EX ANTE INDIVIDUALLY RATIONAL TRADE

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ABSTRACT. We study truthful implementation under an *ex ante* individual rationality constraint. In bilateral trade, we show that efficiency is not always achievable without running a deficit, but is achievable if the buyer's median value is larger than the seller's median value. Using similar ideas, we show that in partnership dissolution, efficiency is achievable without running a deficit regardless of the initial ownership shares.

1. INTRODUCTION

Budget-constrained designers often face a tension between achieving efficient outcomes, and ensuring that it is individually rational for agents to participate in a mechanism after learning their private information—i.e., at the interim stage (Myerson and Satterthwaite, 1983; Güth and Hellwig, 1986; Mailath and Postlewaite, 1990). Against this backdrop, a seminal result of d'Aspremont and Gérard-Varet (1979) and Arrow (1979) shows that efficiency is always achievable without running a deficit if agents must decide whether to participate before learning their private information—i.e., at the *ex ante* stage.

The resulting *ex ante* individual rationality constraint is an economically natural condition when agents commit to participating in a mechanism in the future. For example, in bilateral trade, it may be a reasonable property if a buyer and a seller commit to transacting using a particular bargaining protocol at a point in the future. Similarly, in partnership dissolution, partners may commit to a procedure for dissolving the partnership in the future before uncertainty in the investments of the partnership is resolved. In these cases, it may be costly for agents to renege on their participation commitment after they learn their values.

Unfortunately, the d'Aspremont–Gérard-Varet mechanisms are not truthful (i.e., dominant strategy incentive compatible). In this paper, we study implementation under an *ex ante* individual rationality constraint with truthful mechanisms, focusing on when efficiency can be achieved without running a deficit for the designer in two canonical mechanism design settings. In the case of bilateral trade, we show that efficiency cannot always be achieved, but provide a condition under which it can (Theorem 1). In the case of partnership dissolution, we prove that efficiency can always be achieved (Theorem 2). The proofs of our positive results explicitly construct mechanisms that satisfy all of the required properties.

To motivate our focus on truthful implementation, recall that truthful mechanisms are

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characterized by their robustness to misspecification of agents' higher-order beliefs (Bergemann and Morris, 2005).[1] This property is particularly valuable when individual rationality is imposed only at the *ex ante* stage, as ensuring that the *ex ante* individual rationality constraint holds generally relies on knowledge of agents' higher-order beliefs. In particular, at the *ex ante* stage, an agent's participation decision depend on their prediction of how counterparties will behave during the mechanism, as counterparties' actions affect the payoffs that each agent can expect. These actions depend in turn on counterparties' hierarchy of beliefs after receiving their private information, so an agent's beliefs about counterparties' future hierarchies of beliefs affect their participation decision. If the designer is misspecified about these beliefs, then agents may not choose to participate in the mechanism. Thus, a version of the Wilson (1987) critique applies to the *ex ante* individual rationality constraint.

Under private values, truthful mechanisms simplify each agent's participation decision by making her counterparties' interim beliefs irrelevant to their actions. Hence, each agent can decide whether to participate at the *ex ante* stage using only their first-order beliefs about their counterparties' payoff-relevant private information. That is, truthfulness makes agents' *ex ante* participation decisions strategically simple in the sense of Börgers and Li (2019). From the perspective of the designer, the truthfulness of a mechanism ensures that its *ex ante* individual rationality is robust to misspecification of agents' higher-order beliefs.

In Section 2, we begin by investigating the canonical bilateral trade setting of Myerson and Satterthwaite (1983), in which a seller can sell a good to a buyer, and the agents attribute independent private values to the good. We restrict attention to mechanisms that do not run a deficit, in the sense that the payment of the buyer to the mechanism must always be weakly larger than the payment to the seller. Thus, we allow for the possibility that a broker (e.g., the government) may extract some money from the participants in the mechanism.

Our first main result shows that under the assumption that the median value of the buyer is higher than that of the seller, full efficiency can be achieved by a truthful, *ex ante* individually rational mechanism. By the Green–Laffont–Holmström Theorem, the mechanism must be a Groves (1973) mechanism for an appropriate participation charge. The participation charge we construct turns out to have a simple form: the participation charge of each agent is a piecewise linear function of the value of the other agent. With no participation charges, the mechanism would be a Vickrey–Clarke–Groves (VCG) mechanism, which is *ex post* individually rational but runs a deficit whenever trade is efficient. Our constructed participation charges are large enough to cover the deficit of the VCG mechanism if trade does occur, but

[1]Bergemann and Morris's (2005) characterization generally relies on a separability condition that rules out budget constraints for the designer. With two agents, this foundation for truthful mechanisms applies even when the designer is budget-constrained (Bergemann and Morris, 2005, Section 5). Thus, both our possibility and impossibility results for truthful implementation in bilateral trade apply to robust implementation. Regardless of the number of agents, truthful mechanisms are always robust to misspecification of agents' higher-order beliefs, so our (positive) result for partnership dissolution also applies to robust implementation.

	Interim IR	*Ex ante* IR
Bayesian IC	✗ Myerson and Satterthwaite (1983)	✓ d'Aspremont and Gérard-Varet (1979)
Truthful	✗ Myerson and Satterthwaite (1983)	✓ if $\mathrm{Median}(F_B) \geq \mathrm{Median}(F_S)$ This paper

TABLE 1. (Non)existence of efficient mechanisms for bilateral trade under various combinations of incentive compatibility and individual rationality constraints. We restrict attention to mechanisms that do not run deficits.

small enough to preserve *ex ante* individual rationality.

However, we also show that for general distributions of values, truthful, *ex ante* individually rational mechanisms cannot achieve full efficiency in general. Intuitively, as participation charges must be paid regardless of whether trade occurs, and must be large enough to cover the VCG deficit if trade does occur, the *ex ante* probability of trade must be sufficiently large to be able to achieve *ex ante* individual rationality. The hypothesis that the buyer's median value is higher than the seller's median value is one way of ensuring this property, as it implies that trade is efficient with probability at least $\frac{1}{4}$. Table 1 connects our result to the seminal results on the (non)existence of efficient mechanisms under various combinations of incentive compatibility and individual rationality constraints.

In Section 3, we extend our analysis beyond the case of bilateral trade to the canonical partnership dissolution model of Cramton et al. (1987). Consider N agents who initially share ownership of a partnership, and have independent and identically distributed private values for the partnership. Cramton et al. (1987) provide a necessary and sufficient condition for full efficiency to be achievable by a Bayesian incentive compatible, interim individually rational mechanism—intuitively, sufficiently symmetric initial ownership is necessary and sufficient for efficient disolvability. By contrast, we show that regardless of the initial ownership shares, full efficiency can be achieved by a truthful, *ex ante* individually rational mechanism.

