Restoring Existence and Uniqueness at the Effective Lower Bound with Simple Fiscal Policy*

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Abstract

The presence of an occasionally binding constraint from the effective lower bound (ELB) in New Keynesian models often leads to multiple or no equilibria. The problem stems from a strong feedback loop between expectations (of inflation and output) and current outcomes at the ELB. We show that simple fiscal policy rules can introduce additional stabilising forces that dampen this loop, thereby ensuring the existence and uniqueness of an equilibrium.

Keywords: fiscal policy, existence, uniqueness, rational expectations, effective lower bound

JEL Codes: C62, E4, E61, E62, E63

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1 Introduction

The canonical New Keynesian (NK) model with an occasionally binding effective lower bound (ELB) constraint can admit either no solutions (non-existence) or multiple solutions (non-uniqueness).[1] These findings have been demonstrated by Ascari and Mavroeidis (2022) (henceforce, AM) in a stochastic environment with rational expectations and Holden (2023) under perfect foresight. With non-existence, the economy cannot settle into a consistent, stable outcome; while with multiplicity, non-fundamental shifts in beliefs might trigger belief-driven recessions. To prevent such outcomes, policymakers aim to ensure their actions will produce uniqueness.

This paper demonstrates that simple Ricardian fiscal policy (FP) ensures a unique minimum state variable (MSV) solution. Our key finding is if FP is persistent and reactive to inflation and output fluctuations, it guarantees a unique MSV solution, while also satisfying Blanchard-Kahn (BK) local determinacy conditions. This paper identifies two critical properties for achieving uniqueness of an MSV solution. First, at the ELB, FP stabilises the economy when monetary policy is constrained, establishing an equilibrium path. Second, a countercyclical rule-based FP eliminates belief-driven equilibria when it is sufficiently persistent.

Building on the work of Gourieroux, Laffont, and Monfort (1980) (GLM), AM derive two main results using a system of linearised equations and endogenous regime switching. First, they demonstrate that achieving solution existence in ELB-constrained NK models poses a non-trivial challenge when the Taylor principle is satisfied. Additionally, AM identify conditions that restrict the support of stochastic shocks, necessary to ensure model existence. However, these support restrictions prove cumbersome, and depend on model structural parameters and past realisations of state variables in backward-looking models. Second, even with support restrictions to ensure existence, the model may still exhibit multiple MSV solutions, potentially up to 2^k solutions, where k represents the number of discrete shock states.

This concern extends beyond the conventional scope of the ELB literature, which mainly examined sunspot shocks or belief-driven fluctuations between steady states.[2] However,

1. Referred to as "incoherent" and "incomplete", respectively, in Ascari and Mavroeidis (2022) although the meanings are equivalent.

2. See, for example, Eggertsson and Woodford (2003), Guerrieri and Iacoviello (2015), Kulish, Morley, and Robinson (2017), Aruoba, Cuba-Borda, and Schorfheide (2018), Aruoba et al. (2021), and Angeletos and Lian (2023).

general conditions to ensure existence and uniqueness of MSV solutions in the macroeconomics DSGE literature remain limited, although recent papers have provided sufficiency conditions for MSV equilibrium existence in NK models (Eggertsson, 2011; Christiano, Eichenbaum, and Johannsen, 2018; Nakata, 2018; Nakata and Schmidt, 2019). Compared to this strand of literature, this paper studies both existence and uniqueness.

As highlighted in follow-up work, Ascari, Mavroeidis, and McClung (2023) showed that multiplicity of MSV solutions emerges from the interplay between rational expectations and the inherently non-linear nature of the ELB constraint. In addition, Holden (2024) presents an alternative approach to ensuring existence and uniqueness at the ELB via adjusting the monetary authority's inflation target to be consistent with the Fisher equation with a positive nominal interest rate. Compared to these papers, this paper maintains the FIRE framework and proposes mechanisms specifically emphasising the role of FP to address issues identified by AM and Holden.

Thus, our paper adds to the studies that explore FP, the ELB, and multiple equilibria interactions. Seminal work by Benhabib, Schmitt-Grohé, and Uribe (2001) examined how Ricardian FP with active monetary policy leads to unique convergence to a steady state equilibrium. However, convergence was not always to a unique steady state and could include an unintended liquidity trap steady state. Benhabib, Schmitt-Grohé, and Uribe (2002) extended this to establish convergence to a non-liquidity trap steady state. Both studies assumed perfect foresight environments, while this paper maintains FIRE.[3] Critically, we argue that the problem of non-existence and non-uniqueness stems from the strong feedback loop between expectations (of inflation and output in a canonical NK model) and current realisations at the ELB. The core aim of our paper is to show that certain FP rules can introduce stabilising forces that dampen this feedback loop. This in turn can help ensure not only existence but also uniqueness of equilibrium.

Our paper is closely related to the contributions of Schmidt (2016), Tamanyu (2021), and Nakata and Schmidt (2022), which addressed the aforementioned classical concerns of the literature on the ELB. These studies show how expectations-driven liquidity traps could be avoided with appropriate FP, emphasising fiscal rule variations. Meanwhile, examples of a more policy-focused contribution that our work is related to are Correia et al. (2013) and Seidl and Seyrich (2023) which show that distortionary tax policy can perfectly replicate the unique rational expectations equilibrium without the ELB constraint. While these results were quantitatively demonstrated in a perfect foresight environment

3. See Definition 3 and Propositions 5 and 6 of Benhabib, Schmitt-Grohé, and Uribe (2001).

with agents making expectation errors, our work – using a textbook NK setup – encompasses the mechanisms of their basic model as a special case.

It is notable that the aforementioned literature on the ELB and FP primarily focused on the elimination of a liquidity trap steady state, often assuming restrictions on the shock process or stochastic environment. Our primary contribution is to simultaneously consider existence and uniquness of MSV solutions, as well as local determinacy (BK conditions) concerning the ELB and FP instruments. Additionally, despite the paper looking into fiscal and monetary policy interactions,[4] it refrains from examining fiscal policy potency or fiscal multipliers at the ELB.

The paper is structured as follows: Section 2 provides an overview of existence and uniqueness (E&U) conditions for an MSV solution within the context of an ELB-bound NK model and describes the methodology used to verify these conditions. Section 3 demonstrates how Ricardian FP restores E&U of an MSV solution in a purely forward-looking reference NK model constrained by the ELB. Section 4 assesses E&U conditions for an NK model with FP featuring policy inertia; that is, a model with an endogenous state. Finally, Section 5 concludes the paper.

2 Verifying a Unique MSV Solution of the New Keynesian Model with the ELB

In this section, we provide a sketch of AM's methodology to verify E&U of systems of piecewise-linear equations, applying the methodology to a textbook NK model subject to the ELB. Further explanation and derivation can be found in Appendix A or in AM. For brevity, in this Section and Section 3, we abstract from models that feature endogenous states. We revisit E&U conditions for models with endogenous states in Section 4.

General verification for linear models. Let Y_t be a $n \times 1$ vector of endogenous variables, X_t be a $n_x \times 1$ vector of exogenous state variables, and $s_t \in \{0, 1\}$ be an indicator variable that is equal to 1 when some inequality constraint is slack and 0 otherwise. Additionally, let Ω_t denote the information set, thus allowing us to write: $Y_{t+1|t} = \mathbb{E}_t[Y_{t+1}|\Omega_t]$ and

4. This literature is vast – see, for example, Galí, López-Salido, and Vallés (2007), Davig and Leeper (2011), Eggertsson and Krugman (2012), Billi and Walsh (2022), and Hills and Nakata (2018).

$X_{t+1|t} = \mathbb{E}_t[X_{t+1}|\Omega_t]$. The system can be written in the canonical form

$$\begin{aligned}\mathbf{0} &= \left(\boldsymbol{A}_{s_{t,i}} Y_t + \boldsymbol{B}_{s_{t,i}} Y_{t+1|t} + \boldsymbol{C}_{s_{t,i}} X_t + \boldsymbol{D}_{s_{t,i}} X_{t+1|t}\right), \\ s_{t,i} &= \mathbb{1}\left(\left[\boldsymbol{a}^\top Y_t + \boldsymbol{b}^\top Y_{t+1|t} + \boldsymbol{c}^\top X_t + \boldsymbol{d}^\top X_{t+1|t}\right] > 0\right),\end{aligned} \tag{1}$$

where $\boldsymbol{A}_{s_i}$, $\boldsymbol{B}_{s_i}$, $\boldsymbol{C}_{s_i}$, and $\boldsymbol{D}_{s_i}$ are coefficient matrices, $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$ are coefficient vectors, and $\mathbb{1}(\cdot)$ is an indicator function that is equal to unity if the inequality constraint is slack and zero otherwise. Without loss of generality, we assume that shocks X_t are k-state stationary first-order Markov processes with transition kernel $\boldsymbol{K}$. Collecting all possible states of X_t for states $i = 1, .., k$ into a $n_x \times k$ matrix $\mathbf{X}$. Let $\boldsymbol{e}_i$ denote the i-th column of the $k \times k$ identity matrix $\boldsymbol{I}_k$, such that $\mathbf{X}\boldsymbol{e}_i$, the i-th column of $\mathbf{X}$, is the i-th state of X_t. Moreover, the elements of the transition kernel $\boldsymbol{K}$ are $\boldsymbol{K}_{ij} = \Pr(X_{t+1} = \mathbf{X}\boldsymbol{e}_j | X_t = \mathbf{X}\boldsymbol{e}_i)$.

The existence of MSV solutions of the system (1) cannot be analysed directly using the GLM theorem due to the existence of expectations of the endogenous variables $Y_{t+1|t}$. However, we can convert Equation (1) into a model without expectations of endogenous variables by imposing certain rules for expectations. While there are multiple possible rules, here we focus on FIRE, as in AM.

Under FIRE, the agents' expectation of X_{t+1}, based on Ω_t, should be consistent with the actual outcomes in equilibrium, which implies that $\mathbb{E}_t\left[X_{t+1}|\Omega_t = X_{t+1}|X_t = \mathbf{X}\boldsymbol{e}_i\right] = \mathbf{X}\boldsymbol{K}^\top \boldsymbol{e}_i$.[5] Moreover, define $\mathbf{Y}$ as an $n \times k$ matrix whose i-th column, $\mathbf{Y}\boldsymbol{e}_i$ is the corresponding i-th state value of Y_t, i.e., $Y_t = f(X_t = \mathbf{X}\boldsymbol{e}_i)$ along an MSV solution. Therefore, under rational expectations, the expectation of Y_{t+1} is pinned down by $\mathbb{E}[Y_{t+1}|\Omega_t = Y_{t+1}|Y_t = \mathbf{Y}\boldsymbol{e}_i] = \mathbb{E}[Y_{t+1}|X_t = \mathbf{X}\boldsymbol{e}_i] = \mathbf{Y}\boldsymbol{K}^\top \boldsymbol{e}_i$. Substituting the expectations into Equation (1), we rewrite the problem as

$$\begin{aligned}\mathbf{0} &= \left(\boldsymbol{A}_{s_i}\mathbf{Y} + \boldsymbol{B}_{s_i}\mathbf{Y}\boldsymbol{K}^\top + \boldsymbol{C}_{s_i}\mathbf{X} + \boldsymbol{D}_{s_i}\mathbf{X}\boldsymbol{K}^\top\right)\boldsymbol{e}_i, \\ s_i &= \mathbb{1}\left(\left[\boldsymbol{a}^\top\mathbf{Y} + \boldsymbol{b}^\top\mathbf{Y}\boldsymbol{K}^\top + \boldsymbol{c}^\top\mathbf{X} + \boldsymbol{d}^\top\mathbf{X}\boldsymbol{K}^\top\right]\boldsymbol{e}_i > 0\right), \quad i = 1, ..., k.\end{aligned} \tag{2}$$

The system (2) relates $\mathbf{Y}$ to $\mathbf{X}$, and can be expressed as $F(\mathbf{Y}) = \lambda(\mathbf{X})$, where $\lambda(\cdot)$ is some function of $\mathbf{X}$,[6] and $F(\cdot)$ is a piecewise linear function of $\mathbf{Y}$. To formalise this, consider a subset $J \subseteq \{1, ..., k\}$. For each subset J, define the corresponding cone $\boldsymbol{C}_J$ as the set of

5. For example, suppose $k = 2$. At time t, X_t is in state h, i.e., $X_t = X(k = h)$. With probability p it will stay in state h, and with probability $1 - p$ it will switch to state l. Then, the expectation of X_{t+1} under FIRE should be $\mathbb{E}[X_{t+1}|X_t = X(k = h)] = pX(k = h) + (1 - p)X(k = l)$.

