(3, 3)$,
- bidder 4: $(2, 3)$.

Bidder 2 and bidder 3 (the shill bidder of bidder 2) still win each item, but each now needs to pay $5 - 3 = 2$. However, bidder 2's true value on both items are only 2, as in (i). Thus, bidder 2's effective payoff is $\pi'_2 = 2 - (2 + 2) = -2 < 0 = \pi_2$, which makes him worse off. □

In the above examples, whether a shill bid is profitable or not depends on others' bid profile. We now show that this is true in general. That is, the VCG package auction is not TTD manipulable in terms of shill-bidding. The main idea is similar to Proposition 1, and we show the result formally for the VCG package auction.

Theorem 4. *There is no TTD shill-bidding strategy in the VCG package auction (without externalities).*

To prove Proposition 4, we first introduce several lemmas. Lemma 4 shows the price bound of a package. For intuition, in the simplest case, suppose all goods are heterogeneous, and bidder 1 wins package $A \in Y$. Then, bidder 1's payment cannot be lower than the maximum bid on A of all other bidders. However, this argument should be modified when there is a homogeneous good. For instance, in Example 11-(ii), bidder 2 can still win one unit at zero, even though bidder 3's bid on one item is 3. Considering homogeneous goods as well, Lemma 4 generalizes the price bound. Let $W(\tilde{B})$ denote the set of all bidders who win at least one item with reported profile $\tilde{B}$, e.g., if $Y = \{A, B, C\}$ and bidder 1 wins A and bidder 2 wins B and C with truthful bidding, $W(T) = \{1, 2\}$.

Lemma 4 (Price bound). *In the VCG package auction, suppose bidder i wins package $Z_i \subseteq Y$ only. If there is a bid $b_j^{Z_i} \leq b_i(Z_i)$ of bidder $j \notin W(T)$, then $p_i \geq b_j^{Z_i}$.*

Proof. By Eq. (8), it is sufficient to show that the first term of the RHS of (8) is at least $b_j^{Z_i} + \sum_{j \neq i} v_j(Z_j^*)$. But, the first term of the RHS of (8) is the welfare of $N \backslash \{i\}$ with i's absence. When i is present, $W(T) \backslash \{i\}$ wins $Y \backslash Z_i$. Since $j \notin W(T)$, one way to allocate Y with i's absence is to allocate Z_i to j and to allocate $Y \backslash Z_i$ to $W(T) \backslash \{i\}$. Thus, the welfare of $N \backslash \{i\}$ with i's absence should be at least $b_j^{Z_i} + \sum_{j \neq i} v_j(Z_j^*)$. □

Lemma 5 (No overbidding TTD). *In the VCG package auction, any shill bidding including overbidding cannot be TTD.*

Proof. If overbidding does not change any allocation, the payoff also does not change. Thus, we only need to consider the case when the allocation changes. Suppose bidder i cannot win package Z_i with truthful bidding. First, consider a case where i uses only one shill bidder s who submits $b_s^{Z_i} > v_i(Z_i)$ and wins Z_i. Then, if there is a bid $(v_i(Z_i) + b_s^{Z_i})/2$ of bidder $j \notin W(T)$, the shill bidder s of bidder i wins but pays $(v_i(Z_i) + b_s^{Z_i})/2 > v_i(Z_i)$. Likewise, suppose real bidder i also submits some bid $b_i^{Z_i'}$. Then, if there is a bid $v_i(Z_i')$ of bidder $k \notin W(T) \backslash \{j\}$, then bidder i either loses or pays at least $v_i(Z_i')$. Thus, the effective payoff of bidder i is $\tilde{\pi}_i < 0$. The case when there are multiple shill bidders of i can be proved in the same way. □

The following lemma shows that the same package cannot be won at a lower price by shill bidding. That is, for shill bidding to be profitable, the allocation change is necessary.

Lemma 6 (One price for the same package). *In the VCG package auction, it is impossible to buy the same package at a lower price by shill bidding.*

Proof. Suppose bidder i wins package Z_i at p_i. Let C and D denote the first and the second term in Eq (8) so that $p_i = C - D$. Since there are no externalities, using shill bidding can only weakly increase C by monotonicity. Thus, to win Z_i at a lower price, D should be increased. But then bidder i's shill bidder needs to win some nonempty package $Z' \subseteq Y \backslash Z_i$, which leads to an allocation change. □

By using Lemmas 4 to 6, we now prove Proposition 4.