To construct an efficient, *ex ante* individually rational, truthful mechanism, we first suppose that the partnership is initially owned by one agent. Thus, the design problem boils down to one with a single seller and several buyers with independent and identically distributed values. As there are several buyers among whom the partnership (if traded) will be allocated to the one with the highest value, it is as if the buyers as a whole have stochastically higher values than the seller. We can then construct a Groves mechanism by extending our construction from the bilateral trade case. We conclude by first randomly allocating ownership of the entire partnership in proportion to the initial ownership shares.

The closest papers to ours are Hagerty and Rogerson (1987) and Athey and Miller (2007), who studied truthful mechanisms for bilateral trade. Hagerty and Rogerson (1987) showed that if the mechanism can run neither a deficit nor a surplus and is *ex post* individually rational, then it must be a (randomized) posted price (see also Čopič and Ponsatí (2016)). As we allow the mechanism to run surpluses and impose only *ex ante* individual rationality, we can move beyond (random) posted prices and, under certain conditions, achieve full efficiency. Athey and Miller (2007) allowed for surpluses, but excluded the designer's revenue from welfare. They showed that full efficiency cannot then be achieved, and characterized the second best. We instead consider efficient allocation of the good, thereby implicitly assuming that revenue for the designer is weighted equally to money left in the agents' hands.

2. Bilateral Trade

We begin by considering a bilateral trade setting, where a buyer has a private value θ_B for the good, and a seller has a private value θ_S for the good, which are distributed independently according to distributions F_B and F_S, respectively. For simplicity, we assume that both distributions have support on $[0,1]$ and admit positive densities f_B and f_S, respectively.

By the Revelation Principle, we can restrict attention to direct revelation mechanisms. A mechanism (q,t) specifies the probability of trade $q(\theta) = q(\theta_B, \theta_S) \in [0,1]$, and payments $t(\theta) = t(\theta_B, \theta_S) = (t_B(\theta_B, \theta_S), t_S(\theta_B, \theta_S))$.[2] Normalizing the utility of the outside option to 0, if the types are θ_B, θ_S and both parties report truthfully, the buyer obtains utility $u_B(\theta_B, \theta_S) = \theta_B q(\theta) - t_B(\theta)$ and the seller obtains utility $u_S(\theta_B, \theta_S) = -\theta_S q(\theta) - t_S(\theta)$.

We next recall the four standard properties of mechanisms that feature in our analysis.

- A mechanism (q,t) is *truthful* if for all types θ_B, θ_S and reports $\hat{\theta}_B, \hat{\theta}_S$, we have

$$\theta_B q(\theta_B, \theta_S) - t_B(\theta_B, \theta_S) \geq \theta_B q(\hat{\theta}_B, \theta_S) - t_B(\hat{\theta}_B, \theta_S)$$
$$-\theta_S q(\theta_B, \theta_S) - t_S(\theta_B, \theta_S) \geq -\theta_S q(\theta_B, \hat{\theta}_S) - t_S(\theta_B, \hat{\theta}_S).$$

- A mechanism (q,t) is *efficient* if for all types θ_B, θ_S with $\theta_B > \theta_S$ (resp. $\theta_B < \theta_S$), we have that $q(\theta_B, \theta_S) = 1$ (resp. $q(\theta_B, \theta_S) = 0$).
- A mechanism (q,t) has *no deficit* if for all types θ_B, θ_S, we have that

$$t_B(\theta_B, \theta_S) + t_S(\theta_B, \theta_S) \geq 0.$$

- A mechanism (q,t) is *ex ante individually rational* if $\mathbb{E}_{\theta_B,\theta_S}[u_B(\theta_B, \theta_S)] \geq 0$ and $\mathbb{E}_{\theta_B,\theta_S}[u_S(\theta_B, \theta_S)] \geq 0$.

Our first main result shows that these four properties are not simultaneously achievable in general, but are if the median value of the buyer is higher than that of the seller.

Theorem 1. *(a) There exist F_B, F_S such that there does not exist an efficient, truthful, ex ante individually rational mechanism that has no deficit.*

(b) If $\text{Median}(F_B) \geq \text{Median}(F_S)$*, there exists an efficient, truthful, ex ante individually rational mechanism that has no deficit.*

[2] We assume that $q(\theta)$ and $t(\theta)$ are measurable, and that $t(\theta)$ is uniformly bounded, so expected utilities exist.

Remark 1. If we required exact *ex post* budget balance instead of the no deficit property, an impossibility result would follow from Hagerty and Rogerson's (1987) characterization of truthful, exactly budget-balanced mechanisms as (randomized) posted prices (see also Čopič and Ponsatí (2016)). By contrast, if we considered *ex ante* exact budget balance instead of our (*ex post*) no deficit property, combining a VCG mechanism with lump-sum transfers to the designer to cover the *ex ante* VCG deficit would achieve full efficiency regardless of the distributions of values. Theorem 1(a) holds despite allowing the mechanism to run a surplus.

To understand Theorem 1, recall that the Green–Laffont–Holmström Theorem tells us that a mechanism is efficient and truthful if and only if it is a Groves (1973) scheme for some participation charges $h_B(\theta_S)$ and $h_S(\theta_B)$. Here, a Groves scheme with participation charges $h_B(\theta_S)$ and $h_S(\theta_B)$ is defined by any efficient allocation rule $q(\theta_B, \theta_S)$ and payment rule

$$(1)\quad t_B(\theta_B,\theta_S) = \underbrace{\theta_S q(\theta_B,\theta_S)}_{\text{VCG payment}} + \underbrace{h_B(\theta_S)}_{\text{participation charge}} \quad \text{and} \quad t_S(\theta_B,\theta_S) = \underbrace{-\theta_B q(\theta_B,\theta_S)}_{\text{VCG payment}} + \underbrace{h_S(\theta_B)}_{\text{participation charge}}.$$

The first step of our analysis is to characterize which Groves schemes have no deficit and are *ex ante* individually rational in terms of relationships between the participation charges and the gains from trade. The *ex post* gains from trade are $\mathbb{1}(\theta_B \geq \theta_S)(\theta_B - \theta_S)$. The *ex ante* gains from trade are given by $GT = \mathbb{E}_{\theta_B,\theta_S}\left[\mathbb{1}(\theta_B \geq \theta_S)(\theta_B - \theta_S)\right]$.

Lemma 1. *In bilateral trade, a Groves scheme with participation charges $h_B(\theta_S)$ and $h_S(\theta_B)$:*

(a) has no deficit if and only if $h_B(\theta_S) + h_S(\theta_B) \geq \mathbb{1}(\theta_B \geq \theta_S)(\theta_B - \theta_S)$ for all types θ_B, θ_S;

(b) is ex ante individually rational if and only if $\mathbb{E}_{\theta_S}[h_B(\theta_S)] \leq GT$ and $\mathbb{E}_{\theta_B}[h_S(\theta_B)] \leq GT$.

Intuitively, a Groves scheme has no deficit if the participation charges always sum to at least the VCG deficit, which is the *ex post* gains from trade. On the other hand, a Groves scheme is *ex ante* individually rational if the expected participation charges are not so large to exhaust the expected VCG surplus of each agent, which is the *ex ante* gains from trade.

