The Empirical Content of Expected Utility

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Abstract

We characterize the empirical content of expected utility. We show that the empirical content of the expected utility theory is contained in reflexivity, transitivity, and strong independence axioms. However, under commonly used weaker forms of independence axiom, the continuity axiom adds empirical content in the expected utility theory. This formalizes and makes exact the ubiquitous claim that the continuity axiom is a technical axiom without empirical content in the expected utility theory.

1 Introduction

The expected utility theory is one of the most influential theories in economics. It was axiomatized by completeness, transitivity, independence, and Archimedean continuity axioms by von Neumann and Morgenstern (1947). However, it is commonly regarded that the continuity axiom is a technical assumption without empirical content (Fishburn, 1988, p. 47). In this paper, we formalize this common claim with a caveat: It only holds under Samuelson's (1983b) strong independence axiom and not under commonly used weaker forms of independence axiom.

We show that the empirical content of the expected utility is contained in reflexivity, transitivity, and strong independence axioms from Samuelson (1983b) assuming that for all lotteries P, Q, R and $\alpha \in (0,1)$,

$$P \succsim Q \iff \alpha P + (1-\alpha)R \succsim \alpha P + (1-\alpha)R.$$

However, contrary to common thought (Fishburn, 1988, p. 47), we show that the continuity axiom adds empirical content under weaker forms of independence axiom as introduced by Jensen (1967) stating that for all lotteries P, Q, R and $\alpha \in (0,1)$,

$$P \succ Q \Longrightarrow \alpha P + (1-\alpha)R \succ \alpha P + (1-\alpha)R,$$

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The author thanks Artur Dolgopolov for valuable discussions. This work was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) — Project-ID 317210226 — SFB 1283.

as introduced by Samuelson (1983a) stating that for all lotteries P, Q, R and $\alpha \in (0,1)$,

$$P \succsim Q \Longrightarrow \alpha P + (1-\alpha)R \succsim \alpha P + (1-\alpha)R,$$

or as introduced by Luce and Raiffa stating that for all lotteries P, Q, R, S and $\alpha \in (0,1)$,

$$\alpha P + (1-\alpha)R \succsim \alpha Q + (1-\alpha)R \Longrightarrow \alpha P + (1-\alpha)S \succsim \alpha Q + (1-\alpha)S.$$

Based on this, we recommend using the strong independence axiom for the expected utility theory, in which case the continuity axiom is purely a technical axiom without empirical content. Some advanced microeconomics textbooks use the strong independence axiom for the expected utility theory such as Mas-Colell et al. (1995), Rubinstein (2006), and Muñoz-Garcia (2017). However, most advanced microeconomics textbooks use a weaker form of independence axiom such as Fishburn (1970), Kreps (1988; 1990; 2013), Luenberger (1995), Silberberg and Suen (2000), Jehle and Reny (2011), Varian (2014), and Wang (2018).

More specifically, we consider a finite data set of weak and strict preferences. We show that this finite data set is rationalizable by an expected utility if and only if the data set is rationalizable by reflexive, transitive, and strong independence axiom satisfying preferences. This shows that the Archimedean continuity axiom and the completeness axiom do not add empirical content to the expected utility and are purely a technical axiom. Chambers et al. (2014) observed that the completeness axiom alone does not have empirical content and we show that it does not have empirical content under strong independence axiom and transitivity either. Additionally, we show that this characterization is tight and the strong independence axiom cannot be weakened in this theorem.

This paper contributes to the literature on studying the empirical content of representations. Fishburn (1975) characterized the empirical content of expected utility with a testable condition based on solving a system of linear inequalities. Border (1992) considers lotteries with monetary prizes and connects rationalization of choices with expected utility to first-order stochastic dominance violations. Kim (1996) generalizes Fishburn (1975) by considering lotteries over compact metric spaces. In contrast, we connect the empirical content of expected utility to standard axioms of reflexivity, transitivity, and independence axiom.