Proof of Proposition 4. Suppose bidder i wins package $Z_i \subseteq Y$ (which may also be $\emptyset$) with truthful bidding that makes $\pi_i \geq 0$, and also suppose there exists a TTD shill-bidding strategy. By Lemma 6, for a profitable shill bidding, bidder i and his shill bidders should win some package $Z \neq Z_i$. But by Lemma 5, any overbidding shill-bidding strategy cannot be TTD. Thus, for any possible package $Z \neq Z_i$ for which bidder i and his shill bidders win, by Lemma 4, if there is a bid $v_i(Z)$ of bidder $j \notin W(T)$, then the effective payment is at least $v_i(Z)$; thus, the effective payoff becomes $\tilde{\pi}_i \leq 0 \leq \pi_i$. □

4 Other Applications

This section provides more applications of truth-telling dominance.

4.1 Matching markets

The Deferred Acceptance algorithm (Gale and Shapley, 1962) is widely used in matching markets. While it is strategy-proof from the proposing side, there are many well-known manipulations from the proposed side: *truncation strategy* (Roth and Rothblum, 1999), *manipulation via capacities* (Sönmez, 1997). We show that for both manipulation, TTD strategies do not exist. We use the same model in Sönmez (1997). Since the proof is quite simple, we do not state the model in detail. The model has n students and m hospitals. Each student has a strict preference ranking over hospitals, and vice versa. Each hospital h has a capacity $q_h \geq 1$.[17]

Interestingly, to prove the nonexistence of a TTD strategy in each case, only *individual rationality* is needed: a mechanism is *individually rational* if it does not assign i to j when i is unacceptable to j. That is, a mechanism need not be *pairwise stable* for our results, Proposition 5 and 6. Thus, we only consider *individually rational* mechanisms.

Truncation strategy A *truncation strategy* is a strategy that truncates some least preferred choices, e.g., let $\succ_{h_1}$: s_1, s_2, s_3 denote h_1's true preference such that h_1 prefers s_1 to s_2 and then s_2 to s_3, then $\succ'_{h_1}$: s_1 which truncates s_2 and s_3 is a truncation strategy of h_1. While the Deferred Acceptance algorithm is not immune to the manipulation via truncation, any individually rational mechanism is not TTD manipulable via truncation.

Proposition 5. *Any individually rational matching mechanism is not TTD manipulable via truncation.*

[17] For our results, Proposition 5 and 6, each student can also has a capacity greater than 1.

Proof. For any realization of truncation strategies of hospital h where some student s is unacceptable by truncation, there exists a preference profile of students where h is acceptable to student s only. $\square$

Manipulation via capacities A *manipulation via capacities* is a strategy such that hospital h misreports its capacity $q'_h < q_h$. We assume that hospitals prefer a matching with more students than with less students, e.g., three students are more preferred than two students regardless of the combination of students. The Deferred Acceptance algorithm is not immune to the manipulation via capacities, but any individually rational machanism is not TTD manipulable via capacity-manipulation.

Proposition 6. *Any individually rational matching mechanism is not TTD manipulable via capacity-manipulation.*

Proof. Consider hospital h with a capacity $q_h \geq 2$. For any realization of strategies with a reduced capacity $q'_h < q_h$, there exists a preference profile of other players such that q_h students could have been assigned to h with the original capacity q_h, but only up to q'_h students are assigned with the reduced capacity q'_h, e.g., h is only acceptable to some q_h students, and all those q_h students are acceptable to h, and h is the only acceptable hospital to those q_h students. $\square$

Another widely used and studied mechanism, the Boston mechanism is obviously manipulable (Troyan and Morrill, 2020), but not TTD manipulable.