Proof. To prove Part (a), note that for a Groves scheme, we have

$$\begin{aligned} t_B(\theta_B,\theta_S) + t_S(\theta_B,\theta_S) &= h_B(\theta_S) + h_S(\theta_B) - (\theta_B - \theta_S)q(\theta_B,\theta_S) \\ &= h_B(\theta_S) + h_S(\theta_B) - \mathbb{1}(\theta_B \geq \theta_S)(\theta_B - \theta_S), \end{aligned}$$

which is nonnegative if and only if $h_B(\theta_S) + h_S(\theta_B) \geq \mathbb{1}(\theta_B \geq \theta_S)(\theta_B - \theta_S)$.

To prove Part (b), note that for a Groves scheme, we have

$$\mathbb{E}_{\theta_B,\theta_S}[u_B(\theta_B,\theta_S)] = \mathbb{E}_{\theta_B,\theta_S}[\theta_B q(\theta_B,\theta_S) - \theta_S q(\theta_B,\theta_S) - h_B(\theta_S)] = GT - \mathbb{E}_{\theta_S}[h_B(\theta_S)],$$

which is nonnegative if and only if $\mathbb{E}_{\theta_S}[h_B(\theta_S)] \leq GT$. A similar calculation shows that $\mathbb{E}_{\theta_B,\theta_S}[u_S(\theta_B,\theta_S)] \geq 0$ if and only if $\mathbb{E}_{\theta_B}[h_S(\theta_B)] \leq GT$. □

We now turn to the proof of Theorem 1. We start by proving Theorem 1(a). For this part, we exhibit two distributions for which no Groves scheme can both have no deficit and be *ex ante* individually rational. Intuitively, the no deficit property requires that the participation charges be sufficiently large, which has a tension with the requirement that expected participation charges be sufficiently small, as entailed by *ex ante* individual rationality. For some distributions, the two conditions are mutually incompatible.

More precisely, we construct distributions in which the buyer has a low value with high probability and a high value with low probability, while the seller has a high value with high probability and a low value with high probability. Thus, we ensure that the gains from trade are large when trade is efficient, but the expected gains from trade are low due to trade being efficient with low probability. We show that this leads to a conflict between the conditions of Parts (a) and (b) of Lemma 1, and hence an impossibility result.

Proof of Theorem 1(a). Define the densities of the distributions F_B, F_S respectively by

$$f_B(\theta_B) = \begin{cases} 9-9\delta & \text{if } \theta_B < 0.1 \\ \delta & \text{if } 0.1 < \theta_B < 0.9 \\ 1+\delta & \text{if } \theta_B > 0.9 \end{cases} \quad \text{and} \quad f_S(\theta_S) = \begin{cases} 1+\delta & \text{if } \theta_S < 0.1 \\ \delta & \text{if } 0.1 < \theta_S < 0.9 \\ 9-9\delta & \text{if } \theta_S > 0.9 \end{cases},$$

where $\delta \in (0,1)$. We claim that for δ small enough, there does not exist an efficient, truthful, *ex ante* individually rational mechanism that has no deficit.

Suppose that such a mechanism exists. The Green–Laffont–Holmström Theorem implies that it must be a Groves mechanism, say with participation charges $h_B(\theta_S)$ and $h_S(\theta_B)$. By Lemma 1(a), the no deficit property implies that

$$\begin{cases} h_B(\theta) + h_S(\theta) \geq 0 & \text{for all } 0 \leq \theta \leq 1 \\ h_B(\theta_S) + h_S(\theta_B) \geq 0.8 & \text{for } \theta_S \leq 0.1 \text{ and } \theta_B \geq 0.9 \\ h_B(\theta_S) + h_S(\theta_B) \geq 0 & \text{for } \theta_S \geq 0.9 \text{ and } \theta_B \leq 0.1. \end{cases} \tag{2}$$

The *ex ante* gains from trade are $GT = \frac{3}{250} + O(\delta)$. By Lemma 1(b), *ex ante* individual rationality implies that $\mathbb{E}_{\theta_S}[h_B(\theta_S)] \leq GT$ and $\mathbb{E}_{\theta_B}[h_S(\theta_B)] \leq GT$—i.e.,

$$\int_0^{0.1} h_B(\theta_S)(1+\delta)\,d\theta_S + \int_{0.1}^{0.9} h_B(\theta_S)\delta\,d\theta_S + \int_{0.9}^{1} h_B(\theta_S)(9-9\delta)\,d\theta_S \leq \frac{3}{250} + O(\delta)$$
$$\int_0^{0.1} h_S(\theta_B)(9-9\delta)\,d\theta_B + \int_{0.1}^{0.9} h_S(\theta_B)\delta\,d\theta_B + \int_{0.9}^{1} h_S(\theta_B)(1+\delta)\,d\theta_B \leq \frac{3}{250} + O(\delta)$$

Summing these inequalities and grouping terms yields that

$$\frac{3}{125} + O(\delta) \geq \delta \int_{0.1}^{0.9} (h_B(\theta) + h_S(\theta))\,d\theta + (1-\delta)\left[\int_0^{0.1} h_B(\theta_S)\,d\theta_S + \int_{0.9}^{1} h_S(\theta_B)\,d\theta_B\right]$$
$$+ (9-9\delta)\left[\int_0^{0.1} h_S(\theta_B)\,d\theta_S + \int_{0.9}^{1} h_B(\theta_S)\,d\theta_B\right].$$

By (2), it follows that

$$\frac{3}{125} + O(\delta) \geq 0 + (1-\delta)\cdot\frac{8}{100} + 0 = \frac{10}{125} + O(\delta).$$

This condition fails for δ small enough, so for such δ, there does not exist an efficient, truthful, *ex ante* individually rational mechanism that has no deficit. □

By contrast, when the *ex ante* gains from trade GT are large enough, participation charges can be set to overcome the VCG deficit without compromising *ex ante* individual rationality. One way of ensuring that GT is large enough is to ensure that the sale probability be large, which is what the median buyer value being above the median seller value (as assumed in Theorem 1(b)) ensures. We formalize this property in terms of the market-clearing quantity in a replica economy with infinitely many buyers and sellers.

Figure 1 depicts market equilibrium in the replica economy. We plot two cumulative distribution functions F_B, F_S. The large-market supply curve is then F_S^{-1} and the large-market demand curve is $(1-F_B)^{-1}$. The replica economy competitive equilibrium is the intersection of these two curves—i.e., the equilibrium price p^* and equilibrium quantity q^* satisfy $F_S^{-1}(q^*) = (1-F_B)^{-1}(q^*) = p^*$. Thus, p^* is the solution to $F_B(p^*) + F_S(p^*) = 1$.[3] As the following lemma shows, the hypothesis that $\text{Median}(F_B) \geq \text{Median}(F_S)$ implies $q^* \geq \frac{1}{2}$.