6. Appendix A.1.3 shows that the function is unbounded for the class of models considered in this paper.

matrices $\mathbf{Y} \in \mathbb{R}^{n\times k}$ for which the regime indicators s_i satisfy:

$$\boldsymbol{C}_J = \{\mathbf{Y}|\mathbf{Y} \in \mathbb{R}^{n\times k}, s_i = 1 \text{ if } i \in J \text{ and } s_i = 0 \text{ if } i \notin J\}.$$

For example, the cone of the subset $J = 1$, $\boldsymbol{C}_{\{1\}}$ implies that $s_1 = 1$ but all others $s_{i\neq 1} = 0$.[7] There are 2^k such cones. Let us associate an invertible linear mapping $\boldsymbol{\mathcal{A}}_J$ with each cone. Then the piecewise-linear function $F(\mathbf{Y})$ can be expressed as:

$$F(\mathbf{Y}) = \sum_J \boldsymbol{\mathcal{A}}_J\, \mathbb{1}_{\boldsymbol{C}_J} \operatorname{vec}(\mathbf{Y}), \tag{3}$$

where $\mathbb{1}\big(\boldsymbol{C}_J|\mathbf{Y} \in \mathbb{C}_J\big) = 1$ and $\mathbb{1}\big(\boldsymbol{C}_J|\mathbf{Y} \notin \boldsymbol{C}_J\big) = 0$,[8] and $\operatorname{vec}(\cdot)$ is the vector operator function.[9]

If $F(\mathbf{Y})$ in (3) is invertible, then the linear system has a unique MSV solution. A necessary and sufficient condition for the invertibility of $F(\cdot)$, as stipulated in GLM, is that all the determinants of $\boldsymbol{\mathcal{A}}_J$, $J \subseteq \{1, \ldots, k\}$ have the same sign. Failure of this requirement implies that either no MSV solution exists or that there may be multiple MSV solutions:

Theorem 1 (GLM)**.** *Suppose that the mapping $F(\cdot)$ defined in (3) is continuous. A necessary and sufficient condition for $F(\cdot)$ to be invertible is that all the determinants* $\det \mathcal{A}_J, J \subseteq \{1, \ldots, k\}$ *have the same sign.*

Application to a canonical New Keynesian model with the ELB. Consider the canonical NK model as in, for example, Galí (2015). The model in its linearised in terms of log-deviations from steady state can be written in three equations, the dynamic IS equa-

7. Using the state-space formulation, $\boldsymbol{C}_{\{1\}}$ means

$$0 = \left(\boldsymbol{A}_1\mathbf{Y} + \boldsymbol{B}_1\mathbf{Y}\boldsymbol{K}^\top + \boldsymbol{C}_1\mathbf{X} + \boldsymbol{D}_1\mathbf{X}\boldsymbol{K}^\top\right)\boldsymbol{e}_i, \quad \text{if } \left(\boldsymbol{a}^\top\mathbf{Y} + \boldsymbol{b}^\top\mathbf{Y}\boldsymbol{K}^\top + \boldsymbol{c}^\top\mathbf{X} + \boldsymbol{d}^\top\mathbf{X}\boldsymbol{K}^\top\right)\boldsymbol{e}_i > 0, \quad i = 1,$$

$$\text{and } 0 = \left(\boldsymbol{A}_0\mathbf{Y} + \boldsymbol{B}_0\mathbf{Y}\boldsymbol{K}^\top + \boldsymbol{C}_0\mathbf{X} + \boldsymbol{D}_0\mathbf{X}\boldsymbol{K}^\top\right)\boldsymbol{e}_i, \quad \text{if } \left(\boldsymbol{a}^\top\mathbf{Y} + \boldsymbol{b}^\top\mathbf{Y}\boldsymbol{K}^\top + \boldsymbol{c}^\top\mathbf{X} + \boldsymbol{d}^\top\mathbf{X}\boldsymbol{K}^\top\right)\boldsymbol{e}_i \leq 0, \quad i = 2, \ldots, k.$$

8. This indicator tells you which segment you are in. If you are in the cone $\boldsymbol{C}_J$, then F calls the mapping $\boldsymbol{\mathcal{A}}_J$.

9. The transformation of (2) into (3) is generally non-trivial (in which the expressions of $\boldsymbol{\mathcal{A}}_J$ require Kronecker product operations) as it presents a Sylvester equation in $\mathbf{Y}$. See, for example, Kolmogorov and Fomin (1957). However, there are two exceptions that allow straightforward computation of the $\boldsymbol{\mathcal{A}}_J$: $n = 1$ and $n = k > 1$. We make use of this simplifying assumption both in this example and the analytical derivation in Appendix A.

tion (DISE), New Keynesian Phillips Curve (NKPC), and the Taylor rule (TR):

$$\text{DISE: } \hat{y}_t = \mathbb{E}_t \hat{y}_{t+1} - \frac{1}{\sigma}(\hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1}) + \varepsilon_t, \tag{4a}$$

$$\text{NKPC: } \hat{\pi}_t = \beta \mathbb{E}_t \hat{\pi}_{t+1} + \kappa \hat{y}_t + u_t, \tag{4b}$$

$$\text{TR: } \hat{i}_t = \max\left\{-\mu, \phi_\pi \hat{\pi}_t + \phi_y \hat{y}_t\right\}, \tag{4c}$$

and where ε_t is a demand shock, u_t is a cost-push shock, $\hat{y}_t$ is the output gap, $\hat{\pi}_t$ is inflation, and $\hat{i}_t$ is the nominal interest rate. The parameters of interest in the model are: σ, the coefficient of relative risk aversion; β, the representative household's subjective discount factor; κ, the slope of the NKPC; $\mu = \ln(r\pi^*)$, the ELB of the nominal interest rate in deviation from the steady state, where $r = 1/\beta$ is the steady state gross real interest rate and π^* is the gross inflation target of the monetary authority; ϕ_y, the monetary authority's response parameter to output fluctuations; and ϕ_π, the monetary authority's responsiveness to inflation. In particular, we consider an active monetary policy rule: $\phi_\pi > 1$. We can write the model in canonical form (1) with $Y_t = \begin{bmatrix} \hat{\pi}_t & \hat{y}_t \end{bmatrix}^\top$ and $X_t = \begin{bmatrix} u_t & \varepsilon_t & \mu \end{bmatrix}^\top$, and with coefficient matrices given in Appendix A.1.1.

To simplify the analysis, we assume $u_t = 0, \forall t$, $\phi_y = 0$, and that ε_t follows a two-state Markov process $(k = 2)$ with states $\varepsilon_t = (\varepsilon^T, 0)$ and a transition kernel:

$$\boldsymbol{K} = \begin{bmatrix} p & 1-p \\ 1-q & q \end{bmatrix}, \tag{5}$$

where p is the probability of remaining in the first state and q is the probability of remaining in the second state.

In this model, the expectation of the endogenous state variable under FIRE is given by $\mathbb{E}_t[Y_{t+1}|X_t = \mathbf{X}\boldsymbol{e}_i] = \mathbf{Y}\mathbf{K}^\top \boldsymbol{e}_i$. Given this, the E&U of an MSV solution can be checked following the strategy outlined earlier in this section. First, under FIRE, the model can be written in canonical form (2). Then write the piecewise linear function $F(\mathbf{Y})$, where the mappings are given by $\mathcal{A}_J$:[10]

$$
\begin{aligned}
\mathcal{A}_{J_1} &= \boldsymbol{I}_2 \otimes \boldsymbol{A}_1 + \boldsymbol{K} \otimes \boldsymbol{B}_1 && J_1 = \{1,2\}, \\
\mathcal{A}_{J_2} &= \boldsymbol{e}_1\boldsymbol{e}_1^\top \otimes \boldsymbol{A}_0 + \boldsymbol{e}_1\boldsymbol{e}_1^\top \boldsymbol{K} \otimes \boldsymbol{B}_0 + \boldsymbol{e}_2\boldsymbol{e}_2^\top \otimes \boldsymbol{A}_1 + \boldsymbol{e}_2\boldsymbol{e}_2^\top \boldsymbol{K} \otimes \boldsymbol{B}_1 && J_2 = \{2\}, \\
\mathcal{A}_{J_3} &= \boldsymbol{e}_1\boldsymbol{e}_1^\top \otimes \boldsymbol{A}_1 + \boldsymbol{e}_1\boldsymbol{e}_1^\top \boldsymbol{K} \otimes \boldsymbol{B}_1 + \boldsymbol{e}_2\boldsymbol{e}_2^\top \otimes \boldsymbol{A}_0 + \boldsymbol{e}_2\boldsymbol{e}_2^\top \boldsymbol{K} \otimes \boldsymbol{B}_0 && J_3 = \{1\}, \\
\mathcal{A}_{J_4} &= \boldsymbol{I}_2 \otimes \boldsymbol{A}_0 + \boldsymbol{K} \otimes \boldsymbol{B}_0, && J_4 = \emptyset,
\end{aligned}
\tag{6}
$$

where $\boldsymbol{e}_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^\top$ and $\boldsymbol{e}_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^\top$, and matrices $(\boldsymbol{A}_0, \boldsymbol{B}_0)$ and $(\boldsymbol{A}_1, \boldsymbol{B}_1)$ are defined in Appendix A.1.1, corresponding to the coefficient matrices of Y_t and $Y_{t+1|t}$ when the inequality constraint is binding and slack, respectively. $\otimes$ is the Kronecker product. J follows the same definition as before. Specifically, $J_1 = \{1,2\}$ implies that the constraints are slack in both states (i.e., $s_i = 1$ for $i = \{1,2\}$). $J_2 = \{2\}$ implies that the constraint is slack in state $i = 2$ but binding in state $i = 1$ (i.e., $s_2 = 1$ and $s_1 = 0$). $J_3 = \{1\}$ implies that the constraint is slack in state $i = 1$ but binding in state $i = 2$ (i.e., $s_1 = 1$ and $s_2 = 0$). $J_4 = \emptyset$ implies that the constraint is binding in both states (i.e., $s_i = 0$ for $i = 1, 2$).

We solve the determinants of $\mathcal{A}_J$ and the expressions can be found in the Appendix A.1.2. Under an active monetary policy rule, the determinants do not have the same sign for some values of p and q. This implies that the system does not generally have a unique MSV solution. We show the results analytically and explain the intuition through the following special case where $q = 1$.

A special case ($q = 1$). We consider the special case where $p < 1$ (transitory state) and $q = 1$ (absorbing state), with the support of ε_t equal to ε^T and 0, respectively. Note that $q = 1$ implies that once the system enters this state where $\varepsilon_t = 0$, the shock vanishes forever. This simplifying assumption allows us to isolate the transitory state equilibrium, without needing to account for expected transitions. Under this assumption, the model admits two absorbing states when $\varepsilon_t = 0$: a positive interest rate (PIR) steady state, where $\{\hat{\pi}, \hat{y}, \hat{i}\} = \{0, 0, 0\}$, and a zero interest rate (ZIR) steady state, $\{\hat{\pi}, \hat{y}, \hat{i}\} = \{-\mu, -\mu(1 - \beta)/\kappa, -\mu\}$.[11]

In the transitory states, the equilibrium is characterised by $(\hat{\pi}^T, \hat{y}^T)$ and with probability $1 - p$ the equilibrium transitions to the absorbing state, which can be either a PIR or

10. The matrix form of the mappings can be found in the Appendix A.1.
11. For a derivation see Appendix A.1.4.

ZIR equilibrium. Here, we consider the dynamic system around the PIR absorbing state.[12] The AS and AD relations can be written as:

$$\hat{\pi}^T = \frac{\kappa}{1 - p\beta}\hat{y}^T \quad AS, \tag{7a}$$

$$\hat{\pi}^T = \begin{cases} \frac{\sigma(1-p)}{p-\phi_\pi}\hat{y}^T - \frac{\sigma}{(p-\phi_\pi)}\varepsilon^T & AD^{TR} \text{ for } \hat{\pi}^T \geq -\frac{\mu}{\phi_\pi}, \\ \frac{\sigma(1-p)}{p}\hat{y}^T - \frac{\mu}{p} - \frac{\sigma}{p}\varepsilon^T & AD^{ELB} \text{ for } \hat{\pi}^T \leq -\frac{\mu}{\phi_\pi}. \end{cases} \tag{7b}$$

AD^{TR} is downward-sloping when $\phi_\pi > 1$, while both AS and AD^{ELB} are upward-sloping. Define θ as the ratio of the slopes of the AD^{ELB} and AS,

$$\theta \equiv \frac{\sigma(1-p)(1-p\beta)}{p\kappa}, \tag{8}$$

where it is decreasing in p, the probability of staying in the transitory state. For a small p, θ can be greater than 1, which implies that the slope of AD^{ELB} is steeper than the slope of AS.[13] For large values of p, $\theta < 1$, i.e., the slope of AD^{ELB} is flatter than the slope of AS. These two possibilities are illustrated in Figure 1. As discussed by AM, for the case where $\theta > 1$, there is always a solution for any value of ε^T. But when $\theta \leq 1$, there are two solutions when ε^T is small and no solution if the shock ε^T is large. For the existence of the solution, we need to impose a lower bound on the shock ε^T:

$$\varepsilon^T \geq -\frac{\mu p}{\sigma}\left(\frac{\theta}{\phi_\pi} + \frac{\phi_\pi - p}{p\phi_\pi}\right). \tag{9}$$

Put simply, these support restrictions ensure that a negative shock to AD does not lead it shifting too far to the left or above of AS such that there is no intersection, as shown in $AD_1^{TR,ELB}$ of Subfigure 1b.