Proposition 7. *The Boston mechanism is not TTD manipulable with a canonical strategy space.*

Proof. The proof is almost by definition. Suppose there exists a TTD strategy $\sigma_i(\cdot)$ for agent i. By the definition of TTD, $\sigma_i(\theta_i) \neq \theta_i$ for some θ_i. But then there exists a profile of others θ_{-i} to which i's strictly best response is truth-telling, e.g., i could have been assigned to her first choice by truth-telling (i.e., the first choice was not competitive as expected), but ended up with the second choice by misreporting. $\square$

4.2 Referral network

Jeong and Lee (2024) study mechanism design in the *referral network*, an environment where an agent can only participate via referrals. In the auction context, each bidder $i \in N = \{1, 2, ..., n\}$ and the seller, denoted by 0, has a type $\theta_i = (v_i, r_i) \in V_i \times R_i$, where v_i is a *valuation*, and $r_i \subseteq N \backslash \{i\}$ is a *connection*, i.e., a set of referable bidders. The seller's

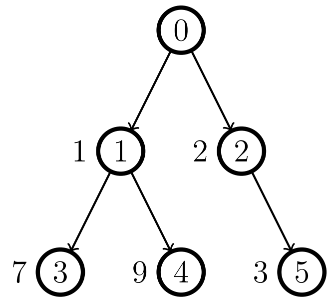


Figure 3: Auction with referral network

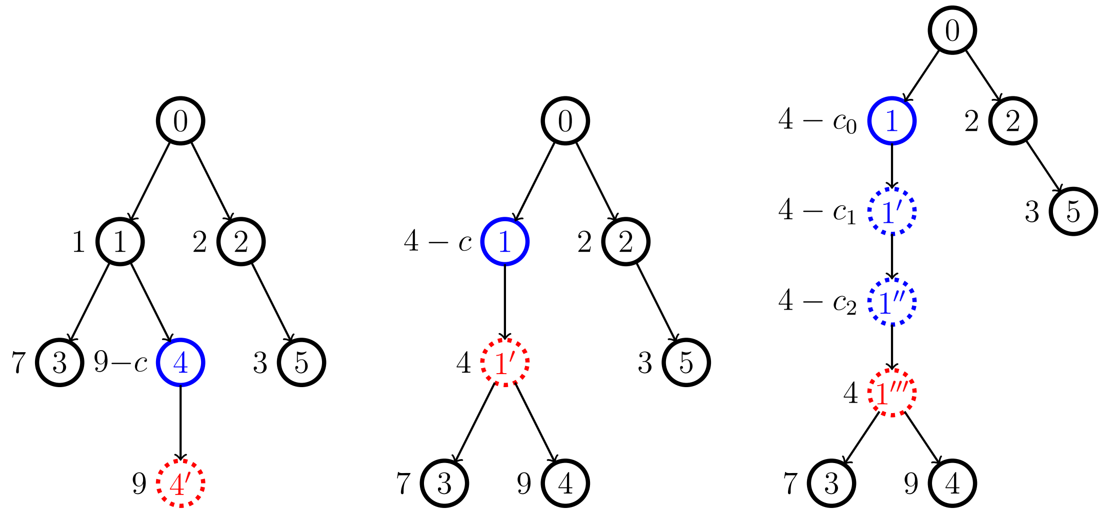
valuation is normalized to $v_0 = 0$, and by Myerson and Satterthwaite (1983) we assume that the seller is not a strategic player, i.e., the seller reports (v_0, r_0) truthfully. Thus, r_0 denotes the bidders who can participate without any referral by other bidders. All other bidders can participate if they are (directly or indirectly) referred by other bidders. For instance, in Figure 3, bidder 3 (who is not directly referred by the seller) can still participate if the seller refers bidder 1, and bidder 1 refers bidder 3. Each bidder can misreport his value $v_i' \neq v_i$ and his referral $r_i' \neq r_i$ but with a restriction $r_i' \subseteq r_i$. That is, i can only refer someone whom i "knows."

The usual second-price auction does not provide a *referral reward*, i.e., payment to a bidder. Thus, there is no incentive for bidders to refer other bidders, since referring only intensifies the competition. That is, in second-price auctions, only bidders in r_0 participate and still bid truthfully, which does not always achieve an efficient outcome.