Lemma 2. *Let p^* be the solution to $F_B(p^*) + F_S(p^*) = 1$. If $\text{Median}(F_B) \geq \text{Median}(F_S)$, then we have that $q^* = F_S(p^*) \geq \frac{1}{2}$.*

Proof. The hypothesis that $\text{Median}(F_S) \leq \text{Median}(F_B)$ implies that

$$F_B(\text{Median}(F_S)) + F_S(\text{Median}(F_S)) \leq F_B(\text{Median}(F_B)) + F_S(\text{Median}(F_S)) = \frac{1}{2} + \frac{1}{2} = 1.$$

[3]This equation has a unique solution. Indeed, as $F_B + F_S : [0,1] \to [0,2]$ is a continuous function and we have $F_B(0) + F_S(0) = 0$ and $F_B(1) + F_S(1) = 2$, the Intermediate Value Theorem implies that there exists p^* with $F_B(p^*) + F_S(p^*) = 1$. The uniqueness of p^* follows from the fact that $F_B + F_S$ is strictly increasing.

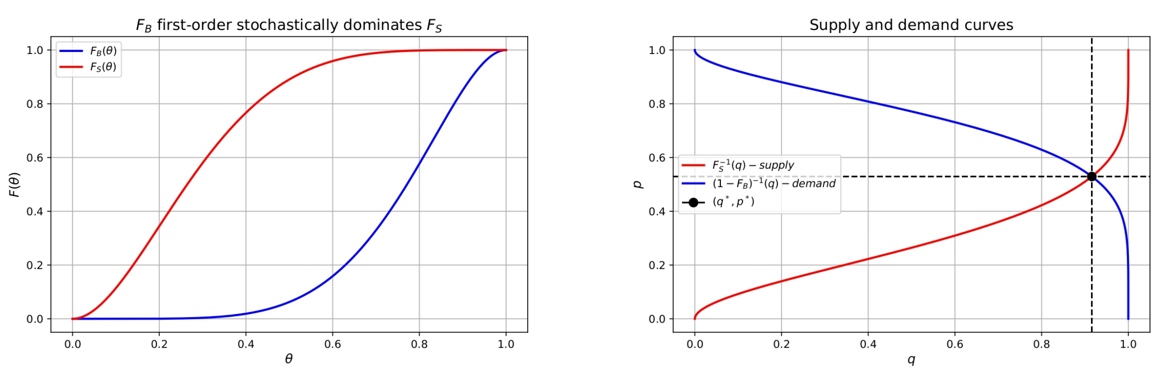

FIGURE 1. Determination of the replica economy market-clearing price.

As $F_B + F_S$ is strictly increasing, it follows that $p^* \geq \text{Median}(F_S)$. Hence, we have that

$$q^* = F_S(p^*) \geq F_S(\text{Median}(F_S)) = \frac{1}{2},$$

as claimed. □

To prove Theorem 1(b), note again that by the Green–Laffont–Holmström Theorem, we must use a Groves mechanism. The buyer's (resp. seller's) participation charge must depend only on the seller's (resp. buyer's) type. We construct the buyer's (resp. seller's) participation charge to be the producer (resp. consumer) surplus in competitive equilibrium in the replica economy given the seller's (resp. buyer's) type, plus a lump-sum transfer between the agents.

In particular, our participation charges are piecewise linear in the other agent's type with a kink point at type p^*. They require the buyer (resp. seller) to make payments to the mechanism if the seller has a low value (resp. buyer has a high value), which eliminates the deficit. Intuitively, by Lemma 1(a), we need the participation charges to be large when the difference in values is large, which is what our construction achieves. We derive *ex ante* individual rationality in aggregate across the two agents using Lemma 1(b) by noting that the expected participation charges sum to the gains from trade in the replica economy; we bound that quantity above by twice the expected gains from trade using Lemma 2. We then use a lump-sum transfer to ensure *ex ante* individual rationality for each agent.

Proof of Theorem 1(b). Let p^* be the solution to $F_B(p^*) + F_S(p^*) = 1$. We consider the Groves mechanism with efficient allocation rule $q(\theta) = \mathbb{1}(\theta_B \geq \theta_S)$ and participation charges

$$h_B(\theta_S) = \max\{p^* - \theta_S, 0\} + \kappa$$
$$h_S(\theta_B) = \max\{\theta_B - p^*, 0\} - \kappa$$

with a constant lump-sum transfer κ to be chosen. We will show that this mechanism is efficient, truthful, *ex ante* individually rational, and has no deficit. For all κ, efficiency and truthfulness hold because the mechanism is a Groves mechanism.

To show that the mechanism has no deficit regardless of the choice of κ, using Lemma 1(a), we must show that for all types θ_B,θ_S, we have

$$h_B(\theta_S) + h_S(\theta_B) \geq \mathbb{1}(\theta_B \geq \theta_S)(\theta_B - \theta_S).$$

To show this inequality, we consider four cases in turn:

(1) If $\theta_S \geq p^*$ and $\theta_B \geq p^*$, then we have $h_B(\theta_S)+h_S(\theta_B) = \theta_B - p^* \geq \mathbb{1}(\theta_B \geq \theta_S)(\theta_B - \theta_S)$.
(2) If $\theta_S \geq p^*$ and $\theta_B < p^*$, then we have $h_B(\theta_S) + h_S(\theta_B) = \mathbb{1}(\theta_B \geq \theta_S)(\theta_B - \theta_S) = 0$.
(3) If $\theta_S < p^*$ and $\theta_B < p^*$, then we have $h_B(\theta_S)+h_S(\theta_B) = p^* - \theta_S \geq \mathbb{1}(\theta_B \geq \theta_S)(\theta_B - \theta_S)$.
(4) If $\theta_S < p^*$ and $\theta_B \geq p^*$, then we have

$$h_B(\theta_S) + h_S(\theta_B) = p^* - \theta_S + \theta_B - p^* = \theta_B - \theta_S = \mathbb{1}(\theta_B \geq \theta_S)(\theta_B - \theta_S).$$

It remains to choose κ and show that the mechanism is *ex ante* individually rational. With a view to applying Lemma 1(b), we can write the expected gains from trade as

$$\begin{aligned} GT &= \mathbb{E}_{\theta_B,\theta_S}\left[(\theta_B-\theta_S)\mathbb{1}(\theta_B \geq \theta_S)\right] \\ &= \int_0^1 \int_0^{\theta_B} (\theta_B-\theta_S)\, dF_S(\theta_S)\, dF_B(\theta_B) \\ &= \int_0^1 \left[\theta_B F_S(\theta_B) - \int_0^{\theta_B} \theta_S\, dF_S(\theta_S)\right] dF_B(\theta_B) \\ &= \int_0^1 \left[\int_0^{\theta_B} F_S(\theta_S)\, d\theta_S\right] dF_B(\theta_B) = \int_0^1 F_S(y)(1-F_B(y))\, dy, \end{aligned}$$

where the fourth equality is obtained by integration by parts of the inner integral, while the fifth is obtained by integration by parts of the outer integral. Similar calculations show that the expected participation charges are given by

$$\mathbb{E}_{\theta_B}\left[h_S(\theta_B)\right] = \int_{p^*}^1 (1-F_B(\theta_B))\, d\theta_B - \kappa \quad \text{and} \quad \mathbb{E}_{\theta_S}\left[h_B(\theta_S)\right] = \int_0^{p^*} F_S(\theta_S)\, d\theta_S + \kappa. \tag{3}$$