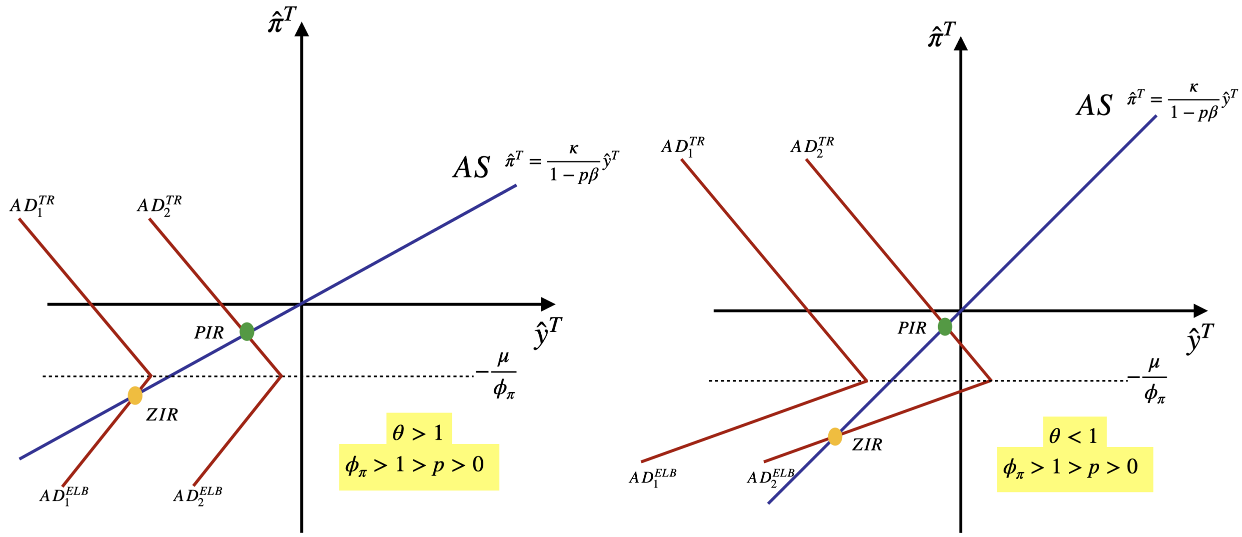
We now explain the intuition behind the non-existence of a solution for a large negative demand shock ε^T. First, note that output gap in the transitory state when $\varepsilon_t = \varepsilon^T$ is given by:

$$\hat{y}_t = \Lambda\mathbb{E}_t\hat{y}_{t+1} - \frac{1}{\sigma}\max\left\{\phi_\pi\frac{\kappa}{1-\beta p}\hat{y}_t, -\mu\right\} + \varepsilon^T,$$

12. We relegate the discussion of the ZIR absorbing state to Appendix A.1.4.

13. If we instead assume perfect foresight with $\hat{\pi}_{t+1} = p\hat{\pi}_t$ and $\hat{y}_{t+1} = p\hat{y}_t$, then the condition $\theta > 1$ exactly matches the $M_{11} > 0$ condition in Holden (2023). Moreover, if we further assume $p = 0$, i.e., the economy jumps to the PIR steady state in the next period, then both the condition $\theta > 1$ and the condition $M_{11} > 0$ are always satisfied.

Figure 1: Transitory States of the New Keynesian Model ($\phi_\pi > 1$)



(a) PIR absorbing states ($\theta > 1$) (b) PIR absorbing states ($\theta < 1$)

Note: The figure shows how adverse demand shocks of different size, ε^T, shift the demand curve. Left panel shows the case where the equilibrium is unique. Right panel illustrates non-existence and non-uniqueness.

where

$$\Lambda \equiv \left[1 + \frac{\kappa}{\sigma(1-\beta p)}\right] > 1.$$

For small values of $\varepsilon^T < 0$, the ELB constraint is slack, and the above equation is stable. However, for sufficiently large $\varepsilon^T < 0$ (very negative, to be precise), the ELB will be binding and the output gap will be given by:

$$\hat{y}_t = \Lambda \mathbb{E}_t \hat{y}_{t+1} + \frac{\mu}{\sigma} + \varepsilon^T, \tag{10}$$

where $\mathbb{E}_t \hat{y}_{t+1} = p\hat{y}_t$ under rational expectations. We focus on the feedback loop between $\hat{y}_t$ and $\mathbb{E}_t \hat{y}_{t+1}$. When a sufficiently large negative demand shock eventuates, the contemporaneous output gap becomes negative ($\hat{y}_t < 0$). This leads to $\mathbb{E}_t \hat{y}_{t+1} < 0$. If $p\Lambda > 1$, Equation (10) implies an even more negative output gap and thus an even more negative $\mathbb{E}_t \hat{y}_{t+1}$. The feedback loop implies that the output gap (and its expectation) fall indefinitely. This leads to no fixed point where expectations settle into an equilibrium, and thus the solution does not exist.[14]

Another way to interpret the non-existence of the solution in this system is that the equilibrium is inconsistent with the ELB constraint. To see this, set $\hat{y}_t = \hat{y}^T$ and substitute $\mathbb{E}_t \hat{y}_{t+1} = p\hat{y}^T$ into Equation (10), to write the output gap as:

$$\hat{y}^T = \frac{1}{1 - p\Lambda}\left(\frac{\mu}{\sigma} + \varepsilon^T\right). \tag{11}$$

If again $p\Lambda > 1$ and there is a large negative realisation of ε^T, then $\hat{y}^T$ and $\hat{\pi}^T$ can potentially be positive following ε^T, which precludes existence of a solution in which the ELB binds. Therefore, for a solution to exist we require either:

(i) p to be small such that $p\Lambda < 1$, which the solution exists for any ε^T, this corresponds to the condition $\theta > 1$ and is illustrated in Figure 1a; or

(ii) For $p\Lambda > 1$ (corresponding to $\theta < 1$), instead we restrict ε^T to be small such that the equilibrium exists, as illustrated in Figure 1b.

Either way, both options require restrictions on the support of the demand shock. The interpretation of such restrictions on p and on ε^T is straightforward. The negative demand shock ε^T increases real interest rates when nominal interest rates are at the ELB. As real interest rates rise, intertemporal substitution effects induce households to save more and consume less, which puts downward pressure on inflation and output. Conversely, income effects exert upward pressure on inflation and output. Since $\Lambda > 1$, the income effect is strong at the ELB: current output $\hat{y}_t$ responds by proportionally more than an increase in $\mathbb{E}_t \hat{y}_{t+1}$. For high values of p such that $p\Lambda > 1$, the income effect dominates the substitution effect, leading to a positive inflation and output gap for sufficiently large negative values of ε^T. This implies the potential non-existence of a solution at the ELB.

From the discussion above, the non-existence of a solution in such models arises from the strong income effect at the ELB, as evident from Equation (10). One way to resolve this issue is to relax the assumption of rational expectations. In models with non-rational expectations, even when the income effect is strong, i.e., $\Lambda > 1$, the expectation $\mathbb{E}_t \hat{y}_{t+1}$ is dampened. As a result, despite the strong feedback mechanism, the impact on $\hat{y}_t$ is muted. This is investigated in detail in Ascari, Mavroeidis, and McClung (2023). Alternatively, we

14. Such feedback loop does not exist under perfect foresight as y_{t+1} is known with certainty, therefore, the standard three-equation NK model with ELB has a unique equilibrium under perfect foresight, see Holden (2023).

could introduce additional stabilising forces to reduce the feedback loop between $\mathbb{E}_t \hat{y}_{t+1}$ and $\hat{y}_t$, i.e., a smaller Λ. In what follows, we show that a simple fiscal policy rule can ensure an MSV solution.

3 Fiscal Policy and Existence and Uniqueness

In this section, we show how fiscal policy that consists of government spending financed through lump-sum taxes can restore uniqueness of MSV solution in a baseline NK model subject to the ELB.

Model. We augment the baseline NK model with a simple FP setup. The model is otherwise standard, and derivation is given in Appendix B. In what follows, we show that under simple fiscal feedback rules, the model can generate a unique MSV solution despite the ELB under certain restrictions on FP. The model is described by the DISE, NKPC, TR, and the natural interest rate given by:

$$\hat{x}_t = \mathbb{E}_t \hat{x}_{t+1} - \frac{s_c}{\sigma}(\hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1} - \hat{r}^n_t), \tag{12a}$$

$$\hat{\pi}_t = \beta \mathbb{E}_t \hat{\pi}_{t+1} + \kappa_y \hat{x}_t, \tag{12b}$$

$$\hat{i}_t = \max\left\{-\mu, \phi_\pi \hat{\pi}_t + \phi_y \hat{x}_t\right\}, \tag{12c}$$

$$\hat{r}^n_t = -\Gamma \mathbb{E}_t \Delta \hat{g}_{t+1} + \sigma \varepsilon_t, \tag{12d}$$

where $s_c \equiv C/Y$ is the steady-state consumption-output ratio, $s_g \equiv G/Y$ is the steady-state government expenditure-output ratio, $\kappa_y = \frac{(1-\gamma)(1-\gamma\beta)}{\gamma}(\varphi + \sigma/s_c)$ is the slope of the NKPC, $\Gamma \equiv (s_g \sigma \varphi)/(s_c \varphi + \sigma)$ and ε_t is a demand shock. Other parameters are defined in the same way as in Section 2.

Permanent fiscal policy change. The model in (12) is closed with a rule for government expenditure given by

$$\hat{g}_{t+1} - \hat{g}_t = \psi_\pi \hat{\pi}_t + \psi_y \hat{x}_t, \tag{13}$$

where ψ_π and ψ_y denote the sensitivity of the government spending growth rate to deviations of inflation and the output gap, respectively. Throughout this section, we assume that future government spending depends on the current realisations of endogenous variables such that the model is purely forward-looking. This assumption allows us to check if the model possesses a unique MSV solution analytically. We relax this assumption in Section 4.

Similar to Section 2, define $Y_t = \begin{bmatrix} \hat{\pi}_t & \hat{x}_t \end{bmatrix}^\top$, and $X_t = \begin{bmatrix} \varepsilon_t & \mu \end{bmatrix}^\top$, so the system can be written as the canonical form (2). The coefficient matrices can be found in Appendix B.3. For analytical tractability, we assume $\phi_y = \psi_y = 0$. Moreover, ε_t follows a two-state Markov chain process ($k = 2$) with states $\varepsilon_t = (\varepsilon^T, 0)$, with a transition kernel as in Equation (5).

We can write the piecewise linear function $F(\mathbf{Y})$, where the mappings follow the same formulation as in Equation (6). If all the determinants ($\det \mathcal{A}_J, J \subseteq \{1, \ldots, k\}$) have the same sign, then the model exists a unique MSV solution. Suppose the BK condition is satisfied, i.e., $\phi_\pi + \Gamma\psi_\pi > 1$, the results are summarised in Proposition 1.

Proposition 1. *When the Blanchard-Kahn condition is satisfied, the New Keynesian model with fiscal policy as defined in* (12) *has a unique minimum state variable solution if*

$$\Gamma\psi_\pi > \max\left\{1, \Phi^g_{p,q,\beta,\kappa} + \frac{\phi_\pi (1-q)\left[1 + \frac{\sigma(1-\beta p - \beta q + \beta)}{\kappa_y s_c}\right]}{\Gamma\psi_\pi + \phi_\pi - 1}\right\}. \tag{14}$$

Proof: *Appendix B.4.*

From Proposition 1, if the state $i = 2$ is absorbing, i.e., $q = 1$, the second term in the max operator simplifies to $\Phi^g_{p,q,\beta,\kappa}$, which is bounded from above by 1. We can characterise the existence of a solution in the special case in Corollary 1.

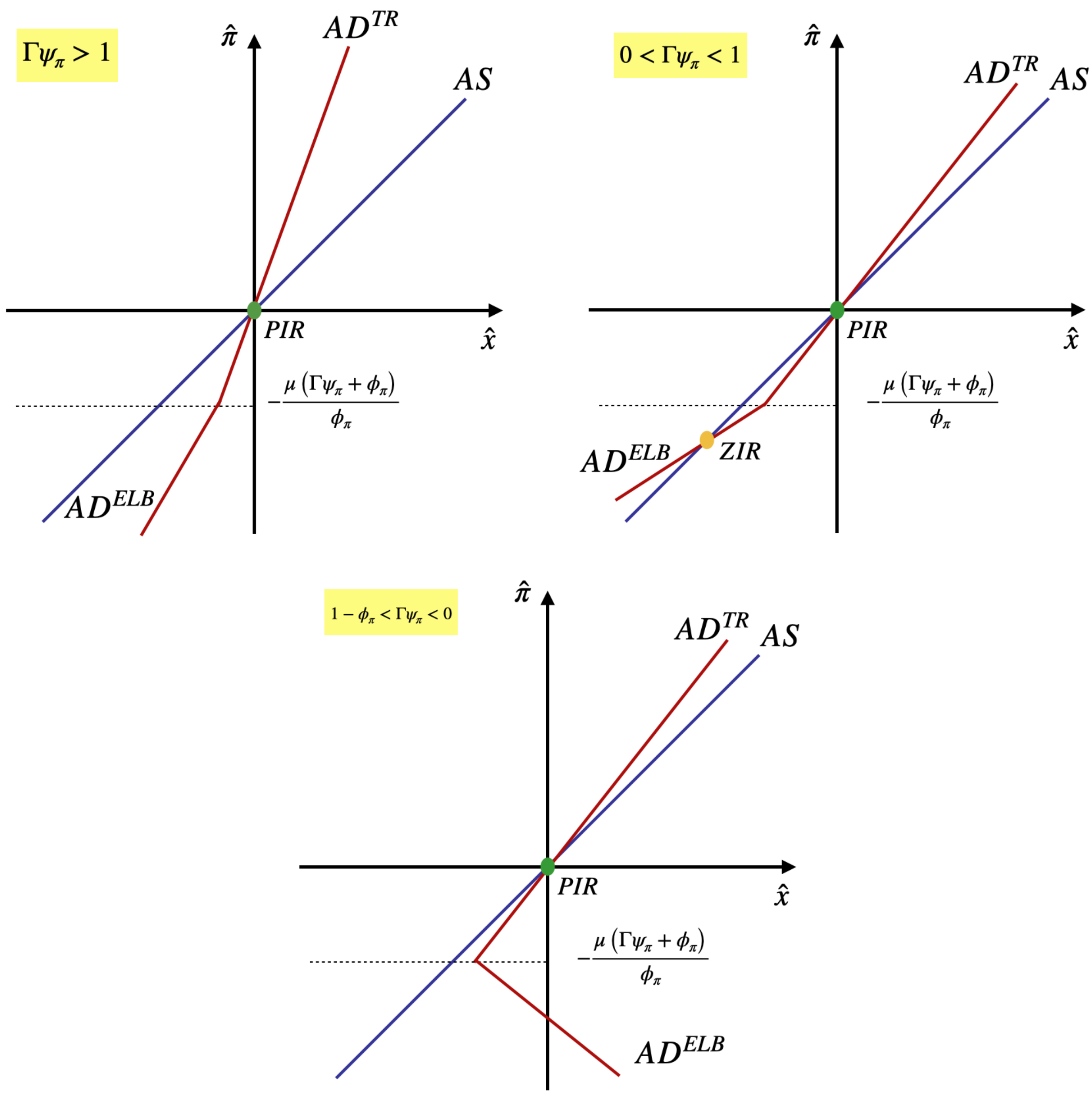
Corollary 1. *Suppose state $i = 2$ is absorbing ($q = 1$), the baseline New Keynesian model with fiscal policy as defined in* (12) *has a unique minimum state variable solution if*

$$\Gamma\psi_\pi > 1, \text{ where } \Gamma \equiv \frac{s_g \sigma \varphi}{(s_c \varphi + \sigma)}. \tag{15}$$

Here the parameter Γ captures how the natural rate responds to the expected growth in government spending (see Equation (12)). For large values of Γ, the restriction on ψ_π is relatively relaxed.