4.2.1 The VCG mechanism

In the referral network model, the VCG mechanism is still efficient, strategy-proof, and individually rational. Note that what bidder i pays in the VCG mechanism in general is the amount of the externality that bidder i imposes, i.e., the social welfare of others with i's absence $-$ the social welfare of others with i's presence. Let w and w_{-i} respectively denote the winner and the winner with i's absence (after ties are broken), i.e., w_{-i} is the bidder with the highest value among bidders who can participate without i. Then, if bidder i loses but the winner cannot participate without i, then i's payment is $p_i = v_{w_{-i}} - v_w \leq 0$ (< 0 if no ties), i.e., bidder i receives a referral reward. However, VCG is not budget feasible, i.e., the seller may have a negative revenue $\pi_0 < 0$.

TTD shill bidding Considering the motivation where the seller does not know the network beyond her own connection r_0, shill bidding may be more plausible, but the VCG mechanism is not immune to shill bidding, as in other settings such as Section 3. Moreover, Proposition



(a) Shill bidding of bidder 4 (b) Shill bidding of bidder 1 (c) Iteration of shill bidding

Figure 4: TTD shill bidding in VCG (Example 12)

8 and Example 12 below show that i-TTD shill-bidding strategies exist.

Example 12 (Shill Bidding)**.** As shown in Figure 3, consider a single-item auction with 5 bidders $N = \{1, 2, 3, 4, 5\}$ with values $v = (1, 2, 7, 9, 3)$ and connections $r_0 = \{1, 2\}$, $r_1 = \{3, 4\}$, $r_2 = \{5\}$, and $r_3 = r_4 = r_5 = \emptyset$.

- *Second-price auction*: Bidders 1 and 2 do not refer any other bidder, and bid truthfully. Thus, bidder 2 wins and pays 1, which is an inefficient outcome.

- *VCG*: Bidders report truthfully, and bidder 4 wins and pays $p_1^V = v_3 - 0 = 7$. But bidder 1 pays $p_1^V = v_5 - v_4 = 3 - 9 = -6$, i.e., the seller should pay 6 to bidder 1. Bidders 2, 3, and 5 pay nothing. Thus the revenue is $\pi_0 = 7 - 6 = 1$. Note that, however, if $v_3 < 6$, then $p_4^V < 6$, or if $v_4 > 10$, then $p_1^V < -7$; thus, the seller's revenue becomes $\pi_0 < 0$, i.e., *budget infeasible.*

- *VCG with shill bidding*: As shown in Figure 4a, suppose the winner, bidder 4, introduces a shill bidder $4'$ and submits $v'_{4'} = v_4 = 9$, $r'_{4'} = \emptyset$, and $v'_4 = v_4 - c$ for some $0 < c \leq v_4$ and $r'_4 = \{4'\}$. Then, $p'_4 = v_3 - v'_{4'} = v_3 - v_4 = -2$ and $p'_{4'} = \max\{v_3, v'_4\} - 0 \leq v_4 - c$; thus, bidder 4's effective payment is $p'_4 + p'_{4'} = v_3 - c = 7 - c < 7 = v_3 = p_1^V$, and therefore, bidder 4 is better off.

  Likewise, a loser who receives a referral reward can also be better off by using shill bidding in the same way. For instance, in Figure 4b, bidder 1 can use a shill bidder $1'$ with any $v'_1 < v_1 = 4$. Moreover, as shown in Figure 4c, bidder 1 can use such a shill-bidding strategy iteratively, which turns out to be i-TTD (Proposition 8).

Neither the existence nor the nonexistence of a TTD strategy may not be trivial. For instance, the shill bidding that i refers $r_i \cup \{s\}$ for a shill bidder s with some (v_s, r_s) may not work depending on other bidders' reports.

Proposition 8 below shows that the shill-bidding strategy in Example 12—bidder i's shill bidder i' reports as if he is i, and i bids any bid less than his true value and refers his shill bidder—is indeed i-TTD. In other words, the i-TTD strategy is to insert a "shill" bidder (who is in fact perceived as a real bidder to other bidders) between himself and his referrals. By doing this, the last shill bidder (colored in red, or $4'$, $1'$, and $1'''$ in Figure 4a, 4b, and 4c, respectively) pays the same as before, or may pay more but even in that case, the additional referral reward due to shill bidding dominates the additional payment of the last shill bidder. If the i-TTD strategy is used iteratively, the seller eventually ends up with a budget deficit.