Now, observe that

$$\begin{aligned} GT = \int_0^1 F_S(y)(1-F_B(y))\, dy &= \int_0^{p^*} F_S(y)(1-F_B(y))\, dy + \int_{p^*}^1 F_S(y)(1-F_B(y))\, dy \\ &\geq (1-F_B(p^*)) \int_0^{p^*} F_S(y)\, dy + F_S(p^*) \int_{p^*}^1 (1-F_B(y))\, dy, \end{aligned}$$

where the inequality holds because F_S (resp. $1-F_B$) is non-decreasing (resp. non-increasing). By Lemma 2, we have $F_S(p^*) = 1-F_B(p^*) = q^* \geq \frac{1}{2}$. Hence, we obtain an aggregate *ex ante* individual rationality condition

$$2GT \geq \int_0^{p^*} F_S(y)\, dy + \int_{p^*}^1 (1-F_B(y))\, dy = \mathbb{E}_{\theta_B}\left[h_S(\theta_B)\right] + \mathbb{E}_{\theta_S}\left[h_B(\theta_S)\right]. \tag{4}$$

To ensure *ex ante* individual rationality, we define the lump-sum transfer κ by

$$\kappa = \frac{1}{2}\left[\int_{p^*}^1 (1-F_B(\theta_B))\, d\theta_B - \int_0^{p^*} F_S(\theta_S)\, d\theta_S\right].$$

By (3) and (4), we then have that

$$\mathbb{E}_{\theta_S}[h_B(\theta_S)] = \mathbb{E}_{\theta_B}[h_S(\theta_B)] = \frac{1}{2}\left[\int_{p^*}^1 (1-F_B(\theta_B))\, d\theta_B + \int_0^{p^*} F_S(\theta_S)\, d\theta_S\right] \leq GT.$$

Therefore, by Lemma 1(b), the mechanism is *ex ante* individually rational. □

In the argument, we use the fact that the equilibrium quantity q^* in the replica economy is at least $\frac{1}{2}$ when the buyer's median value is higher than the seller's (Lemma 2) to bound the total expected participation charge by twice the expected gains from trade—see (4).

3. Partnership Dissolution

We now conduct a similar analysis for the canonical partnership dissolution model of Cramton et al. (1987). There are N agents who are involved in a partnership. Each agent i initially owns share r_i of the partnership, with $r_i \geq 0$ and $\sum_i r_i = 1$; the initial stakes are common knowledge. Agents have independent and identically distributed values for (shares of) the partnership, which we assume are drawn from a distribution F supported on $[0, 1]$ with strictly positive density f. Let θ_i denote the type of agent i.

By the Revelation Principle, we can restrict attention to direct revelation mechanisms. A direct revelation mechanism $M = (q, t)$ specifies, for each agent i, a payment $t_i(\theta)$ and the probability $q_i(\theta) \in [0, 1]$ with which they receive the partnership, as a function of the reports of all the agents in the mechanism. The feasibility constraint is that $\sum_i q_i(\theta) = 1$ for all type profiles θ. The utility that the mechanism delivers to agent i under truthtelling is $\theta_i q_i(\theta) - t_i(\theta)$, while agent i's reservation utility is $\theta_i r_i$. We can therefore consider agent i's net utility under the mechanism, which is defined by $u_i(\theta) = (q_i(\theta) - r_i)\theta_i - t_i(\theta)$. The definitions of truthfulness, efficiency, no deficit, and *ex ante* individual rationality then extend easily.

- A mechanism (q, t) is *truthful* if for all agents i, type profiles θ, and reports $\hat{\theta}_i$, we have
$$\theta_i q_i(\theta_i, \theta_{-i}) - t_i(\theta_i, \theta_{-i}) \geq \theta_i q_i(\hat{\theta}_i, \theta_{-i}) - t_i(\hat{\theta}_i, \theta_{-i}).$$
- A mechanism (q, t) is *efficient* if for all agents i and type profiles θ with $\theta_i < \max_j \theta_j$, we have that $q_i(\theta) = 0$.
- A mechanism (q, t) has *no deficit* for all type profiles θ, we have that $\sum_i t_i(\theta) \geq 0$.
- A mechanism (q, t) is *ex ante individually rational* if $\mathbb{E}_\theta[u_i(\theta)] \geq 0$ for all agents i.

Our second main result shows that all partnerships can be efficiently dissolved.

Theorem 2. *For all initial ownership shares $(r_i)_{1 \leq i \leq N}$, there exists an efficient, truthful, ex ante individually rational mechanism that has no deficit.*

As discussed by Cramton et al. (1987), the results of Myerson and Satterthwaite (1983) imply that *ex ante* individual rationality cannot be strengthened to interim individual rationality in Theorem 2 without imposing a condition on initial ownership shares. The results of d'Aspremont and Gérard-Varet (1979) imply that Theorem 1 holds if truthfulness is relaxed to Bayesian incentive compatibility. Our contribution here is to provide a positive result for truthful mechanisms that holds for all initial ownership shares.

To prove Theorem 2, we prove a version of Theorem 1 that allows for multiple buyers, but imposes that all agents' values are drawn from the same distribution. This corresponds to the case of Theorem 2 for which there exists an agent j with $r_j = 1$.

Proposition 1. *If there exists an agent j with $r_j = 1$, then there exists an efficient, truthful, ex ante individually rational mechanism that has no deficit.*

We prove Proposition 1 in the sequel. Intuitively, the result holds for similar reasons to Theorem 1 because the distribution of the highest value of the potential buyers first-order stochastically dominates the distribution of the value of the seller/owner (which in particular, ensures that the comparison of medians holds). However, the detailed argument is much more involved due to the presence of multiple buyers.

Using Proposition 1, the proof of Theorem 2 is straightforward.

Proof of Theorem 2. Let $(r_i)_{1\leq i\leq N}$ be any initial ownership shares. Consider the following two stages. First, a lottery is run to give the full property of the partnership to one of the agents, with probabilities given by the initial ownership shares $(r_i)_{1\leq i\leq N}$. The partnership is then fully owned by one of the agents. By Proposition 1, for this single-owner partnership, there exists an efficient, truthful, *ex ante* individually rational mechanism that has no deficit—this mechanism comprises the second stage. The grand mechanism comprised of the lottery and the second-stage mechanism is clearly efficient, truthful, and has no deficit.