Similar to the baseline NK case, below we characterise the conditions for existence of an MSV solution in the special case for this model with FP where $p < 1$ (transitory state) and $q = 1$ (absorbing state), with the support of ε_t given by ε^T and 0, respectively.

Figure 2: Existence and Uniqueness with Simple Fiscal Rules (Absorbing State; $\varepsilon_t = 0$)



Note: Figure shows range for $\Gamma\psi_\pi$ for existence and uniqueness of a unique minimum state variable solution (left and bottom figures). Right panel shows a case of multiplicity.

Absorbing state. First, considering the absorbing state of the model, where $\varepsilon_t = 0$. In the absorbing state we have $\hat{\pi}_t = \hat{\pi}_{t+1} = \hat{\pi}$ and $\hat{x}_t = \hat{x}_{t+1} = \hat{x}$. Hence, we can write the following AS and AD relations:

$$\hat{\pi} = \frac{\kappa_y}{1-\beta}\hat{x} \quad AS, \tag{16a}$$

$$\hat{\pi} = \begin{cases} (\phi_\pi + \Gamma\psi_\pi)\frac{\kappa_y}{1-\beta}\hat{x} & AD^{TR}, \text{ for } \hat{\pi} > -\frac{\mu(\Gamma\psi_\pi+\phi_\pi)}{\phi_\pi}, \\ -\mu + \Gamma\psi_\pi\frac{\kappa_y}{1-\beta}\hat{x} & AD^{ELB}, \text{ for } \hat{\pi} \leq -\frac{\mu(\Gamma\psi_\pi+\phi_\pi)}{\phi_\pi}. \end{cases} \tag{16b}$$

We plot AS (16a) and AD (16b) in Figure 2. When the BK condition holds (i.e., $\phi_\pi + \Gamma\psi_\pi > 1$), the slope of AD^{TR} must be positive. Then E&U of the solution depends on the FP rule coefficient ψ_π. When ψ_π is positive, to ensure a unique intersection between AD and AS we require the slope of the AD^{ELB} to be steeper than the that of AS, as shown in the left panel of Figure 2. From Equations (16a) and (16b), this condition is satisfied if $\Gamma\psi_\pi > 1$. In this case, we end up with one unique PIR equilibrium, $\{\hat{\pi}, \hat{y}, \hat{i}\} = \{0, 0, 0\}$.[15] However, when ψ_π is positive but less than one, multiple equilibria arise, as illustrated in the right panel of Figure 2.

Note that the BK condition requires $\phi_\pi + \Gamma\psi_\pi > 1$ to hold, and consequently if $\phi_\pi > 1$, ψ_π can take on negative values while still satisfying the BK condition. In this case, the slope of AD^{ELB} becomes negative and AD and AS also intersects uniquely at a PIR equilibrium.[16] This is shown in the bottom panel of Figure 2.

To conclude, the E&U of an MSV solution in the absorbing state requires either $\Gamma\psi_\pi > 1$ or $1 - \phi_\pi < \Gamma\psi_\pi < 0$.[17] When $\Gamma\psi_\pi > 1$, the slope of AD^{ELB} is steeper than that of AS, ensuring a single intersection and a unique equilibrium. When $1 - \phi_\pi < \Gamma\psi_\pi < 0$, the slope of AD^{ELB} is negative, again leading to a unique equilibrium. In contrast, when $0 < \Gamma\psi_\pi < 1$, multiple equilibria emerge.

Transitory states. We proceed with analysing the transitory equilibria with $\varepsilon_t = \varepsilon^T$. As before, the economy remains in a transitory state with probability p, and with probability

15. For the unique solution to be the PIR equilibrium, we also require $-\mu \leq 0$, which holds if $(r\pi^*)^{-1} \leq 1$.
16. However, if $\phi_\pi \leq 1$, ψ_π cannot take on negative values and this case is ruled out.
17. The second condition exists only if $\phi_\pi > 1$.

$1-p$ jumps to the PIR absorbing state. AS and AD are given by:

$$\hat{\pi}^T = \frac{\kappa_y}{1-p\beta}\hat{x}^T \quad AS, \tag{17a}$$

$$\hat{\pi}^T = \begin{cases} \frac{\sigma(1-p)}{s_c(p-\phi_\pi-\Gamma\psi_\pi)}\hat{x}^T - \frac{\sigma}{(-\phi_\pi+p-\Gamma\psi_\pi)}\varepsilon^T & AD^{TR} \text{ for } \hat{\pi}^T \geq -\frac{\mu}{\phi_\pi}, \\ \frac{\sigma(1-p)}{s_c(p-\Gamma\psi_\pi)}\hat{x}^T - \frac{\mu}{(p-\Gamma\psi_\pi)} - \frac{\sigma}{(p-\Gamma\psi_\pi)}\varepsilon^T & AD^{ELB} \text{ for } \hat{\pi}^T \leq -\frac{\mu}{\phi_\pi}. \end{cases} \tag{17b}$$

First, if the BK condition holds, the slope of AD^{TR} is always negative. If the slope of AD^{ELB} is also negative (which implies $\Gamma\psi_\pi > p$), it should be flatter than AD^{TR}. In this case, there is only one intersection between the AS and AD curves, as shown in the left panel of Figure 3. If the slope of AD^{ELB} is instead positive, i.e., $\Gamma\psi_\pi < p$, then there is a unique equilibrium if the slope of AD^{ELB} is steeper than AS. Define $\tilde{\theta}$ as the ratio of the slope of AD^{ELB} to the slope of AS:

$$\tilde{\theta} \equiv \frac{\sigma(1-p)(1-\beta p)}{s_c\kappa_y(p-\Gamma\psi_\pi)}. \tag{18}$$

Then E&U of an MSV solution requires $\tilde{\theta} > 1$, which implies

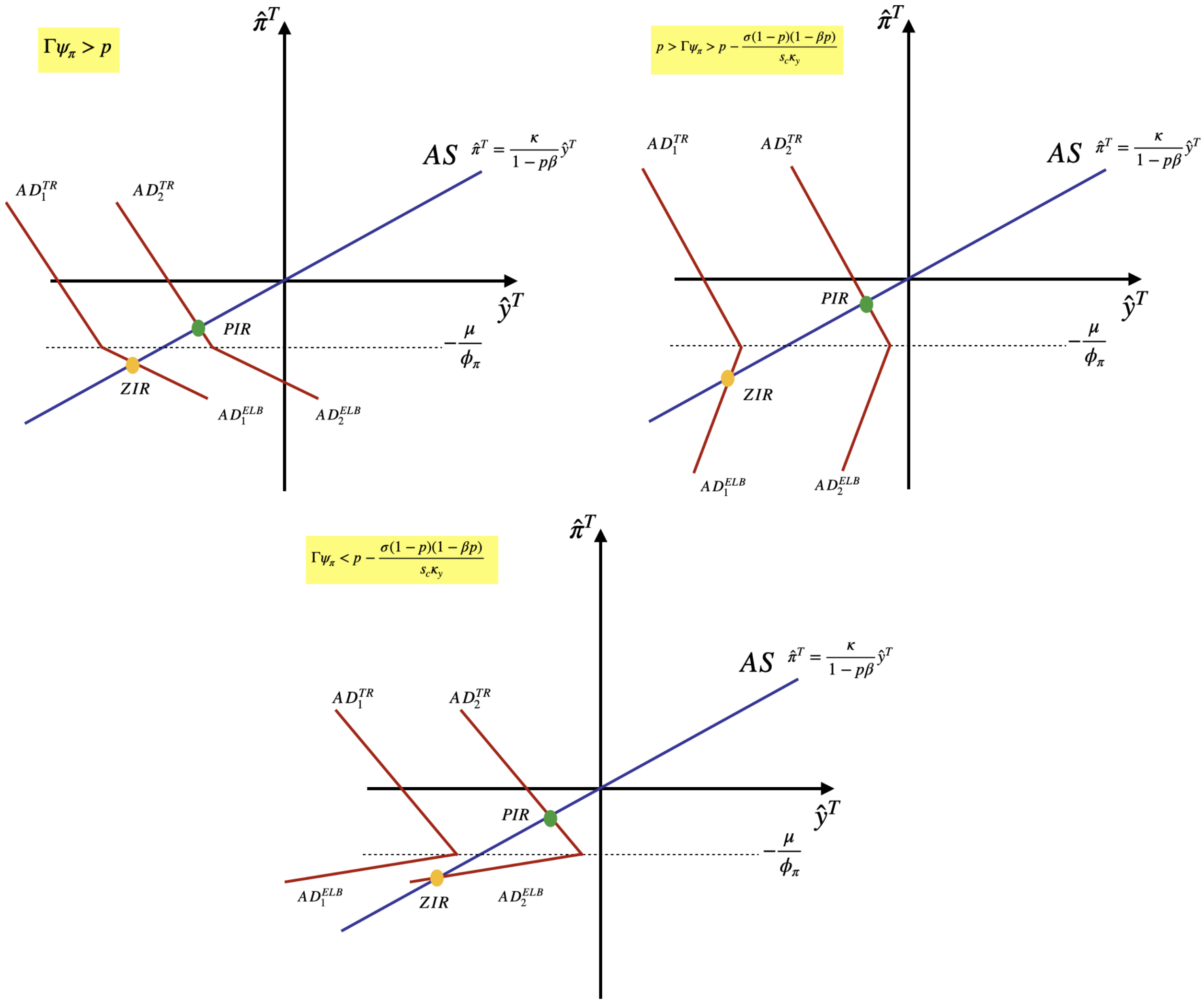
$$\Gamma\psi_\pi > \underbrace{p - \frac{\sigma(1-p)(1-\beta p)}{s_c\kappa_y}}_{\equiv \Phi^g_{p,q=1,\beta,\kappa_y}}. \tag{19}$$

This condition coincides with the second term in Equation (14) when $q = 1$. Note that condition (19) is less restrictive than $\Gamma\psi_\pi > p$. The solution is plotted in the right panel of Figure 3. For $\tilde{\theta} < 1$, where the slope of AD^{ELB} is flatter than AS, there is either no solution or multiple solutions as illustrated in the bottom panel of Figure 3.

To conclude, E&U of an MSV solution with an absorbing state requires either $\Gamma\psi_\pi > 1$ or $1-\phi_\pi < \Gamma\psi_\pi < 0$, and the E&U condition in a transitory state requires $\Gamma\psi_\pi > \Phi^g_{p,q=1,\beta,\kappa}$. The overlap between the two cases is the requirement of $\Gamma\psi_\pi > 1$, as outlined in Corollary 1.

Comparison to baseline NK model and the importance of a forward-looking rule. Thus far we have characterised the requirements for FP such that there exists a unique MSV solution when both $\phi_y = \psi_y = 0$. Before we extend to the special case to where ϕ_y and ψ_y are different from zero, we first discuss the intuition why such FP can guarantee a unique

Figure 3: Transitory States under Active and Passive Fiscal Policy



Note: Top row shows transitory state with a positive interest rate absorbing state with active fiscal policy. Top left panel shows procyclical fiscal policy. Top right shows countercyclical fiscal policy. Bottom panels shows passive procyclical fiscal policy regime. Passive fiscal policy in general implies non-existence of solution or two solutions as a special case. Active fiscal policy implies existence of unique solution.

solution in NK models with the ELB.

For the baseline NK model considered in Section 2, the E&U of an MSV solution requires θ as defined in Equation (8) to be greater than unity, i.e.,

$$\theta > 1.$$

This condition simply stated that AD^{ELB} has to be steeper than AS. However, as the slope of AD^{ELB} depends on exogenous uncertainty p, so too does the ratio θ. From the expression, for some values of p, the condition $\theta > 1$ does not hold. Therefore, the baseline NK model does not generally have a unique MSV solution.

Analogously, for the NK model with FP following (13), the relative slope between AD^{ELB} and AS was given in Equation (18). Then, E&U requires $\tilde{\theta}$ to satisfy the following conditions:

$$\text{either } \tilde{\theta} < 0, \text{ or } \tilde{\theta} > 1. \tag{20}$$

That is, AD^{ELB} can have a negative slope or have a positive slope and be steeper than AS. In contrast to the baseline NK model, the slope of AD^{ELB} in the NK model with FP also depends on the fiscal policy rule, and so too does $\tilde{\theta}$. With certain fiscal policy satisfying Proposition 1, the condition (20) always holds regardless the value of p. Thus, one can argue that if the effects of uncertainty p can be counteracted by FP, the model will have a unique MSV solution thus satisfying the E&U conditions. Our work is thus complimentary to the results of Nakata and Schmidt (2022), particularly the role of FP in ruling out sunspot equilibria.