Proposition 8 (TTD Shill Bidding). *The VCG mechanism is i-TTD manipulable via shill-bidding. Any bidder $i \in N$ has an i-TTD shill-bidding strategy that reports any $\theta'_i = (v'_i \in [0, v_i), r'_i = \{i'\})$ and $\theta'_{i'} = (v'_{i'} = v_i, r'_{i'} = r_i)$, where i' is a shill bidder. In addition, no other bidders are worse off.*

Proof. First, suppose i is the winner without shill bidding. Then, the winner's payment is $p_i = v_{w_{-i}}$. Note that $v_{w_{-i}} < v_w$ generically. Now, if i uses a shill bid with some $v'_i < v_i$, then

$$p'_i = v_{w_{-i}} - v_w,$$
$$p'_{i'} = \max\{v'_i, v_{w_{-i}}\} \leq v_w.$$

Thus, i's effective payment is $p'_i + p'_{i'} \leq v_{w_{-i}} = p_i$, and $\pi'_i \geq \pi_i$ (strictly unless $v_{w_{-i}} = v_w$).

Second, suppose i is a loser who receives a referral reward. Note that $p_i = v_{w_{-i}} - v_w < 0$. Now, if i uses a shill bid with some $v'_i < v_i$, then

$$p'_i = v_{w_{-i}} - v_w = p_i,$$
$$p'_{i'} = \max\{v'_i, v_{w_{-i}}\} - v_w < 0.$$

Thus, i's effective payment is $p'_i + p'_{i'} < p_i$, and $\pi'_i > \pi_i$.

Finally, suppose i is a loser without a referral reward, then using the shill bid does not change any allocation (up to ties) and payments. Note also that, as shown in Proposition 1, the shill-bidding strategy is not only TTD but also i-TTD. The shill-bidding strategy does not change the winner (since there is no overbidding) and only reduces the effective payment of the bidder who uses shill bidding. $\square$

TTD fake referral Jeong (2020b) shows that the VCG mechanism is *referral monotone*,

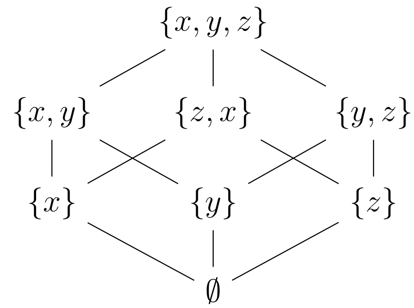


Figure 5: The lattice induced by the fake referral i-TTD strategy (Proposition 9) of bidder i when $N \backslash r_i = \{x, y, z\}$ in Example 13.

i.e., i's payoff increases as the reported referral r_i' expands. That is, the more refer, the better the payoff is.

Definition 6 (Referral Monotonicity)**.** A mechanism is *referral monotone* if any bidder i's payoff increases as r_i expands, i.e., $\pi_i(v', r'; v, r'') \leq \pi_i(v', r''; v, r'')$ for all $i \in N$, all $(v, r) \in V \times R$, all $v_i' \in V_i$, and all $r_i', r_i'' \in R_i$ such that $r_i' \subseteq r_i''$.

If bidder i can do "*fake referral*", i.e., reporting $r_i' \not\subseteq r_i$, then the VCG mechanism is no longer strategy-proof. If fake referral is allowed, reporting true value and the *full fake-referral*, i.e., $(v_i, N \backslash \{i\})$, is a weakly dominant strategy, and is also TTD but not i-TTD by Proposition 3.

In addition, Proposition 9 below shows that there exists a bit more general (i.e., referring one more instead of everyone) and iterative TTD referral strategy that induces a lattice, as illustrated in Example 13 with Figure 5.

Example 13. Suppose $N \backslash r_i = \{x, y, z\}$. Bidder i is better off reporting true value but referring one more bidder $j \in \{x, y, z\}$ in addition to r_i. Figure 5 shows the lattice structure induced by the iteration of this fake referral.