From the point of view of agent i, the grand mechanism averages one *ex ante* individually rational mechanism in which they are the seller, and $N-1$ mechanisms in which they are a potential buyer. The first case occurs with probability r_i and provides the agent with an *ex post* utility of at least $(1-r_i)\mathbb{E}_\theta[\theta_i]$, while the second case occurs with probability $1-r_i$ and provides the agent with a utility of at least $-r_i\mathbb{E}_\theta[\theta_i]$. Adding expected utilities over these two events, we can see that the grand mechanism is *ex ante* individually rational. □

4. Proof of Proposition 1

Note again that by the Green–Laffont–Holmström Theorem, we must use a Groves mechanism. Analogously to the proof of Theorem 1(b), we construct the seller's participation charge to be consumer surplus in a replica economy given the the buyers' types at a price p^*. There are two key complications in our argument. First, p^* cannot be taken to be a replica economy competitive equilibrium price. Second, we need to make the buyer's participation charges depend on the other buyers' values—in addition to the seller's value. We construct the participation charge of each buyer to be piecewise linear (with a kink point at p^*) in the larger of the seller's value and the second-highest of the other buyers' values.

Given the constructed participation charges, we show first that the mechanism has no deficit. The key to the proof is to show that the mechanism is *ex ante* individually rational in the aggregate. (We can then ensure *ex ante* individual rationality for each agent by adding appropriate lump-sum transfers between agents as in the proof of Theorem 1(b).) The proof of this part goes in four steps. In the first step, we decompose the sum of the *ex ante* expected utilities as the sum of three terms: the ex-ante gains from trade GT, the expected deficit from the VCG payment of everyone in the Groves scheme VCG_{def}, and the total expected participation charge TPC. In the second and third step, we calculate these three terms in turn. To obtain comparable expressions, it proves convenient to work in the quantile space. The last step is to show that $GT + VCG_{def} \geq TPC$. We do so by proving a technical lemma that provides the desired inequality in the quantile space.

In the remainder of this section, we first discuss some preliminaries about order statistics that we use to reduce to the quantile space, and then prove the proposition. As the distribution F has a density, ties occur with probability 0, and we hence ignore them hereafter.

Some Preliminaries on Order Statistics. We use the following notation. For any set $Z \subset \{1, \ldots, N\}$, we let $\theta_{-Z} = (\theta_i)_{i \notin Z}$. Given any vector of random variables, we use a superscript $\cdot^{(1)}$ (resp $\cdot^{(2)}$) to denote the first (resp. second) order statistic of the variables. Thus, for example, we write $\theta^{(1)} = \max_{1 \leq i \leq N} \theta_i$ and $\theta^{(1)}_{-N} = \max_{1 \leq i \leq N-1} \theta_i$.

For each type θ_i, let $U_i = F(\theta_i)$. The random variables $U_1, \ldots, U_N$ are independent, and (by the Probability Integral Transform) each distributed uniformly on $[0, 1]$. We remind the reader of two facts. First, for each $Z \subset \{1, \ldots, N\}$, we have $\theta^{(1)}_{-Z} = F^{-1}(U^{(1)}_{-Z})$. Second, a property of order statistics of uniforms is that $U^{(k)} \sim \text{Beta}(n - k + 1, k)$. Hence, for all quantiles $u \in [0, 1]$, the densities of $U^{(1)}_{-i}$ and $U^{(2)}_{-i}$ are, respectively, given by

$$
\begin{aligned}
&f_{U^{(1)}_{-i}}(u) = (N-1)u^{N-2} \\
&f_{U^{(2)}_{-i}}(u) = (N-1)(N-2)u^{N-3}(1-u) = (N-1)(N-2)u^{N-3} - (N-1)(N-2)u^{N-2}.
\end{aligned}
$$

Construction of the Mechanism. Without loss of generality, we suppose that $r_N = 1$.

We let p^* be such that $(N-2)F(p^*)^{N-2} - (N-4)F(p^*)^{N-1} = 1$.[4] We consider a Groves scheme (q, t), where the allocation rule is $q_i(\theta) = \mathbb{1}(\theta_i \geq \max_{j \neq i} \theta_j)$ and the payment rule is

$$
\begin{aligned}
t_i(\theta) &= \theta^{(1)}_{-i} q_i(\theta) + \frac{1}{N-1} \max\left\{p^* - \max\left\{\theta^{(2)}_{-\{i,N\}}, \theta_N\right\}, 0\right\} + \kappa \qquad \text{for } 1 \leq i \leq N-1 \\
t_N(\theta) &= -\theta^{(1)}_{-N}(1 - q_N(\theta)) + \max\left\{\theta^{(1)}_{-N} - p^*, 0\right\} - (N-1)\kappa,
\end{aligned}
$$

with a constant lump-sum transfer κ to be chosen.

To understand the intuition behind our constructed Groves payments, note that the first term in each payment is the VCG payment and the second term is a participation charge. For the seller (agent N), the payment is similar to the bilateral trade case. The VCG payment is 0 if they do not sell, while they receive the buyer's value $\theta^{(1)}_{-N}$ if they sell. The participation charge is a piecewise linear function of the highest potential buyers' value $\theta^{(1)}_{-N}$ with a kink point at p^*. The buyers who do not end up buying the good have a VCG payment of zero, while if one buyer ends up buying the good, they have a VCG payment of the highest value among other agents. Unlike the bilateral trade case, the buyers' participation charges are not a function of the seller's value, but also depend on other buyers' values. In particular,

[4]The existence of such a p^* follows from the Intermediate Value Theorem. With $N = 2$, we have $2F(p^*) = 1$, which recovers the replica economy competitive equilibrium price in the case of bilateral trade with $F_S = F_B$.

the participation charge of each potential buyer is piecewise linear in an auxiliary quantity, namely the larger of the seller's value and the second-highest of the other buyers' values.

We examine the four desired properties of our mechanism in turn. First, as the mechanism is a Groves mechanism, efficiency and truthfulness are automatic.

Second, we show that the mechanism has no deficit. If the seller retains the good, then

$$\sum_{j=1}^{N} t_j(\theta) = \max\left\{\theta_{-N}^{(1)} - p^*, 0\right\} + \max\left\{p^* - \theta_N, 0\right\} \geq 0.$$

On the other hand, if the seller sells the good, there exists an agent $i \in \{1, \ldots, N-1\}$ such that $q_i(\theta) = 1$. We then have that

$$\sum_{j=1}^{N} t_j(\theta) = -\theta_{-N}^{(1)} + \max\left\{\theta_{-N}^{(1)} - p^*, 0\right\} + \theta_{-i}^{(1)} + \frac{1}{N-1} \max\left\{p^* - \max\left\{\theta_{-\{i,N\}}^{(2)}, \theta_N\right\}, 0\right\}$$
$$+ \frac{1}{N-1} \sum_{j \neq i, N} \max\left\{p^* - \max\left\{\theta_{-\{j,N\}}^{(2)}, \theta_N\right\}, 0\right\}.$$

Here, if $N \leq 3$, we take $\theta_{-\{i,N\}}^{(2)} = \theta_{-\{i,N\}}^{(2)} = 0$. It follows that

$$\sum_{j=1}^{N} t_j(\theta) \geq -\theta_i + \max\left\{\theta_i - p^*, 0\right\} + \theta_{-i}^{(1)} + \max\left\{p^* - \theta_{-i}^{(1)}, 0\right\} \geq 0,$$

where we use that $\theta_{-\{j,N\}}^{(2)} \leq \theta_{-i}^{(1)}$ for $j \notin \{i, N\}$, that $\theta_{-N}^{(1)} = \theta_i \geq \theta_{-i}^{(1)}$, and that $\theta_{-\{i,N\}}^{(2)} \leq \theta_{-i}^{(1)}$.