The importance of persistence implied by (13) cannot be overstated and is a key point of this paper. To highlight this, consider the case where the fiscal targeting rule is given in deviations and not in growth rates, i.e., $\hat{g}_t = \psi'_\pi \hat{\pi}_t$. This will imply the following AD^{ELB} and AS slope ratio:

$$\hat{\theta} \equiv \frac{\sigma (1-p)(1-\beta p)}{s_c \kappa_y \left[p - \Gamma \psi'_\pi (p-1)\right]}.$$

To satisfy the E&U conditions, the value of $\hat{\theta}$ can be either negative or greater than unity. However, this does not hold for sufficiently high values of p. Therefore, the model with FP specified as $\hat{g}_t = \psi'_\pi \hat{\pi}_t$ does not generally satisfy conditions for E&U for an MSV solution.

The intuition for why a contemporaneous FP rule would fail lies in the fact that the strength of the FP would depend on the uncertainty parameter p. In particular, in the transitory state, when $\hat{g}_t = \psi'_\pi \hat{\pi}^T$, we have $\mathbb{E}_t \hat{g}_{t+1} = \psi'_\pi \mathbb{E}_t \hat{\pi}_{t+1} = \psi'_\pi p \hat{\pi}^T$. Therefore, the natural rate would depend on p,

$$r_t^n = \Gamma \psi'_\pi (1-p) \hat{\pi}.$$

Hence, the effect of FP depends on the value of p. And for large value of p, ψ'_π can be unbounded. For example, when $p = 1$, FP has no effect on the system. This highlights the

Table 1: Model Calibration

Parameter	Value	Description
σ	3	Coefficient of relative risk-aversion
φ	2	Frisch elasticity of labour supply
β	0.99	Discount factor
γ	3/4	Calvo probability
ϵ	10	Elasticity of substitution between goods
s_c	0.7	Fraction of consumption in output
s_g	1-s_c	Fraction of government spending in output
ϕ_π	1.5	Weight on inflation, Taylor rule
ϕ_y	0.2	Weight on output gap, Taylor rule
κ_y	$\frac{(1-\gamma)(1-\gamma\beta)}{\gamma}(\varphi + \sigma/s_c)$	Slope of NKPC

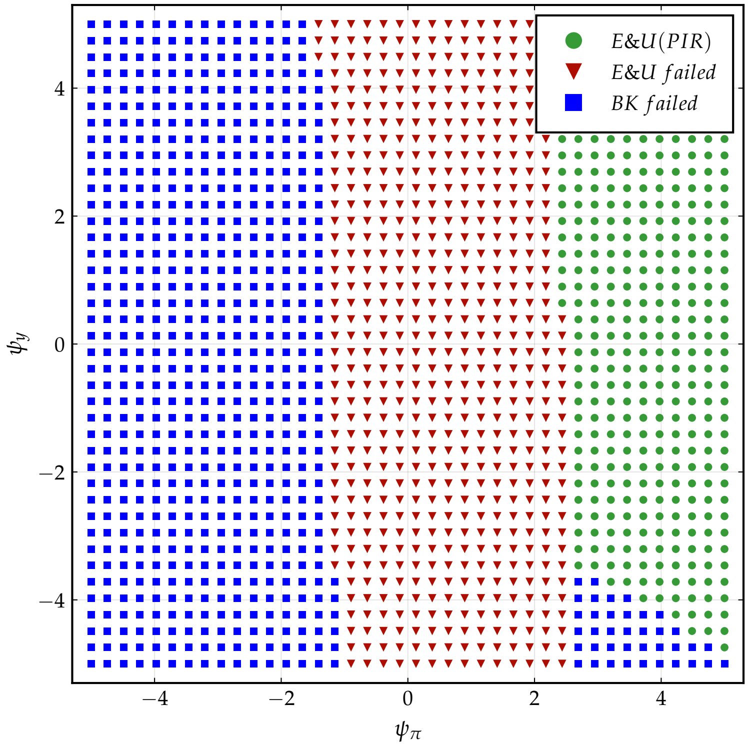
importance of commitment to future changes in policy that depend on contemporaneous deviations of endogenous variables as in, for example, (13).

Besides the aforementioned link with Nakata and Schmidt (2022), other approaches in the literature – such as price level targeting (PLT) in Holden (2023), implicit inflation target adjustment in Holden (2024), and unconventional monetary policy in AM and Ikeda et al. (2021) – rely on a similar mechanism to guarantee uniqueness. As argued in Holden (2023), PLT rules can restore uniqueness in the presence of an occasionally binding ELB constraint as such a policy implies a promise about future inflation given inflation today. If monetary policy is committed to a given price level path, the monetary authority promises that a period of low inflation today will be followed by a period of high inflation in the future. Thus, agents expecting high prices in the future increase their consumption in periods of low inflation and, by implication, the system has a unique solution around the PIR absorbing state. The commitment to higher inflation in the future delivers sufficient information about the expected dynamics of the system that alleviates uncertainty that would otherwise engender multiplicity and, by implication, pins down the unique solution similar to persistent fiscal policy.

Output gap targeting. Here we relax the assumption $\phi_y = \psi_y = 0$ and find the E&U conditions quantitatively. The model is calibrated according to the values in Table 1. These parameter values are standard in the NK DSGE literature.

The region for which the model satisfies the E&U conditions as a function of the fiscal

Figure 4: Existence and Uniqueness Region for Inflation and Output Gap Fiscal Rule (Equation (13))



Note: Green circles denote regions where the solution is unique and locally determinate. Red triangles denote region where the solution is either non-unique or non-existent. Blue squares denote regions where the solution is not locally determinate.

authority's reaction parameters, ψ_π and ψ_y, are shown in Figure 4. Using our baseline calibration, we see that the model generally satisfies the E&U conditions in the negative orthant of ψ_π and ψ_y space $\mathbb{R}^2_-$, and when ψ_π is sufficiently large. Mechanically, a strong enough reaction on the part of the fiscal authority to inflation and output deviations leads to a unique MSV solution by ensuring an intersection between AD and AS.[18] Furthermore, we note that the degree of reaction on the part of the fiscal authority to the output gap is largely irrelevant as to whether or not the model satisfies the E&U conditions. Moreover, the rule in Equation (13) nests the special case where FP can fully replicate monetary policy as considered in Correia et al. (2013) and Seidl and Seyrich (2023), who termed this as "unconventional fiscal policy". This is the case if FP activates at the ELB, and its feedback coefficients are set such that they exactly mirror the effects of the counterfactual

18. This is illustrated for a simple case in Figure 2.

unconstrained monetary policy. Further details and derivations of this special case are provided in Appendix B.5.

Existence and uniqueness with distortionary taxes. As shown above, if FP reacts to exogenous disturbances aggressively enough, the model satisfies the E&U conditions through standard aggregate demand channels. But satisfaction of E&U conditions is not exclusive to the fiscal setup we have discussed.

Consider the case similar to Correia et al. (2013) and Seidl and Seyrich (2023) where the fiscal authority levies consumption subsidy and wage taxes, τ^c and τ^w, respectively. The expression for the natural rate in the model in (12) becomes

$$\hat{r}_t^n = \sigma \varepsilon_t - \Psi^c \Delta \hat{\tau}_{t+1}^c, \tag{21}$$

where $\Delta \hat{\tau}_{t+1}^c$ is the consumption subsidy growth rate. If the fiscal authority sets $\Delta \hat{\tau}_{t+1}^c$ according to a rule in (13), a unique MSV solution can be achieved under similar parametric restrictions on feedback coefficients.[19]

4 New Keynesian Model with Less Persistent Fiscal Policy

In Section 3, we showed that a permanent change in government spending ensures the existence of a unique MSV solution conditional on satisfying a certain set of conditions. For analytical and computational tractability, the model was purely forward-looking. But in this section we augment the fiscal expenditure rule (13) so that the model contains an endogenous state variable. More precisely, we replace (13) with:

$$\hat{g}_{t+1} - g_t = \rho_g \left(g_t - g_{t-1}\right) + \left(1 - \rho_g\right)\left(\psi_\pi^* \hat{\pi}_t + \psi_y^* \hat{x}_t\right), \tag{22}$$

where $\rho_g \in (-1, 0]$ is a mean-reversion parameter, and where the rest of the model is as given in (12). Notably, if $\rho_g \neq 0$ the system has an endogenous state variable Δg_t, and if $\rho_g = 0$, the government spending nests the fiscal rule in Section 3. The persistence of government spending is given by $1 + \rho_g \leq 1$, which decreases as ρ_g becomes more negative.[20] In addition, the parameters $\psi_\pi \equiv \left(1 - \rho_g\right)\psi_\pi^*$ and $\psi_y \equiv \left(1 - \rho_g\right)\psi_y^*$ in (22)

19. See Appendix B.2 for model derivation details.

20. Government spending can be rewritten as $\hat{g}_{t+1} = (1 + \rho_g) g_t + \left(1 - \rho_g\right)\left(\psi_\pi^* \hat{\pi}_t + \psi_y^* \hat{x}_t\right)$ when $g_{t-1} = 0$.

capture the overall responsiveness of the growth of government spending $\Delta\hat{g}_{t+1}$ to current inflation $\hat{\pi}_t$ and the current output gap $\hat{x}_t$, respectively.

Define $Y_t = \begin{bmatrix} \hat{\pi}_t & \hat{x}_t & \mathbb{E}_t\Delta\hat{g}_{t+1} \end{bmatrix}^\top$ and $X_t = \begin{bmatrix} \varepsilon_t & \mu \end{bmatrix}^\top$, then we can write the model in canonical form,

$$\begin{aligned} \mathbf{0} &= \left(\boldsymbol{A}_{s_{t,i}} Y_t + \boldsymbol{B}_{s_{t,i}} Y_{t+1|t} + \boldsymbol{C}_{s_{t,i}} X_t + \boldsymbol{D}_{s_{t,i}} X_{t+1|t} + \boldsymbol{H}_{s_{t,i}} Y_{t-1} \right), \\ s_{t,i} &= \mathbb{1}\left(\left[\boldsymbol{a}^\top Y_t + \boldsymbol{b}^\top Y_{t+1|t} + \boldsymbol{c}^\top X_t + \boldsymbol{d}^\top X_{t+1|t} + \boldsymbol{h}^\top Y_{t-1} \right] > 0 \right), \end{aligned} \tag{23}$$

where the coefficient matrices given in Appendix C.1. We also assume as before that ε_t follows a 2-state Markov chain with transition kernel $\boldsymbol{K}$ $(k = 2)$.

With an endogenous state variable, the MSV solutions are of the form $Y_t = f(Y_{t-1}, X_t)$. The key difference compared to Section 3 is that the support of Y_t will vary endogenously over time through the evolution of Y_{t-1}, and thus, it can no longer be characterised by a constant matrix $\mathbf{Y}$. Hence, when there are endogenous state variables, we have

$$\mathbb{E}\left[Y_{t+1} | Y_t = \mathbf{Y}_t \boldsymbol{e}_i, X_t = \mathbf{X}\boldsymbol{e}_i\right] = \mathbf{Y}^i_{t+1} \boldsymbol{K}^\top \boldsymbol{e}_i,$$

where $\mathbf{Y}^i_{t+1} \in \mathbb{R}^{n\times k}$ gives the support of Y_{t+1} when Y_t is in the i-th state. Therefore, from this expression, we can see that the support of Y_t grows at 2^t for any given initial condition Y_0, thence, the MSV solution cannot be represented by any finite-dimensional system of piecewise-linear equations. This implies that we cannot check the E&U condition directly using the GLM Theorem.

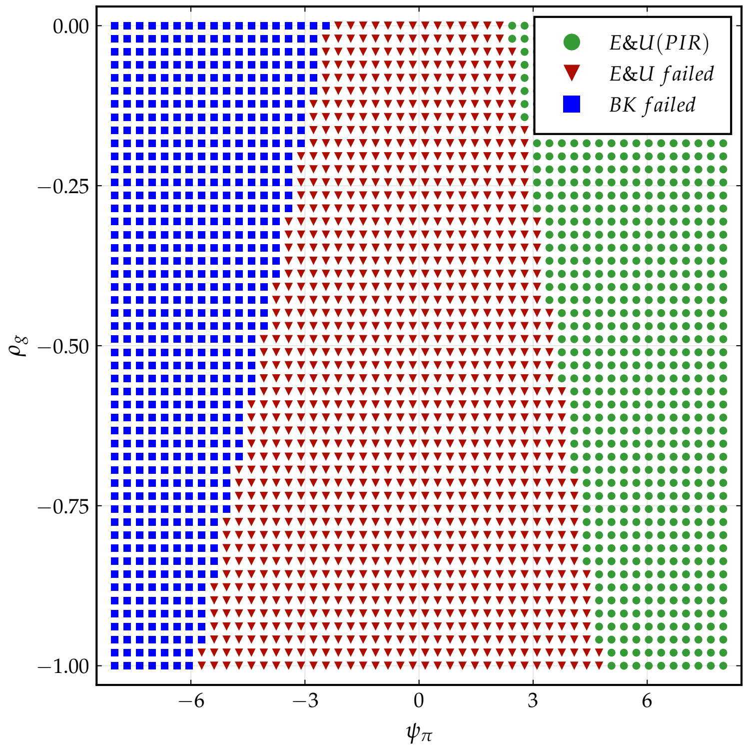
Since there is only one endogenous state variable in this model, for ease of notation, define $y_t \equiv g^\top \boldsymbol{Y}_t$, where $\boldsymbol{g} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\top$. Moreover, we can write $\boldsymbol{H}_{s_{t,i}} \boldsymbol{Y}_{t-1} = \boldsymbol{h}_{s_{t,i}} y_{t-1}$. Suppose that the support at time t can be represented in the form $\mathbf{Y}_t = \boldsymbol{G} y_{t-1} + \boldsymbol{Z}$, where matrix $\boldsymbol{G}$ captures how the support $\mathbf{Y}_t$ depends on the endogenous state y_{t-1}, and matrix $\boldsymbol{Z}$ represents the part of $\mathbf{Y}_t$ that depends on exogenous variable X_t.[21] The matrix $\boldsymbol{G}$ is an $n \times k$ matrix with each column being the coefficient of y_{t-1} for each different state of X_t.