Proposition 9 (TTD Fake Referral)**.** *In the VCG mechanism, $\sigma_i(v_i, r_i) = (v_i, r_i \cup \{j\})$ for some $j \in N \backslash r_i$ is i-TTD.*[18] *In addition, $\big(\cup_{k \geq 0} \operatorname{supp}(\sigma_i^k((v_i, r_i))), \leq\big)$ is a Jordan-Dedekind lattice.*

Proof. By referral monotonicity (Definition 6), referring additional agent is always weakly better off, and the existence of a type profile of others for which strictly better off is obvious, i.e., bidder i is strictly better off if the additional referral includes a bidder with a higher value than all the bidders already connected. This is also true when i is the winner, i.e.,

[18]If $N \backslash r_i = \emptyset$, then $\sigma_i(v_i, r_i) = (v_i, r_i)$, i.e., truth-telling, which has no problem due to Definition 1 (ii).

additionally referring such a bidder provides i a greater payoff (referral reward) than winning. Formally, this happens when $v_i - v_{w_{-i}} < -(v_{w_{-i}} - v_k)$, where k is the bidder additionally referred by i. That is, $v_{w_{-i}} - v_k$ is i's payment with the additional referral, but this amount is negative, i.e., $-(v_{w_{-i}} - v_k)$ is the referral reward. The same logic applies to a loser. Thus, $\sigma_i(v_i, r_i)$ is TTD, and due to referral monotonicity, until r_i' becomes $N \backslash \{i\}$, i can use $\sigma_i(\cdot)$ iteratively; thus, $\sigma_i(v_i, r_i)$ is an iterative TTD strategy.

For the Jordan-Dedekind chain condition, first note that the usual set inclusion partial order is well defined in the collection of the realized set of referred agents (see Figure 5, for instance). Since the cardinality of the referred agents can only increase by one, for any two distinct elements in $\cup_{k \geq 0} \text{supp}(\sigma_i^k(v_i, r_i))$, all maximal chains have the same length. $\square$

Considering the motivation where the seller does not know the network beyond her own connection r_0, shill bidding or fake referral may be a real concern. While fake referral can be prevented if the bidders' identities are kept secret, shill bidding cannot be prevented under the assumption that each bidder's type (v_i, r_i) is private information. Jeong (2020b) shows that the fake referral strategy due to referral monotonicity can be seen as a connection-expansion incentive, i.e., given an initial network, bidders have an incentive to expand their connection. Jeong (2020b) shows that if every bidder expands his connection, the seller is better off and bidders are worse off, which is called the *Referrer's Dilemma*. That is, (the outcome due to) the Referrer's Dilemma is better for the seller, whereas shill bidding is worse for the seller. But Jeong (2020b) shows that the effect of the TTD shill-bidding dominates the Referrer's Dilemma in the sense that the seller has a budget deficit. Thus, the existence of a TTD shill-bidding strategy may be quite undesirable for the VCG mechanism to be used in this application.

4.2.2 Groupwise-Pivotal Referral mechanism

Jeong and Lee (2024) introduce the Groupwise-Pivotal Referral (GPR) mechanism that is budget feasible, efficient, individually rational. GPR is also *referral strategy-proof*, i.e., referring truthfully is the best response to any reported profiles of others and her own bid. GPR has weakly higher ex-post revenue than both the VCG and the second-price auctions. As its name suggests, the GPR mechanism's payment rule can be interpreted as a level-by-level payment, i.e., each bidder i pays $p_i^G = v_{w_{-i}}$ to her parent in the *contribution tree*, a directed rooted tree induced from marginal effective referrals. As in Figure 3, when there is no bidder who is referred by two bidders, the referral network r itself is a tree, so it is clear which referral is effective for a bidder's participation. However, a referral network may not be a tree in general. Let $G_i(r)$ denote i's *group*, i.e., the set of bidders who cannot participate without i. Then, the set of all distinct groups becomes a partially ordered set with the usual set inclusion, which induces the contribution tree.