It remains to choose κ and show that the mechanism is *ex ante* individually rational.

Proof of *Ex Ante* Individual Rationality. As in the proof of Theorem 1(b), we first prove that our mechanism is *ex ante* individually rational in aggregate—i.e., $\mathbb{E}_\theta\left[\sum_{i=1}^{N} u_i(\theta)\right] \geq 0$. We show this by decomposing the sum of *ex ante* expected utilities as in the proof of Theorem 1 (Step 1); evaluating each term in the decomposition in the quantile space using properties of order statistics (Steps 2 and 3); and then comparing them (Step 4).

Step 1. Decomposing the Sum of Ex Ante Utilities. Note that

$$(q_N(\theta) - 1)\theta_N + \theta_{-N}^{(1)}(1 - q_N(\theta)) + \sum_{i=1}^{N-1} q_i(\theta)\left(\theta_i - \theta_{-i}^{(1)}\right)$$

can be rewritten by observing the following facts:

(1) if the seller sells the good ($q_N(\theta) = 0$), which happens with probability $\frac{N-1}{N}$ independently of the values of the order statistics $\theta^{(k)}$, then we have

$$\left(q_N(\theta) - 1\right)\theta_N + \theta_{-N}^{(1)}\left(1 - q_N(\theta)\right) + \sum_{i=1}^{N-1} q_i(\theta)\left(\theta_i - \theta_{-i}^{(1)}\right) = -\theta_N + \theta^{(1)} + \theta^{(1)} - \theta^{(2)}.$$

(2) if alternatively the seller keeps the good ($q_N(\theta) = 1$), then we have

$$\big(q_N(\theta) - 1\big)\theta_N + \theta^{(1)}_{-N}\big(1 - q_N(\theta)\big) + \sum_{i=1}^{N-1} q_i(\theta)\big(\theta_i - \theta^{(1)}_{-i}\big) = -\theta_N + \theta^{(1)} = 0.$$

Hence, we can rewrite

$$\mathbb{E}_\theta\Big[\big(q_N(\theta) - 1\big)\theta_N + \theta^{(1)}_{-N}\big(1 - q_N(\theta)\big) + \sum_{i=1}^{N-1} q_i(\theta)\big(\theta_i - \theta^{(1)}_{-i}\big)\Big]$$
$$= \mathbb{E}_\theta\big[\theta^{(1)} - \theta_N\big] + \frac{N-1}{N}\mathbb{E}_\theta\big[\theta^{(1)} - \theta^{(2)}\big].$$

Thus, we can decompose the sum of *ex ante* expected utilities as

$$\mathbb{E}_\theta\Big[\sum_{i=1}^{N} u_i(\theta)\Big] = GT + VCG_{def} - TPC, \tag{5}$$

where $GT = \mathbb{E}_\theta\left[\theta^{(1)} - \theta_N\right]$ are the expected gains from trade, $VCG_{def} = \frac{N-1}{N}\mathbb{E}_\theta\left[\theta^{(1)} - \theta^{(2)}\right]$ is the expected VCG deficit, and

$$TPC = \mathbb{E}_\theta\Big[\max\big\{\theta^{(1)}_{-N} - p^*, 0\big\} + \sum_{i=1}^{N-1}\frac{1}{N-1}\max\big\{p^* - \max\big\{\theta^{(2)}_{-\{N,i\}}, \theta_N\big\}, 0\big\}\Big]$$

is the expected total participation charge.

Step 2. Calculating the Expected Gains from Trade and VCG Deficit. We next derive expressions for the quantities GT and VCG_{def} in the quantile space. For the former, note that

$$GT = \mathbb{E}_\theta\left[\theta^{(1)} - \theta_N\right] = \int_0^1 (Nx^{N-1} - 1)F^{-1}(x)\,dx = \int_0^1 (x - x^N)\,dF^{-1}(x), \tag{6}$$

where the first inequality is by definition, the second is by using properties of order statistics, and the third is obtained by integration by parts.

Using similar techniques, we can obtain that the expected VCG deficit is given by

$$VCG_{def} = \frac{N-1}{N}\int_0^1 (Nx^{N-1} - N(N-1)x^{N-2}(1-x))F^{-1}(x)\,dx$$
$$= (N-1)\int_0^1 (Nx^{N-1} - (N-1)x^{N-2})\,dF^{-1}(x)$$
$$= (N-1)\int_0^1 (x^{N-1} - x^N)\,dF^{-1}(x) \tag{7}$$

Step 3. Calculating the Expected Total Participation Charge. We next calculate TPC by calculating the expected participation charge of each agent. For the seller, we have that

$$\begin{aligned}\mathbb{E}_\theta\big[\max\big\{\theta^{(1)}_{-N} - p^*, 0\big\}\big] &= \mathbb{E}_U\big[\max\big\{F^{-1}(U^{(1)}_{-N}) - p^*, 0\big\}\big]\\ &= \int_{F(p^*)}^{1} (F^{-1}(x) - p^*)(N-1)x^{N-2}\,dx\\ &= (1-p^*) - \int_{F(p^*)}^{1} x^{N-1}\,dF^{-1}(x) = \int_{F(p^*)}^{1}(1-x^{N-1})\,dF^{-1}(x),\end{aligned}$$

where the third inequality is obtained by integration by parts.

To calculate the expected participation charges of a buyer i, we first find the distribution of $Z_i = \max\big\{U^{(2)}_{-\{i,N\}}, U_N\big\}$. If U_N is the highest realization among all realizations except U_i, which occurs with probability $\frac{N-2}{N-1}$ independently of the order statistics $U^{(j)}_{-i}$, we have that $Z_i = U^{(1)}_{-i}$. If not, which occurs with probability $\frac{1}{N-1}$ independently of the order statistics $U^{(j)}_{-i}$, we have that $Z_i = U^{(2)}_{-i}$. Hence, Z_i has the same distribution as a mixture of $U^{(1)}_{-i}$ and $U^{(2)}_{-i}$ with weights $\frac{1}{N-1}$ and $\frac{N-2}{N-1}$, respectively. Hence, the density of Z_i is given by

$$f_{Z_i}(z) = z^{N-2} + (N-2)^2 z^{N-3} - (N-2)^2 z^{N-2} \quad = (N-2)^2 z^{N-3} - (N-1)(N-3)z^{N-2},$$

so the cumulative distribution function of Z_i is given by

$$F_{Z_i}(z) = (N-2)z^{N-2} - (N-3)z^{N-1}.$$

It follows that the expected participation charge of buyer i is

$$\begin{aligned}&\mathbb{E}_\theta\left[\frac{1}{N-1}\max\big\{p^* - \max\big\{\theta^{(2)}_{-\{N,i\}}, \theta_N\big\}, 0\big\}\right]\\ =&\mathbb{E}_Z\left[\frac{1}{N-1}\max\big\{p^* - F^{-1}(Z_i), 0\big\}\right]\\ =&\frac{1}{N-1}\int_0^{F(p^*)}(p^* - F^{-1}(z))\big[(N-2)^2 z^{N-3} - (N-1)(N-3)z^{N-2}\big]\,dz\\ =&\frac{1}{N-1}\int_0^{F(p^*)}\big[(N-2)z^{N-2} - (N-3)z^{N-1}\big]\,dF^{-1}(z),\end{aligned}$$

where the third equality is obtained by integration by parts and the definition of p^*.