Relatedly, Payró (2019) studied the empirical content of having an expected utility over a subjective states space as in Kreps (1979) and Dekel et al. (2001). Payró offers a similar condition for rationalization as in Fishburn (1975) and shows that the condition can be formulated as solving a non-linear system of inequalities.

The empirical content of additive representations has been often studied in Pfanzagl (1968), Adams et al. (1970), Luce et al. (1990), and discussed in Wakker (1988). Additionally, Adams (1992) studied the empirical content of theories of subjective probability.

Chambers et al. (2014) provided a general axiomatization for the empirical content of a theory by UNCAF axioms[1]. However, adding axioms that are not UNCAF form to other axioms can increase the empirical content of the theory. We show that this does not happen under the strong independence axiom but happens under weaker forms of the independence axiom. Additionally, Chambers et al. (2017) studies when a revealed preference theory has an axiomatization with universal sentences.

The remainder of the paper proceeds as follows: Section 2 characterizes the empirical content of the expected utility under strong independence axiom. Section 2.2 provides counterexamples for weaker forms of independence axiom. Section 3 concludes.

2 Setting

We follow the definitions for empirical content from Chambers et al. (2014) when applied to lotteries. The set of outcomes is X and we consider (simple) lotteries on X, $\Delta(X)$. We assume that the lotteries are reduced to single-stage lotteries and the mixtures of lotteries are defined prizewise: We define for all $\alpha \in [0,1]$, $P, Q \in \Delta(X)$, and $x \in X$,

$$\Big(\alpha P + (1-\alpha)Q\Big)(x) = \alpha P(x) + (1-\alpha)Q(x).$$

A data set is $\mathcal{D} = (D, \succsim_{\mathcal{D}}, \succ_{\mathcal{D}})$ where $D \subset \Delta(X)$ is a finite set and $\succsim_{\mathcal{D}}$ and $\succ_{\mathcal{D}}$ are (possibly incomplete) binary relations on D. Here, each $P \succsim_{\mathcal{D}} Q$ is an observation for weak preference and $P \succ_{\mathcal{D}} Q$ is an observation for strict preference.

[1]A universal negation of a conjunction of atomic formulae (UNCAF) axiom is a string of the form

$$\forall v_1 \forall v_2 \dots \forall v_n \neg(\varphi_1 \wedge \varphi_2 \cdots \wedge \varphi_m)$$

where $\varphi_1, \varphi_2, \dots, \varphi_m$ are atomic formulae with variables from $v_1, \dots, v_n$.

A theory $\mathcal{T}$ is a collection of pairs of binary relations $(\succsim, \succ)$ on $\Delta(X)$.[2]

The theory of expected utility $\mathcal{T}_{\text{EU}}$ is the collection of pairs of binary relations $(\succsim, \succ)$ on $\Delta(X)$ such that $\succ$ is the asymmetric part[3] of $\succsim$ and there exists an affine $u : \Delta(X) \to \mathbb{R}$ with for all $P, Q \in \Delta(X)$,

$$P \succsim Q \iff u(P) \geq u(Q).$$

The theory of reflexive transitive order satisfying the strong independence axiom $\mathcal{T}_{\text{Tr-Ind}}$ is the collection of pairs of binary relations $(\succsim, \succ)$ on $\Delta(X)$ such that $\succ$ is the asymmetric part of $\succsim$ and the following three conditions hold.

(1) For all $P \in \Delta(X)$, $P \succsim P$.

(2) $\succsim$ is transitive.

(3) For all $P, Q, R \in \Delta(X)$ and $\alpha \in (0, 1)$

$$P \succsim Q \iff \alpha P + (1-\alpha)R \succsim \alpha Q + (1-\alpha)R.$$

We say that a theory rationalizes a data set if the theory includes a pair of binary relations that contain the observed binary relations.