With these definitions, we can rewrite the Equation (23) as

$$\begin{aligned} \mathbf{0} = &\left(\boldsymbol{A}_{s_{t,i}} \boldsymbol{G} \boldsymbol{e}_i + \boldsymbol{B}_{s_{t,i}} \boldsymbol{G} \boldsymbol{K}^\top \boldsymbol{e}_i \boldsymbol{g}^\top \boldsymbol{G} \boldsymbol{e}_i + \boldsymbol{h}_{s_{t,i}} \right) y_{t-1} \\ &+ \left(\boldsymbol{A}_{s_{t,i}} \boldsymbol{Z} + \boldsymbol{B}_{s_{t,i}} \boldsymbol{G} \boldsymbol{K}^\top \boldsymbol{e}_i \boldsymbol{g}^\top \boldsymbol{Z} + \boldsymbol{B}_{s_{t,i}} \boldsymbol{Z} \boldsymbol{K}^\top + \boldsymbol{C}_{s_{t,i}} \mathbf{X} + \boldsymbol{D}_{s_{t,i}} \mathbf{X} \boldsymbol{K}^\top \right) \boldsymbol{e}_i \end{aligned} \tag{24}$$

21. In the case where the system does not have endogenous states (such as in Section 2), we have $\boldsymbol{G} = 0$, and the support $\mathbf{Y}_t$ is thus time-invariant.

Figure 5: Existence and Uniqueness Region with Persistent Fiscal Rule



Note: Green circles denote the parameter space where the solution is unique and locally determinate. Red triangles denote the region where the solution is either non-unique or non-existent. Blue squares denote the region where the solution is locally indeterminate.

for all $i = 1, \ldots, k$. For each subset $J \subseteq \{1, ..., k\}$, we can solve for the corresponding $\boldsymbol{G}$ and $\boldsymbol{Z}$ using the method of undetermined coefficients. For example, for subset $J = \{1, 2\}$, the constraints are slack in both states (i.e., $s_{t,1} = s_{t,2} = 1$), we can solve for the corresponding $\boldsymbol{G}$ and $\boldsymbol{Z}$. However, the system is not piecewise linear in $\boldsymbol{G}$ and $\boldsymbol{Z}$, and so we cannot use the GLM theorem to check for E&U. In the fashion of AM, we have to check all possible 2^k regime configurations J. Moreover, since y_{t-1} is endogenous, we need to solve the system backwards from some terminal condition y_T, and then check E&U for all possible values of y_T.

We first present computational results for E&U conditions for the parameters ρ_g and ψ_π in (22). These are illustrated in Figure 5. As government spending becomes less persistent ($\rho_g \to -1$), a stronger contemporaneous fiscal response to inflation is required for E&U. As we decrease inertia in the government spending rule ($\rho_g \to 0$), E&U of a unique MSV solution is easier to satisfy (i.e., requires a lower value of ψ_π).

We now provide some intuition through the derivation of analytical results.[22] Suppose that the economy starts in a ZIR transitory state equilibrium where where $\varepsilon_t = \varepsilon^T$ and then converges to a PIR in the absorbing state where $\varepsilon_t = 0$. In other words, we wish to show the restrictions on FP needed for a ZIR equilibrium to exist in the temporary state, given that the agents expect to move to the stable manifold of the PIR equilibrium system as soon as the shock vanishes.

Absorbing state. We proceed as in Section 3 by first considering the absorbing state of the model, when $\varepsilon_t = 0$. We solve the system as in (16):

$$\hat{\pi} = \frac{\kappa_y}{1-\beta}\hat{x} \quad AS, \tag{25a}$$

$$\hat{\pi} = \begin{cases} \left(\phi_\pi + \frac{\Gamma\psi_\pi}{1-\rho_g}\right)\frac{\kappa_y}{1-\beta}\hat{x} & AD^{TR} \text{ for } \hat{\pi} > -\frac{\mu\left[\Gamma\psi_\pi + \left(1-\rho_g\right)\phi_\pi\right]}{\left(1-\rho_g\right)\phi_\pi}, \\ -\mu + \frac{\Gamma\psi_\pi}{1-\rho_g}\frac{\kappa_y}{1-\beta}\hat{x} & AD^{ELB} \text{ for } \hat{\pi} \le -\frac{\mu\left[\Gamma\psi_\pi + \left(1-\rho_g\right)\phi_\pi\right]}{\left(1-\rho_g\right)\phi_\pi}. \end{cases} \tag{25b}$$

If $\rho_g = 0$, the system (25) will have the same expression as in (16). Therefore, the graphical representation would be the similar to Figure 2, with slight differences in the slopes of the curves. When AD^{ELB} is steeper than AS or when AD^{ELB} is negative sloping, i.e., $\Gamma\psi_\pi > 1-\rho_g$ or $\Gamma\psi_\pi < 0$ we have a unique equilibrium, which corresponds to the PIR equilibrium, given by $\left(\hat{\pi}, \hat{x}, \hat{i}, \Delta\hat{g}\right) = (0,0,0,0)$. Whereas for $0 < \Gamma\psi_\pi < 1-\rho_g$, the system admits two equilibria, a PIR equilibrium and a ZIR equilibrium. The ZIR equilibrium is given by:

$$\left(\hat{\pi}, \hat{x}, \hat{i}, \Delta\hat{g}\right) = \left(-\frac{\left(1-\rho_g\right)\mu}{1-\rho_g-\Gamma\psi_\pi}, -\frac{\left(1-\rho_g\right)\left(1-\beta\right)\mu}{\left(1-\rho_g-\Gamma\psi_\pi\right)\kappa_y}, -\mu, -\frac{\left(1-\rho_g\right)\psi_\pi\mu}{1-\rho_g-\Gamma\psi_\pi}\right).$$

The transitory state admits endogenous dynamics because of the presence of the endogenous variable $\Delta\hat{g}$. Off of the steady states, the economy will travel along a stable trajectory that leads to the steady states. We then explore the conditions under which the following hold:

(i) When the shock disappears the economy will converge to the PIR along the stable manifold;

(ii) the solution is an MSV solution in the sense that it depends just on state variables;

22. As in Section 3, assume that $\phi_y = \psi_y = 0$, and ε_t follows a two-state Markov chain process ($k = 2$) with states $\varepsilon_t = (\varepsilon^T, 0)$ with a transition kernel $\boldsymbol{K}$ specified as Equation (5).

(iii) in the transitory state where $\varepsilon_t = \varepsilon^T$, the economy will be in a ZIR.

Under these assumptions, once the shock disappears then we must be on the unique stable manifold that leads to the PIR.[23] Assumption (i) is key as it pins down the expectations in the absorbing state, similar to the proof in Section 3. However, rather than jump to the PIR steady state as when the model is forward-looking, the system will arrive there inertially along the unique stable manifold. To find the MSV solution of the PIR system, we use the method of undetermined coefficients and assume a solution of this form:

$$\begin{aligned}
\hat{\pi}_t &= \gamma_\pi \Delta \hat{g}_t, \\
\hat{x}_t &= \gamma_x \Delta \hat{g}_t, \\
\Delta \hat{g}_{t+1} &= \gamma_g \Delta \hat{g}_t.
\end{aligned}$$

Substituting into the system (12) gives the following equation in γ_g:

$$\begin{aligned}
0 = & \left(1 - \beta\gamma_g\right)\left(\gamma_g - \rho_g\right)\left(\gamma_g - 1\right) \\
& - \frac{s_c \kappa_y}{\sigma}\left[\left(\gamma_g - \rho_g\right)\left(1 - \rho_g\right)\frac{\phi_\pi - \gamma_g}{\psi_\pi} + \Gamma\psi_\pi\Big(\rho_g + \gamma_g - \rho_g\Big)\right].
\end{aligned} \tag{26}$$

If there exists a unique solution within the unit circle, i.e., $|\gamma_g| < 1$, the dynamics along the stable trajectory are given by the recursions:

$$\begin{aligned}
\hat{\pi}_{t+j} &= \gamma_\pi \gamma_g^j \Delta \hat{g}_t, \\
\hat{x}_{t+j} &= \gamma_x \gamma_g^j \Delta \hat{g}_t, \\
\Delta \hat{g}_{t+j+1} &= \gamma_g^{j+1} \Delta \hat{g}_t.
\end{aligned}$$

If $\Delta \hat{g}_t = \nu_g$ then simply

$$\begin{aligned}
\hat{\pi}_t &= \gamma_\pi \nu_g, \\
\hat{x}_t &= \gamma_x \nu_g, \\
\hat{g}_{t+1} &= \gamma_g \nu_g.
\end{aligned}$$

Importantly, if $|\gamma_g| < 1$, the system will never be in a ZIR state when the shock vanishes,

23. Here, we focus solely on convergence to PIR steady state. This is motivated by the fact that the appearance of ZIR steady state necessarily implies the presence of multiple equilibria, which trivially undermines the uniqueness of the solution.

since $\hat{g}_{t+1} = \gamma_g \nu_g < \nu_g$. The value of $|\gamma_g|$ depends on the fiscal policy rule, ρ_g, and ψ_π. Holding other parameters constant, a more negative ρ_g requires higher a ψ_π to ensure that $|\gamma_g| < 1$.

Transitory state. Next, consider the ZIR transitory state. Assume that $\Delta\hat{g}_t = \nu_g$, to eliminate the endogenous state, allowing the system to be written as completely forward-looking. If an MSV solution exists, it will be constant (π^T, x^T) and with probability $(1-p)$ we are back on the manifold of the PIR absorbing state. The expectations are thus:

$$\mathbb{E}_t \hat{\pi}_{t+1} = p\hat{\pi}^T + (1-p)\,\gamma_\pi \nu_g,$$
$$\mathbb{E}_t \hat{x}_{t+1} = p\hat{x}^T + (1-p)\,\gamma_x \nu_g.$$

We can then write the ZIR transitory system as:

$$\hat{\pi}^T = \frac{\kappa_y}{(1-\beta p)}\hat{x}^T + \frac{\beta\,(1-p)\,\gamma_\pi \nu_g}{(1-\beta p)} \quad AS, \tag{27a}$$

$$\hat{\pi}^T = \begin{cases} \frac{\sigma(1-p)}{s_c\left(p-\phi_\pi-\Gamma\psi_\pi\right)}\hat{x}^T - \frac{\sigma(1-p)\gamma_x\nu_g}{s_c\left(p-\phi_\pi-\Gamma\psi_\pi\right)} - \frac{(1-p)\gamma_\pi\nu_g-\Gamma\rho_g\nu_g+\sigma\varepsilon^T}{p-\phi_\pi-\Gamma\psi_\pi} & AD^{TR} \text{ for } \hat{\pi}^T \geq -\frac{\mu}{\phi_\pi}, \\ \frac{\sigma(1-p)}{s_c(p-\Gamma\psi_\pi)}\hat{x}^T - \frac{\sigma(1-p)\gamma_x\nu_g}{s_c(p-\Gamma\psi_\pi)} - \frac{\mu+(1-p)\gamma_\pi\nu_g-\Gamma\rho_g\nu_g+\sigma\varepsilon^T}{p-\Gamma\psi_\pi} & AD^{ELB} \text{ for } \hat{\pi}^T \leq -\frac{\mu}{\phi_\pi}. \end{cases} \tag{27b}$$

We could plot this system and it would be similar to Figure 3, since the slopes are identical but different intercepts. The same reasoning therefore applies. For E&U, FP must satisfy the same condition specified in Equation (19). The results are summarised by Proposition 2.

Proposition 2. *When the Blanchard-Kahn condition is satisfied, the New Keynesian model with fiscal policy as defined in (22) has a unique minimum state variable solution such that the economy is in a zero interest rate transitory state when $\varepsilon_t = \varepsilon^T$ and converges to a positive interest rate absorbing state $(q = 1)$ when $\varepsilon_t = 0$ if*

$$\Gamma\psi_\pi > \max\left\{1-\rho_g, p - \frac{\sigma\,(1-p)\,(1-\beta p)}{s_c \kappa_y}\right\}, \tag{28}$$

and where $|\gamma_g| < 1$ depends on the fiscal policy as specified in Equation (26).

Proposition 2 nests Proposition 1 by setting $\rho_g = 0$. From Proposition 2, inertia in the government spending rule affects the solution in two ways. First, for Equation (28) to hold, a larger value of ψ_π is required as $\rho_g \leq 0$ becomes more negative. That is, if government spending is less persistent (i.e., ρ_g is more negative), then to ensure uniqueness we require a larger fiscal response to current inflation (i.e., a higher ψ_π) than we would otherwise. When $\rho_g = 0$, we are back to the solution in Section 3, i.e., $\Gamma\psi_\pi > 1$.