Hence, the expected total participation charge can be written as

$$\begin{aligned}TPC &= \mathbb{E}_\theta\left[\max\big\{\theta^{(1)}_{-N} - p^*, 0\big\} + \sum_{i=1}^{N-1}\frac{1}{N-1}\max\big\{p^* - \max\big\{\theta^{(2)}_{-\{N,i\}}, \theta_N\big\}, 0\big\}\right]\\ &= \int_0^{F(p^*)}\big[(N-2)x^{N-2} - (N-3)x^{N-1}\big]\,dF^{-1}(x) + \int_{F(p^*)}^{1}(1-x^{N-1})\,dF^{-1}(x).\end{aligned}$$

Note that being the cumulative distribution function of Z_i, the function $(N-2)x^{N-2} - (N-3)x^{N-1}$ is non-decreasing on $[0,1]$. Moroever, the function $1-x^{N-1}$ is strictly decreasing on $[0,1]$. As $(N-2)x^{N-2} - (N-3)x^{N-1}$ and $1-x^{N-1}$ are equal at $F(p^*)$ by the definition of p^*, we have that $(N-2)x^{N-2} - (N-3)x^{N-1} > 1 - x^{N-1}$ for $0 \leq x < F(p^*)$, as well as that $(N-2)x^{N-2} - (N-3)x^{N-1} < 1 - x^{N-1}$ for $1 \geq x > F(p^*)$. It follows that

$$TPC = \int_0^1 \min\left\{(N-2)x^{N-2} - (N-3)x^{N-1}, 1 - x^{N-1}\right\} dF^{-1}(x). \tag{8}$$

Step 4. Proving Aggregate Ex Ante Individual Rationality. To compare the expressions (6) and (7) from Step 2 with the expression (8) from Step 3, we use a technical claim.

Lemma 3. *Let $N \geq 2$. For all $0 \leq x \leq 1$, we have that*

$$x + (N-1)x^{N-1} - Nx^N \geq \min\left\{(N-2)x^{N-2} - (N-3)x^{N-1}, 1 - x^{N-1}\right\}.$$

Proof. It suffices to show that the left-hand side is greater than or equal to a convex combination of the two terms on the right-hand side. Taking a convex combination with weight $1 - x^{N-1}$ on the first term and weight x^{N-1} on the second term, it suffices to show that

$$x + (N-1)x^{N-1} - Nx^N \geq (N-2)x^{N-2} - (N-4)x^{N-1} - (N-2)x^{2N-3} + (N-4)x^{2N-2}$$

for all $x \in [0,1]$. This is equivalent to

$$x + (2N-5)x^{N-1} + (N-2)x^{2N-3} \geq (N-2)x^{N-2} + Nx^N + (N-4)x^{2N-2}. \tag{9}$$

Comparing arithmetic and geometric means yields that $\frac{1}{N-2}x + \frac{N-3}{N-2}x^{N-1} \geq x^{N-2}$, so that

$$x + (N-3)x^{N-1} \geq (N-2)x^{N-2}. \tag{10}$$

To deal with the remaining terms, we consider the function

$$g(x) = (N-2) + (N-2)x^{N-2} - Nx - (N-4)x^{N-1}$$

We claim that $g(x) \geq 0$ for all $x \in [0,1]$. For $N = 2$, we have that $g'(x) = g(x) = 0$. For $N = 3$, we have that $g(x) = (x-1)^2 \geq 0$. For $N \geq 4$, since we have $g(1) = 0$, it suffices to show that $g'(x) \leq 0$ for all $x \in [0,1]$. We have

$$g'(x) = -N + (N-2)^2 x^{N-3} - (N-1)(N-4)x^{N-2}.$$

Comparing arithmetic and geometric means yields that

$$\frac{N}{(N-2)^2} + \frac{(N-1)(N-4)}{(N-2)^2} x^{N-2} \geq x^{\frac{(N-1)(N-4)}{N-2}} \geq x^{N-3},$$

where the last inequality holds as $(N-1)(N-4) \leq (N-2)(N-3)$ and $x \in [0,1]$. It follows that $g'(x) \leq 0$ for all $x \in [0,1]$, and hence that $g(x) \geq 0$ for all $x \in [0,1]$.

Using the fact that $x^{N-1}g(x) \geq 0$ for all $x \in [0,1]$ then yields that

$$(N-2)x^{N-1} + (N-2)x^{2N-3} \geq Nx^N + (N-4)x^{N-1}.$$

Adding this inequality to (10) then yields the desired inequality (9). □

Combining (6), (7), and (8) with Lemma 3, we see that

$$\begin{aligned} GT + VCG_{def} &= \int_0^1 \left[x + (N-1)x^{N-1} - Nx^N\right] dF^{-1}(x) \\ &\geq \int_0^1 \min\{(N-2)x^{N-2} - (N-3)x^{N-1}, 1 - x^{N-1}\}\, dF^{-1}(x) = TPC. \end{aligned}$$

By (5), it follows that aggregate *ex ante* individual rationality holds. As in the proof of Theorem 1(b), we can construct a lump-sum transfer κ to ensure *ex ante* individual rationality.

5. Conclusion

Efficiency can always be achieved even in the presence of private information when participation decisions are made *ex ante* (Arrow, 1979; d'Aspremont and Gérard-Varet, 1979), but existing efficient mechanisms are not robust to the misspecification of higher-order beliefs (in the sense of Wilson (1987) and Bergemann and Morris (2005)). To understand the costs of robustness in this context, we study truthful implementation under an *ex ante* individual rationality constraint. Our main results provide conditions under which efficiency is achievable in bilateral trade, and show that it is always achievable in partnership dissolution. We also delineate the limits of truthful implementation under an *ex ante* individual rationality condition by showing that efficiency is not always achievable in bilateral trade.

The *ex ante* individual rationality condition is economically natural in settings in which agents can commit to participating in the mechanism in advance of learning what is needed to determine their values (e.g., before a partner makes investments). An interesting direction for future work would be to investigate truthful implementation under *ex ante* individual rationality constraints more broadly.

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