Definition Let $\mathcal{T}$ be a theory and $\mathcal{D} = (D, \succsim_{\mathcal{D}}, \succ_{\mathcal{D}})$ be a data set. $\mathcal{T}$ *rationalizes* $\mathcal{D}$ if there exists $(\succsim, \succ) \in \mathcal{T}$ such that $\succsim_{\mathcal{D}} \subset \succsim$ and $\succ_{\mathcal{D}} \subset \succ$.

We say that two theories are data equivalent if they rationalize the same data sets.

Definition Let $\mathcal{T}$ and $\mathcal{T}'$ be two theories. $\mathcal{T}$ *is data equivalent to* $\mathcal{T}'$ if for all data sets $\mathcal{D}$,

$$\mathcal{T} \text{ rationalizes } \mathcal{D} \iff \mathcal{T}' \text{ rationalizes } \mathcal{D}.$$

2.1 The Empirical Content of Expected Utility

Our main result shows that the empirical content of the expected utility is contained in transitivity and strong independence axioms. The Archimedean continuity axiom of the expected utility does not have empirical content.

[2]Our definition of a theory is more general than in Chambers et al. (2014). In contrast, Chambers et al. (2014) consider theories that are closed with respect to permutations of the prizes X: If $(\succsim, \succ)$ is a preference included in the theory and $\varphi : X \to X$ is a bijective function, then the preference order $(\succsim^{\varphi}, \succ^{\varphi})$ defined by for all $P, Q \in \Delta(X)$, $P \succsim^{\varphi} Q$ iff $\sum_x P(x)\delta_{\varphi^{-1}(x)} \succsim \sum_x Q(x)\delta_{\varphi^{-1}(x)}$, where $\delta_{\varphi^{-1}(x)}$ denotes the degenerate lottery for $\varphi^{-1}(x)$, and similarly for the strict part, is also included in the theory.

[3]Formally, for all $x, y \in \Delta(X)$, $x \succ y$ if and only if $x \succsim y$ and $y \not\succsim x$.

Theorem 1 $\mathcal{T}_{\text{EU}}$ is data equivalent to $\mathcal{T}_{\text{Tr-Ind}}$.

Next, we prove this result. We start with a lemma that any $(\succsim, \succ) \in \mathcal{T}_{\text{Tr-Ind}}$ satisfies the finite dominance axiom.

Lemma 2 Assume that $(\succsim, \succ) \in \mathcal{T}_{\text{Tr-Ind}}$ and for each $i \in \{1, \ldots, n\}$, $P_i, Q_i \in \Delta(X)$,$\alpha_i > 0$, and $P_i \succsim Q_i$ with $\sum_{i=1}^n \alpha_i = 1$, then $\sum_{i=1}^n \alpha_i P_i \succsim \sum_{i=1}^n \alpha_i Q_i$. Additionally, if for some $k \in \{1, \ldots n\}$, $P_k \succ Q_k$, then $\sum_{i=1}^n \alpha_i P_i \succ \sum_{i=1}^n \alpha_i Q_i$

Proof. We show first that if $P \succ Q$, then for all $R \in \Delta(X)$, $\alpha \in (0,1)$, $\alpha P + (1-\alpha)R \succ \alpha Q + (1-\alpha)R$. By the definition of $\succ$, the strong independence axiom, and the negation of the strong independence axiom, we have $\alpha P + (1-\alpha)R \succsim \alpha Q + (1-\alpha)R$ and $\alpha Q + (1-\alpha)R \not\succsim \alpha P + (1-\alpha)R$ that shows the claim.

We show the lemma for $n = 2$. The full lemma follows from this by induction. By the strong independence axiom, we have,

$$\alpha_1 P_1 + \alpha_2 P_2 \succsim \alpha_1 Q_1 + \alpha_2 P_2 \text{ and } \alpha_1 Q_1 + \alpha_2 P_2 \succsim \alpha_1 Q_1 + \alpha_2 Q_2.$$

Thus by the transitivity axiom, $\alpha_1 P_1 + \alpha_2 P_2 \succsim \alpha_1 Q_1 + \alpha_2 Q_2$.