Second, to ensure that the system transitions to a PIR equilibrium when the shock vanishes, we require that $|\gamma_g| < 1$. Importantly, the value of $|\gamma_g|$ depends on the fiscal policy rule parameters ρ_g and ψ_π. Fixing other parameter values as in Table 1 and setting $\phi_y = \psi_y = 0$, a more negative ρ_g increases the absolute value of γ_g, implying that more inertial government spending reduces the speed at which the economy transitions to the steady state.[24] On the other hand, a more aggressive response to inflation reduces the absolute value of γ_g and pushes the economy back to the steady state more quickly. Therefore, if government spending exhibits greater inertia (a very negative ρ_g), a larger response in $\Delta\mathbb{E}_t g_{t+1}$ to current inflation (a very high ψ_π) is required. For example, the following two combinations both yield $|\gamma_g| = -0.1$: $(\rho_g, \psi_\pi) = (-0.1, 1.4)$ and $(\rho_g, \psi_\pi) = (-0.2, 25.9)$.

Note that by setting $\rho_g = 0$, we obtain $\gamma_g = 0$. Consequently, in the next period we have $\left(\hat{\pi}, \hat{x}, \hat{i}, \Delta\hat{g}\right) = (0, 0, 0, 0)$. This is because when $\rho_g = 0$ the system no longer has endogenous state variables, therefore, it immediately jumps to the PIR equilibrium rather than arriving there inertially along the unique stable manifold.[25]

5 Conclusion

This paper explores how fiscal policy can restore equilibrium uniqueness in a baseline New Keynesian model subject to an occasionally binding effective lower bound constraint on the interest rate. Our findings suggest that simple fiscal policy can guarantee a unique solution that is also locally determinate. We establish that in order to guarantee a minimum state variable solution and local determinacy, fiscal policy needs to be sufficiently persistent and aggressive in its response to inflation.

First, we analytically verified that if the fiscal authority is able to credibly commit to a

24. If government spending is very persistent (for example, if $\rho_g = -1$), then $|\gamma_g|$ might exceed unity.

25. These results also hold for an inertial rule for government spending in levels,

$$\hat{g}_t = \rho_g \hat{g}_{t-1} + (1 - \rho_g)(\psi_\pi^* \hat{\pi}_t + \psi_y^* \hat{x}_t), \quad \rho_g \in [0, 1].$$

sufficiently strong countercyclical permanent policy change in response to an exogenous disturbance, solution uniqueness in the model is restored. This conclusion is rationalised by the fact that fiscal policy is not constrained by the effective lower bound and provides an active policy response when monetary policy is constrained. Moreover, by committing to a permanent policy change, the fiscal authority is able to alleviate the fundamental uncertainty that engenders multiplicity of equilibria in the baseline New Keynesian model.

Second, we find that the fiscal response need not imply a permanent policy change but rather it has to be sufficiently persistent to guarantee existence and uniqueness of a minimum state variable solution. The persistence property of the policy rule, coupled with it being sufficiently countercyclical, are needed to eliminate belief-driven equilibria and pin down a unique solution. By showing this, we address the main concerns raised by Ascari and Mavroeidis (2022) about New Keynesian models featuring occasionally binding constraints.

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A Verification of MSV Solution Existence and Uniqueness

A.1 Canonical New Keynesian Model

A.1.1 Coefficient Matrices in Baseline NK Model

When the constraint on $\hat{i}_t$ is binding, the system can be rewritten as follows

$$\underbrace{\begin{bmatrix} 1 & -\kappa \\ 0 & 1 \end{bmatrix}}_{\equiv A_0} \begin{bmatrix} \pi_t \\ y_t \end{bmatrix} + \underbrace{\begin{bmatrix} -\beta & 0 \\ -\sigma^{-1} & -1 \end{bmatrix}}_{\equiv B_0} \begin{bmatrix} \mathbb{E}_t \pi_{t+1} \\ \mathbb{E}_t y_{t+1} \end{bmatrix} + \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & -\sigma^{-1} \end{bmatrix}}_{\equiv C_0} \begin{bmatrix} u_t \\ \varepsilon_t \\ \mu \end{bmatrix} = 0 \tag{A1}$$

Whilst when the constraint is slack the system is given by

$$\underbrace{\begin{bmatrix} 1 & -\kappa \\ \sigma^{-1}\phi_\pi & 1 + \sigma^{-1}\phi_y \end{bmatrix}}_{\equiv A_1} \begin{bmatrix} \pi_t \\ y_t \end{bmatrix} + \underbrace{\begin{bmatrix} -\beta & 0 \\ -\sigma^{-1} & -1 \end{bmatrix}}_{\equiv B_1} \begin{bmatrix} \mathbb{E}_t \pi_{t+1} \\ \mathbb{E}_t y_{t+1} \end{bmatrix} + \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}}_{\equiv C_1} \begin{bmatrix} u_t \\ \varepsilon_t \\ \mu \end{bmatrix} = 0 \tag{A2}$$

A.1.2 Analytical Derivation of E&U Conditions

In this section, we use the GLM theorem to test the coherence and completeness in the canonical NK model. In the calculation we assume $\phi_y = 0$ and $u_t = 0$. ε_t follows a two-state Markov Chain process as defined in Equation (5). Given that we have two endogenous variables and two states. The $\mathbf{Y}$ is a 4×1 vector containing the values of $(\hat{\pi}_t, \hat{y}_t)$ in each of the two states. And $F(\cdot)$ is the piecewise linear function, with the mapping defined in the Section 2. The matrix-form of these mappings are given by

$$\mathcal{A}_{J_1} = \begin{bmatrix} A_1 + B_1 p & B_1(1-p) \\ B_1(1-q) & A_1 + B_1 q \end{bmatrix}, \quad J_1 = \{1, 2\}$$
$$\mathcal{A}_{J_2} = \begin{bmatrix} A_0 + B_0 p & B_0(1-p) \\ B_1(1-q) & A_1 + B_1 q \end{bmatrix}, \quad J_2 = \{2\}$$
$$\mathcal{A}_{J_3} = \begin{bmatrix} A_1 + B_1 p & B_1(1-p) \\ B_0(1-q) & A_0 + B_0 q \end{bmatrix}, \quad J_3 = \{1\}$$
$$\mathcal{A}_{J_4} = \begin{bmatrix} A_0 + B_0 p & B_0(1-p) \\ B_0(1-q) & A_0 + B_0 q \end{bmatrix}, \quad J_4 = \emptyset$$

We can solve for the determinant of each $\mathcal{A}_{J_1}$ to $\mathcal{A}_{J_4}$ and check if they have the same sign. The derivation of the determinants is quite complex; therefore, we do not provide the detailed derivation in the Appendix but can share it upon request. Solve gives

$$
\begin{aligned}
\det \boldsymbol{\mathcal{A}}_{J_1} &= \kappa^2\sigma^{-2}\left[\phi_\pi - 1\right]\left\{\phi_\pi - \Phi\right\} \\
\det \boldsymbol{\mathcal{A}}_{J_2} &= \kappa\sigma^{-2}\left(-\left(\phi_\pi - 1\right)\kappa\Phi - \phi_\pi\left(1-q\right)\kappa\left[1 + \frac{1-(p+q-1)\beta}{\kappa}\sigma\right]\right) \\
\det \boldsymbol{\mathcal{A}}_{J_3} &= -\sigma^{-2}\kappa^2\left(\phi_\pi - \Phi\right) - \sigma^{-1}\left(1-q\right)\left[-\kappa\phi_\pi\left(1 + \sigma^{-1}\kappa + \beta\left(1-p-q\right)\right)\right] \\
\det \boldsymbol{\mathcal{A}}_{J_4} &= \kappa^2\sigma^{-2}\Phi
\end{aligned}
$$

where

$$
\Phi_{p,q,\beta,\sigma,\kappa} \equiv p + q - 1 - \frac{\left[1 - p\beta - q\beta + \beta\right]\left(2-p-q\right)\sigma}{\kappa}
$$

note that $\Phi_{p,q,\beta,\sigma,\kappa} < 1$ as the fraction is always positive and the highest value $p + q - 1$ can achieve is 1. Therefore, under an active monetary policy rule $\phi_\pi > 1$, $\det \boldsymbol{\mathcal{A}}_{J_1} > 0$. The GLM theorem requires that all the other determinants to have the same sign. $\det \boldsymbol{\mathcal{A}}_{J_4} > 0$ requires $\Phi_{p,q,\beta,\sigma,\kappa} > 0$. However, in that case, $\det \boldsymbol{\mathcal{A}}_{J_2}$ would be negative, which violates the E&U condition in the GLM theorem. Therefore, a unique MSV solution solution may not exist under an active Taylor rule ($\phi_\pi > 1$) without restrictions on p and q.[26]

A.1.3 Proof of unboundedness of $\lambda(\mathbf{X})$

The canonical form is given by

$$
\begin{aligned}
\mathbf{0} &= \left(\boldsymbol{A}_{s_i}\mathbf{Y} + \boldsymbol{B}_{s_i}\mathbf{Y}\boldsymbol{K}^\top + \boldsymbol{C}_{s_i}\mathbf{X} + \boldsymbol{D}_{s_i}\mathbf{X}\boldsymbol{K}^\top\right)\boldsymbol{e}_i, \\
s_i &= \mathbb{1}\left(\left[\boldsymbol{a}^\top\mathbf{Y} + \boldsymbol{b}^\top\mathbf{Y}\boldsymbol{K}^\top + \boldsymbol{c}^\top\mathbf{X} + \boldsymbol{d}^\top\mathbf{X}\boldsymbol{K}^\top\right]\boldsymbol{e}_i > 0\right), \quad i = 1, \dots, k.
\end{aligned}
$$

The canonical form can be represented as $F(\mathbf{Y}) = \lambda(\mathbf{X})$. Below, we show that $\lambda(\mathbf{X}) = -\left(\boldsymbol{C}_{s_i}\mathbf{X} + \boldsymbol{D}_{s_i}\mathbf{X}\boldsymbol{K}^\top\right)\boldsymbol{e}_i$ is unbounded.

Assume without loss of generality that $\mathbf{X}$ is a matrix the columns of which pertain to states $1, 2, \dots, k$ and rows pertaining to m shocks and l constants

$$
\mathbf{X} = \begin{bmatrix}
\varepsilon_{1,1} & \dots & \varepsilon_{1,k} \\
\varepsilon_{2,1} & \dots & \varepsilon_{2,k} \\
 & \ddots & \\
\varepsilon_{m,1} & \dots & \varepsilon_{m,k} \\
\varrho_{1,1} & \dots & \varrho_{1,k} \\
 & \ddots & \\
\varrho_{l,1} & \dots & \varrho_{l,k}
\end{bmatrix}.
$$

26. See Appendix A.1 for a derivation. The assumption $\phi_y = 0$ is imposed for simplification, but the results can be generalised to the case $\phi_y \neq 0$: see the discussion in Ascari and Mavroeidis (2022).

Trivially, $\mathbf{X}$ can be partitioned into two matrices $\mathbf{X}^s$, that includes the first m rows of $\mathbf{X}$, and $\mathbf{X}^c$, that includes the last l rows of $\mathbf{X}$:$\mathbf{X} = \mathbf{X}^s + \mathbf{X}^c$ with

$$\mathbf{X}^s = \begin{bmatrix} \mathbf{X}_{1:m} \\ \mathbf{0}_{l\times k} \end{bmatrix} \quad , \quad \mathbf{X}^c = \begin{bmatrix} \mathbf{0}_{l\times k} \\ \mathbf{X}_{m+1:m+l} \end{bmatrix}.$$

The function $\lambda(\mathbf{X})$ is generally piecewise-linear. In the special case, where the first m columns of $\boldsymbol{C}_{s_i}$ and $\boldsymbol{D}_{s_i}$ pertaining to $\mathbf{X}^s$ are invariant with respect to s_i, the function $\lambda(\mathbf{X}^s + \mathbf{X}^c)$ is linear in $\mathbf{X}^s$, i.e.

$$\boldsymbol{C}_{s_i} = \begin{bmatrix} \boldsymbol{C}^s & \boldsymbol{C}^c_{s_i} \end{bmatrix} \quad , \quad \boldsymbol{D}_{s_i} = \begin{bmatrix} \boldsymbol{D}^s & \boldsymbol{D}^c_{s_i} \end{bmatrix}.$$

This is the case for all the models considered in this paper.

This is an intuitive result as we do not assume that the coefficient on shocks depend upon whether a given constraint is binding.

$\lambda(\mathbf{X})$ is then unbounded as long as $\boldsymbol{C}^s$ and $\boldsymbol{D}^s$ are not null.

Application to baseline NK model. We illustrate the unboundedness of $\lambda(\mathbf{X})$ in the case of the model described in A.1.1. The function is given by

$$\lambda(\mathbf{X}) = -\boldsymbol{C}_{s_i}\boldsymbol{X}\boldsymbol{e}_i = \begin{cases} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} u_1 & u_2 \\ \varepsilon_1 & \varepsilon_2 \\ \mu & \mu \end{bmatrix} \boldsymbol{e}_i, & \text{if} \quad s_i = 1 \\ \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & -\sigma^{-1} \end{bmatrix} \begin{bmatrix} u_1 & u_2 \\ \varepsilon_1 & \varepsilon_2 \\ \mu & \mu \end{bmatrix} \boldsymbol{e}_i, & \text{if} \quad s_i = 0 \end{cases} \tag{A3}$$

Observe that the coefficients on shocks u_i and ε_i do not depend on i, i.e.