Jeong and Lee (2024) show that GPR is *groupwise collusion-proof*, i.e., any group G_i cannot increase the sum of their payoffs by any misreporting of the group. The groupwise collusion-proofness of GPR is implied by its payment rule, since any misreporting of any bidder in i's group cannot change p_i^G (given the "same" allocation, but if the allocation changes, they are worse off). But this also holds when i (or any bidder in G_i) can use shill bidding, since any shill bidder of i cannot be a parent of i but can only be in i's group, which is summarized as the following lemma.

Lemma 7. *In the GPR mechanism, any shill bidding of bidder i cannot change p_i^G (if the winner is still in the group of the same child of the seller).*

Thus, for any shill bidding of i to be profitable, i should be able to increase the payment that i receives by shill bidding, but this turns out to be impossible as a TTD manipulation.

Proposition 10. *The GPR mechanism is not TTD shill-bidding manipulable.*

Proof. For expositional ease, we assume that there are no ties in values. First, suppose i is the winner without shill bidding. Losing by shill bidding but not receiving a reward reward makes i worse off. By Lemma 7, i cannot decrease p_i^G; thus, for i to be better off by shill bidding, i should lose but receive a referral reward greater than his value. But all other bidders in his group have lower values than his; thus, for i to lose and receive a greater referral reward than v_i, the bid of shill bidder i' must be greater than v_i. This makes, however, i's referral reward come from his own shill bidder. Even when i uses multiple shill bidders, any payments up to i on the new winning path with shill bidding come from i's shill bidders. Thus, those payments do not change i's effective payoff.

Second, suppose i is a loser who receives a referral reward. Since the child of i, denoted by j, should pay $v_{w_{-j}} \geq v_i$ to i,

$$\pi_i \geq v_i - v_{w_{-i}}. \tag{9}$$

There are two cases to be considered: (i) i and his shill bidders lose but increase the effective referral reward; (ii) one of his shill bidders wins (note that i's winning cannot be profitable).

Case (i): If i still loses by shill bidding but increases her referral reward, i can be better off indeed, however, this requires shill bidding including overbidding, i.e., $v'_{i'} > v_i$, since i's referral reward without shill bidding is already at least v_i. Then, there exists profile of others such that $v_i < v_{w_{-i'}} < v'_{i'}$ that make i' win, which makes i's effective payoff $\pi'_i = v_i - w_{-i} \leq \pi_i$ by (9), where the inequality holds strictly when $v_{w_{-j}} > v_i$. Thus, i is worse off with shill bidding. Note that as in the first case, the referral reward for i comes from the payment of

i'; thus, the referral reward does not affect i's effective payoff. Case (ii): If i wins by shill bidding, as in the last part of Case (i), i is worse off. Finally, suppose i is a loser without a referral reward, then using a shill bid does not change any allocation and payments. Thus, (ii) of the second case is the only case where i can ever be better off with shill bidding for a fixed profile of others, but for that case there exists a profile of others for which i is worse off. Therefore, there is no TTD shill-bidding strategy. $\square$

4.3 Other Applications

Jeong (2024) shows that when there are only two bidders in the third-price auction, bidders are better off using shill bidding. In general, in the kth-price auction with $k-1$ bidders, bidders have a TTD shill-bidding strategy such that the real bidder bids truthfully and the shill bidder bids the reserve price.

5 Discussion

Strategy-proof mechanisms may not be used due to constraints (e.g., the core property, budget feasibility, etc) that cannot be satisfied together with strategy-proofness. On the other hand, as in shill bidding, even when a mechanism is strategy-proof, it may still be manipulable by some strategy that cannot be expressed by agent's type in the mechanism considered.

It may even be impossible to prove that all kinds of manipulation are impossible even for one mechanism, since there might be some manipulation beyond our imagination. What we can prove instead may be only the impossibility of the manipulation under certain restrictions. When we restrict our attention to a certain strategy space, it would still be important to show whether the mechanism is (i-)TTD manipulable or not, since if an (i-)TTD strategy exists, the players will be likely to use it. Even if a TTD strategy exists, it may still be nontrivial to show its existence. Likewise, showing the nonexistence of a TTD strategy may be nontrivial, as we show both for the shill bidding in the VCG mechanism in two different environments.

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