Second, assume w.l.o.g. that $P_2 \succ Q_2$. By the above claim

$$\alpha_1 P_1 + \alpha_2 P_2 \succsim \alpha_1 Q_1 + \alpha_2 P_2 \text{ and } \alpha_1 Q_1 + \alpha_2 P_2 \succ \alpha_1 Q_1 + \alpha_2 Q_2.$$

Thus by a direct generalization of Mas-Colell et al. (1995, Proposition 1.B.1.(iii)), $\alpha_1 P_1 + \alpha_2 P_2 \succ \alpha_1 Q_1 + \alpha_2 Q_2$. $\square$

Proof of Theorem 1. Since transitivity and strong independence axioms are necessary conditions for the expected utility theory, we have that for any data set $\mathcal{D}$, if $\mathcal{T}_{\text{EU}}$ rationalizes $\mathcal{D}$, then $\mathcal{T}_{\text{Tr-Ind}}$ rationalizes $\mathcal{D}$. We show next that for any data set $\mathcal{D}$, if $\mathcal{T}_{\text{EU}}$ does not rationalize $\mathcal{D}$, then $\mathcal{T}_{\text{Tr-Ind}}$ does not rationalize $\mathcal{D}$.

Since D is a finite set of simple lotteries, there exists a finite set $Y = \{y_1, \ldots, y_n\} \subset X$ such that $D \subset \Delta(Y)$. Enumerate

$$\succsim^{\mathcal{D}} = \Big\{(P_{1,1}, Q_{1,1}), \ldots, (P_{p,1}, Q_{p,1})\Big\} \text{ and } \succ^{\mathcal{D}} = \Big\{(P_{1,2}, Q_{1,2}), \ldots, (P_{q,2}, Q_{q,2})\Big\}.$$

Define the matrices A^P, A^Q, B^P, B^Q as follows

$$A^P = [P_{i,1}(y_j)]_{(i,j)\in\{1,\ldots,p\}\times\{1,\ldots,n\}}, \qquad A^Q = [Q_{i,1}(y_j)]_{(i,j)\in\{1,\ldots,p\}\times\{1,\ldots,n\}}$$
$$B^P = [P_{i,2}(y_j)]_{(i,j)\in\{1,\ldots,q\}\times\{1,\ldots,n\}}, \qquad B^Q = [Q_{i,2}(y_j)]_{(i,j)\in\{1,\ldots,q\}\times\{1,\ldots,n\}}.$$

By Motzkin's Transposition Theorem (1951), one and only one of the following two conditions holds:[4]

(1) There exists $z \in \mathbb{R}^n$ such that $(B^P - B^Q)z \gg 0$ and $(A^P - A^Q)z \geq 0$.

(2) There exist $z^1 \in \mathbb{R}^p$ and $z^2 \in \mathbb{R}^q$ such that $z^1(A^P - A^Q) + z^2(B^P - B^Q) = 0$, $z^1 \geq 0$, and $z^2 > 0$.

Assume first that 1) holds and there exists $z \in \mathbb{R}^n$ such that $(B^P - B^Q)z \gg 0$ and $(A^P - A^Q)z \geq 0$, then preferences $(\succsim, \succ)$ defined by a utility function $u: X \to \mathbb{R}$ such that for each $i = 1, \ldots, n$, $u(y_i) = z_i$ rationalize $\mathcal{D}$ which is a contradiction. Hence, 2) holds. Denote

$$\tilde{z}^1 = \frac{z^1}{\sum_{i=1}^p z_i^1 + \sum_{i=1}^q z_i^2} \text{ and } \tilde{z}^2 = \frac{z^2}{\sum_{i=1}^p z_i^1 + \sum_{i=1}^q z_i^2},$$

which are well-defined since $z^2 > 0$. Now $\tilde{z}^1(A^P - A^Q) + \tilde{z}^2(B^P - B^Q) = 0$, $\tilde{z}^1 \geq 0$, and $\tilde{z}^2 > 0$. Hence, for each $j \in \{1, \ldots, n\}$,

$$\tilde{z}^1 A^P + \tilde{z}^2 B^P = \tilde{z}^1 A^Q + \tilde{z}^2 B^Q \Longrightarrow \sum_{i=1}^p \tilde{z}_i^1 P_{i,1}(y_j) + \sum_{i=1}^q \tilde{z}_i^2 P_{i,2}(y_j) = \sum_{i=1}^p \tilde{z}_i^1 Q_{i,1}(y_j) + \sum_{i=1}^q \tilde{z}_i^2 Q_{i,2}(y_j).$$

Assume, per contra, that there exists $(\succsim', \succ') \in \mathcal{T}_{\text{Tr-Ind}}$ that rationalizes $\mathcal{D}$. By Lemma 2, since $\tilde{z}^2 > 0$,

$$\sum_{k=1}^p \tilde{z}_k^1 P_{i,1} + \sum_{k=1}^q \tilde{z}_k^2 P_{i,2} \succ' \sum_{k=1}^p \tilde{z}_k^1 Q_{i,1} + \sum_{k=1}^q \tilde{z}_k^2 Q_{i,2}$$

which contradicts reflexivity since

$$\sum_{k=1}^p \tilde{z}_k^1 P_{i,1} + \sum_{k=1}^q \tilde{z}_k^2 P_{i,2} = \sum_{k=1}^p \tilde{z}_k^1 Q_{i,1} + \sum_{k=1}^q \tilde{z}_k^2 Q_{i,2}.$$

□

[4] Let $a \in \mathbb{R}^m$. $a \gg 0$ denotes for each $i = 1, \ldots, m$, $a_i > 0$. $a \geq 0$ denotes for each $i = 1, \ldots, m$, $a_i \geq 0$. $a > 0$ denotes for each $i = 1, \ldots, m$, $a_i \geq 0$ and there exists $j \in \{1, \ldots, m\}$ such that $a_j > 0$. The multiplications denote the matrix product.

2.2 Counterexamples for Weaker Forms of Independence

In this section, we show that under weaker forms of independence axiom, the Archimedean continuity axiom adds empirical content in the expected utility theory. We consider weakening the condition (3) in $\mathcal{T}_{\text{Tr-Ind}}$ first as introduced in Jensen (1967),

(3′) For all $P, Q, R \in \Delta(X)$ and $\alpha \in (0,1)$

$$P \succ Q \Longrightarrow \alpha P + (1-\alpha)R \succ \alpha Q + (1-\alpha)R,$$

second as introduced in Samuelson (1983a),

(3*) For all $P, Q, R \in \Delta(X)$ and $\alpha \in (0,1)$

$$P \succsim Q \Longrightarrow \alpha P + (1-\alpha)R \succsim \alpha Q + (1-\alpha)R,$$

third a stronger equivalent version of the special independence assumption formalized by Samuelson (1950),

(3†) For all $P, Q, R \in \Delta(X)$ and $\alpha \in (0,1)$

$$P \sim Q \iff \alpha P + (1-\alpha)R \sim \alpha Q + (1-\alpha)R,$$

and fourth a substitutability axiom as introduced by Luce and Raiffa (1957),

(3‡) For all $P, Q, R, S \in \Delta(X)$ and $\alpha \in (0,1)$

$$\alpha P + (1-\alpha)R \succsim \alpha Q + (1-\alpha)R \Longrightarrow \alpha P + (1-\alpha)S \succsim \alpha Q + (1-\alpha)S.$$

The condition (3†) implies other indifference independence axioms that have been used to axiomatize the expected utility such as in Marschak (1950), Malinvaud (1952), Herstein and Milnor (1953), and Segal (2023). The below counterexample shows that also under all these weaker axiomatizations, the continuity axiom adds empirical content in the expected utility theory.

Some advanced microeconomics textbooks use the strong independence axiom to characterize the expected utility theory such as Mas-Colell et al. (1995), Rubinstein (2006), and Muñoz-Garcia (2017).

Many advanced microeconomics textbooks use a weaker form of independence axiom to characterize the expected utility theory. Fishburn (1970) and Kreps (1988; 1990) use condition ($3'$). Silberberg and Suen (2000) use condition (3^*). Luenberger (1995), Jehle and Reny (2011), Varian (2014), and Wang (2018) use condition ($3^\dagger$). Kreps (2013) uses condition ($3^\ddagger$).

2.2.1 A Counterexample for ($3'$)

Let $X = [0,1]$ and $V : \Delta(X) \to \mathbb{R}$ be defined by for all $P \in \Delta(X)$

$$V'(P) = \begin{cases} 0, & \text{if } |\operatorname{supp}(P)| = 1 \\ \mathbb{E}(P), & \text{otherwise} \end{cases}$$

where $\operatorname{supp}(P)$ denotes the support of P.[5] Define $\succsim'$ by for all $P, Q \in \Delta(X)$

$$P \succsim' Q \iff V(P) \geq V(Q).$$

$\succ'$ denotes the asymmetric part of $\succsim'$. Now $\succsim'$ is complete and transitive and satisfies ($3'$). However, the data set defined by $D = \{\delta_0, \delta_1, \frac{1}{2}\delta_0 + \frac{1}{2}\delta_{0.5}, \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{0.5}\}$ with observations $\delta_0 \succsim_{\mathcal{D}} \delta_1$ and $\frac{1}{2}\delta_1 + \frac{1}{2}\delta_{0.5} \succ_{\mathcal{D}} \frac{1}{2}\delta_0 + \frac{1}{2}\delta_{0.5}$ is rationalized by $(\succsim', \succ')$ but not by the expected utility theory.

2.2.2 A Counterexample for (3^*)

Let $X = [0,1]$ and $V^* : \Delta(X) \to \mathbb{R}$ be defined by for all $P \in \Delta(X)$

$$V^*(P) = \begin{cases} \mathbb{E}(P), & \text{if } |\operatorname{supp}(P)| = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Define $\succsim^*$ by for all $P, Q \in \Delta(X)$

$$P \succsim^* Q \iff V^*(P) \geq V^*(Q).$$

$\succ^*$ denotes the asymmetric part of $\succsim^*$. Now $\succsim^*$ is complete and transitive and satisfies (3^*). However, the data set defined by $D = \{\delta_0, \delta_1, \frac{1}{2}\delta_0 + \frac{1}{2}\delta_{0.5}, \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{0.5}\}$ with observations $\delta_1 \succ_{\mathcal{D}} \delta_0$ and $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_{0.5} \succsim_{\mathcal{D}} \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{0.5}$ is rationalized by $(\succsim^*, \succ^*)$ but not by the expected utility theory.

[5] That is $\{x \in X | P(x) > 0\}$.

2.2.3 A Counterexample for ($3^{\dagger}$)

Let $X = \{0, 1\}$ and $V^{\dagger} : \Delta(X) \to \mathbb{R}$ be defined by for all $P \in \Delta(X)$

$$V^{\dagger}(P) = \begin{cases} \mathbb{E}(P) + 1, & \text{if } |\operatorname{supp}(P)| = 1 \\ \mathbb{E}(P), & \text{otherwise.} \end{cases}$$

Define $\succsim^{\dagger}$ by for all $P, Q \in \Delta(X)$

$$P \succsim^{\dagger} Q \iff V^{\dagger}(P) \geq V^{\dagger}(Q).$$

$\succ^{\dagger}$ and $\sim^{\dagger}$ denotes the asymmetric and symmetric parts of $\succsim^{\dagger}$ respectively. Now for all $P, Q \in \Delta(X)$, $P \sim^{\dagger} Q$ iff $P = Q$. So $\succsim^{\dagger}$ satisfies ($3^{\dagger}$) trivially. Additionally, $\succsim^{\dagger}$ is complete and transitive. However, the data set defined by $D = \{\delta_0, \delta_1, \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1\}$ with observations $\delta_1 \succ_{\mathcal{D}} \delta_0$ and $\delta_0 \succ_{\mathcal{D}} \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$ is rationalized by $(\succsim^{\dagger}, \succ^{\dagger})$ but not by the expected utility theory.

2.2.4 A Counterexample for ($3^{\ddagger}$)

Let $X = \{0, 1\}$. By Aczél (1966), Theorem 2 there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$ $f(x + y) = f(x) + f(y)$, $f(1) > f(0)$ and there exists $0 < a < b < 1$ such that $f(a) > f(b)$.[6] Define $\succsim^{\ddagger}$ by for all $P, Q \in \Delta(X)$

$$P \succsim^{\ddagger} Q \iff f\Big(P(1)\Big) \geq f\Big(Q(1)\Big).$$

$\succ^{\ddagger}$ and $\sim^{\ddagger}$ denotes the asymmetric and symmetric parts of $\succsim^{\ddagger}$ respectively. Now for all $P, Q, R, S \in \Delta(X)$ and $\alpha \in (0, 1)$,

$$\begin{aligned} &\alpha P + (1-\alpha)R \succsim^{\ddagger} \alpha Q + (1-\alpha)R \iff f\Big(\alpha P(1) + (1-\alpha)R(1)\Big) \geq f\Big(\alpha Q(1) + (1-\alpha)R(1)\Big) \\ &\iff f\Big(\alpha P(1)\Big) + f\Big((1-\alpha)R(1)\Big) \geq f\Big(\alpha Q(1)\Big) + f\Big((1-\alpha)R(1)\Big) \\ &\iff f\Big(\alpha P(1)\Big) + f\Big((1-\alpha)S(1)\Big) \geq f\Big(\alpha Q(1)\Big) + f\Big((1-\alpha)S(1)\Big) \\ &\iff f\Big(\alpha P(1) + (1-\alpha)S(1)\Big) \geq f\Big(\alpha Q(1) + (1-\alpha)S(1)\Big) \iff \alpha P + (1-\alpha)S \succsim^{\ddagger} \alpha Q + (1-\alpha)S. \end{aligned}$$

Additionally, $\succsim^{\ddagger}$ is complete and transitive. However, the data set defined by $D = \{\delta_0, \delta_1, (1-a)\delta_0 + a\delta_1, (1-b)\delta_0 + b\delta_1\}$ with observations $\delta_1 \succ_{\mathcal{D}} \delta_0$ and $(1-a)\delta_0 + a\delta_1 \succ_{\mathcal{D}} (1-b)\delta_0 + b\delta_1$ is rationalized by $(\succsim^{\ddagger}, \succ^{\ddagger})$ but not by the expected utility theory.

[6] This is a non-monotone solution to the Cauchy functional equation.

3 Conclusion

In this paper, we formalized the ubiquitous claim that the Archimedean continuity axiom does not add empirical content to the expected utility theory with a caveat: This claim only holds under the strong independence axiom but not under weaker forms of the independence axiom. Based on this, we recommend using the strong independence axiom for the expected utility theory.

Our theorem for the empirical content of expected utility follows as a corollary of Farkas' (1902) lemma. It is left for future research if this approach to studying empirical content of theories extends beyond expected utility for example to models under uncertainty such as subjective expected utility (Savage, 1954; Anscombe and Aumann, 1963) or maxmin expected utility (Gilboa and Schmeidler, 1989).

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