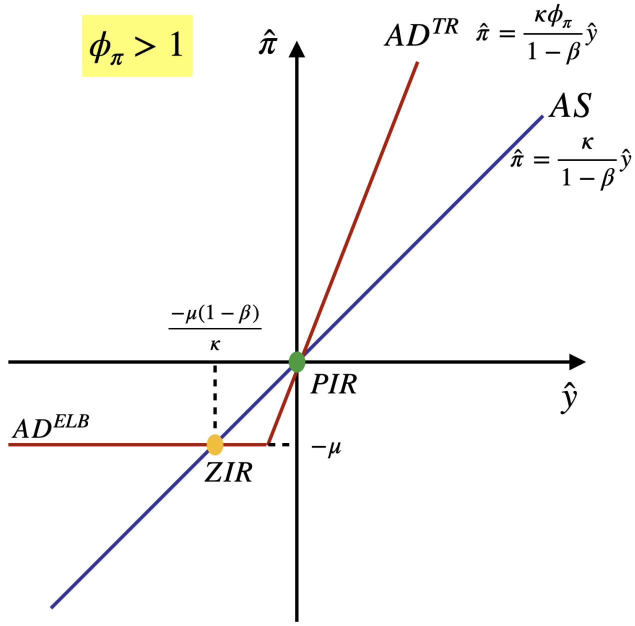
$$\boldsymbol{C}_{s_i} = \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}}_{\boldsymbol{C}^s} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sigma^{-1} s_i \end{bmatrix}. \tag{A4}$$

Thus, in the case of a three-equation NK model, $\lambda(\boldsymbol{X})$ is linear and trivially unbounded.

A.1.4 Derivation for Graphical Representation

First, consider the absorbing state of the model, where $\varepsilon_t = 0$. In the absorbing state we have $\hat{\pi}_t = \hat{\pi}_{t+1} = \hat{\pi}$ and $\hat{y}_t = \hat{y}_{t+1} = \hat{y}$. Hence, the NKPC can be written as the following

Figure 6: Absorbing State of the New Keynesian Model ($\varepsilon_t = 0$)



Note: Existence of two steady states in a standard NK model when the Taylor principle is satisfied.

aggregate supply (AS) relation:

$$\hat{\pi} = \frac{\kappa}{1-\beta}\hat{y} \quad AS. \tag{A5}$$

Meanwhile, the DISE can be written and rearranged to give a piecewise aggregate demand (AD) relation:

$$\hat{\pi} = \max\begin{cases}\frac{\kappa\phi_\pi}{1-\beta}\hat{y} & AD^{TR}, \\ -\mu & AD^{ELB}.\end{cases} \tag{A6}$$

We plot AS (A5) and AD (A6) in Figure 6. The figure shows the non-uniqueness problem when the NK model features an active TR. It is clear that the necessary support restriction for existence of a solution is $\mu \geq 0$, i.e., $(r\pi^*)^{-1} < 1$.[27] And when $\mu \geq 0$, the model admits two absorbing states: a positive interest rate (PIR) equilibrium, $\{\hat{\pi}, \hat{y}, \hat{i}\} = \{0, 0, 0\}$, and a zero interest rate (ZIR) equilibrium, $\{\hat{\pi}, \hat{y}, \hat{i}\} = \{-\mu, -\frac{\mu(1-\beta)}{\kappa}, -\mu\}$.

PIR absorbing state. At time t the economy is in a transitory state. With probability p the economy remains in the transitory state, and with $(1-p)$ the economy moves to the

27. As in the figure, when $\mu < 0$, there is no intersection between the AD and AS curve.

PIR absorbing state where $\{\hat{\pi}, \hat{y}, \hat{i}\} = \{0, 0, 0\}$. From (4b), we can write:

$$\hat{\pi}^T = \kappa \hat{y}^T + p\beta\hat{\pi}^T,$$

where the second term on the RHS comes from the fact that in period $t+1$ you may remain in a transitory state where $\hat{\pi} \neq 0$, and with probability $1-p$ you jump to the PIR absorbing state where $\hat{\pi} = 0$ Thus, AS is:

$$\hat{\pi}^T = \frac{\kappa}{1 - p\beta}\hat{y}^T. \tag{A7}$$

For AD, begin by writing the DISE as:

$$\hat{y}^T = p\hat{y}^T - \frac{1}{\sigma}(\hat{i} - p\hat{\pi}^T) + \varepsilon^T.$$

Rearrange and substitute in (4c) and $\varepsilon = \varepsilon^T$ to get AD:

$$\hat{\pi}^T = \begin{cases} \frac{\sigma(1-p)}{(p-\phi_\pi)}\hat{y}^T - \frac{\sigma}{(p-\phi_\pi)}\varepsilon^T & AD^{TR} \text{ for } \hat{\pi}^T \geq -\frac{\mu}{\phi_\pi}, \\ \frac{\sigma(1-p)}{p}\hat{y}^T - \frac{\mu}{p} - \frac{\sigma}{p}\varepsilon^T & AD^{ELB} \text{ for } \hat{\pi}^T \leq -\frac{\mu}{\phi_\pi}. \end{cases} \tag{A8}$$

ZIR absorbing state. Here in period t the economy is in a transitory state. With probably p the economy can remain in a transitory state, and with $(1-p)$ it can move to a ZIR absorbing state where $\{\hat{\pi}, \hat{y}, \hat{i}\} = \{-\mu, -\frac{\mu(1-\beta)}{\kappa}, -\mu\}$. Therefore, from (4b), AS can be written as:

$$\begin{aligned} \hat{\pi}^T &= \beta[p\hat{\pi}^T + (1-p)(-\mu)] + \kappa\hat{y}^T \\ &= \frac{\kappa}{1-p\beta}\hat{y}^T - \frac{\beta(1-p)}{1-p\beta}\mu. \end{aligned} \tag{A9}$$

To find AD, first begin by writing the DISE as:

$$\hat{y}^T = \left[p\hat{y}^T + (1-p)\left(\frac{-\mu(1-\beta)}{\kappa}\right)\right] - \frac{1}{\sigma}\left[\hat{i} - \left(p\hat{\pi}^T + (1-p)(-\mu)\right)\right] + \varepsilon^T,$$

then substitute in (4c) and the ε to get AD:

$$\hat{\pi}^T = \begin{cases} \frac{\sigma(1-p)}{(p-\phi_\pi)}\hat{y}^T + \frac{(1-p)}{(p-\phi_\pi)}\left[\frac{(1-\beta)\sigma}{\kappa} + 1\right]\mu - \frac{\sigma}{(p-\phi_\pi)}\varepsilon^T & AD^{TR} \text{ for } \pi^T \geq -\frac{\mu}{\phi_\pi}, \\ \frac{\sigma(1-p)}{p}\hat{y}^T + \frac{(1-p)}{p}\left[\frac{(1-\beta)\sigma}{\kappa} + 1\right]\mu - \frac{\mu}{p} - \frac{\sigma}{p}\varepsilon^T & AD^{ELB} \text{ for } \hat{\pi}^T \leq -\frac{\mu}{\phi_\pi}. \end{cases} \tag{A10}$$

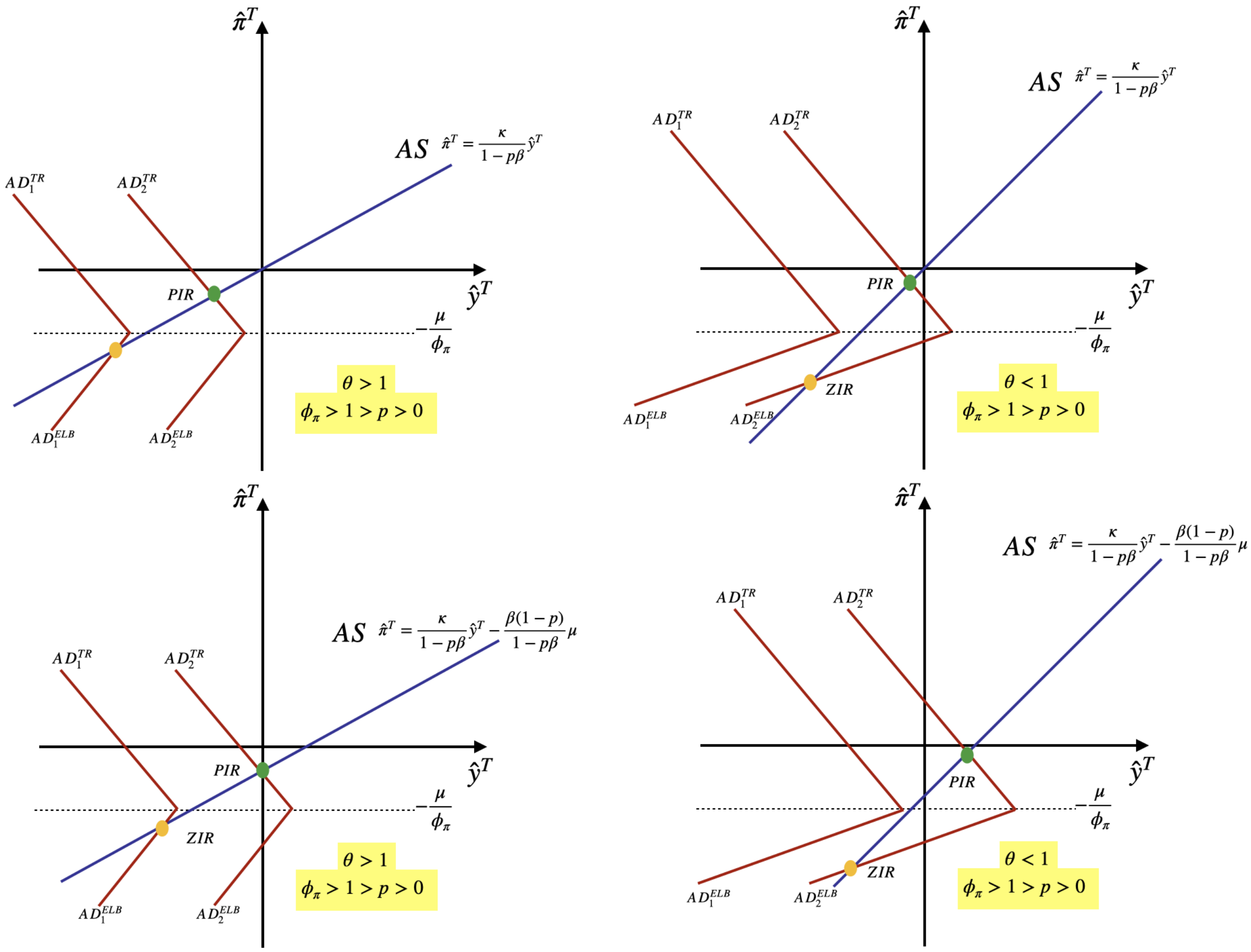


Figure 7: Transitory States of the New Keynesian Model ($\phi_\pi > 1$)

To find θ simply divide the slop of AD^{TR} by the slope of AS:

$$\theta \equiv \frac{slope_{AD-ELB}}{slope_{AS}} = \frac{\sigma(1-p)(1-\beta p)}{p\kappa}$$

We plot AD and AS when the economy is in the transitory state and with PIR and ZIR absorbing states for $\theta > 1$ and $\theta < 1$ in Figure 7. Under ZIR and similar to the PIR case in the main text, if $\theta < 1$ there are either multiple solutions if shocks are sufficiently small or no solutions for shocks that are sufficiently large.

A.2 Models with an Endogenous State

We no longer assume that $\boldsymbol{H}_{s_t} = \boldsymbol{O}$ and $\boldsymbol{h} = \boldsymbol{0}$ in the canonical form (23), but maintain the assumption that $\boldsymbol{X}_t$ follows a k-state stationary Markov process. This implies that, as before, the i-th column of $\mathbf{X}$ gives the value of $\boldsymbol{X}_t$ for a given state i. However, as stipulated by AM, with endogenous states the support of $\boldsymbol{Y}_t$ will vary endogenously over time along the MSV solution given by $\boldsymbol{Y}_t = f(\boldsymbol{Y}_{t-1}, \boldsymbol{X}_t)$. This implies that the solution can no longer be characterised by a time invariant matrix $\mathbf{Y}$. In other words, despite the variables $\boldsymbol{X}_t$ being time invariant (by definition as they are purely forward looking), the support of $\boldsymbol{Y}_t$ must now be a function of $\boldsymbol{Y}_{t-1}$, too. With endogenous states, along an MSV solution we have:

$$\mathbb{E}_t\left[\boldsymbol{Y}_{t+1}|\boldsymbol{Y}_t = \mathbf{Y}_t\boldsymbol{e}_i, \boldsymbol{X}_t = \mathbf{X}\boldsymbol{e}_i\right] = \mathbf{Y}^i_{t+1}\boldsymbol{K}^\top\boldsymbol{e}_i,$$

Starting from terminal date, T, the model solution is:

$$\mathbf{Y}_T = \boldsymbol{G}_{J_0}y_{T-1} + \boldsymbol{Z}_{J_0}, \tag{A11}$$

where $\boldsymbol{G}_{J_0}$ and $\boldsymbol{Z}_{J_0}$ can be solved from (24):

$$\mathbf{0} = \boldsymbol{A}_{s_{t,i}}\boldsymbol{G}\boldsymbol{e}_i + \boldsymbol{h}_{s_{t,i}} + \boldsymbol{B}_{s_{t,i}}\boldsymbol{G}\boldsymbol{K}^\top\boldsymbol{e}_i\boldsymbol{g}^\top\boldsymbol{G}\boldsymbol{e}_i, \tag{A12}$$
$$\mathbf{0} = \left(\boldsymbol{A}_{s_{t,1}}\boldsymbol{Z} + \boldsymbol{B}_{s_{t,i}}\boldsymbol{G}\boldsymbol{K}^\top\boldsymbol{e}_i\boldsymbol{g}^\top\boldsymbol{Z} + \boldsymbol{B}_{s_{t,i}}\boldsymbol{Z}\boldsymbol{K}^\top + \boldsymbol{C}_{s_{t,i}}\mathbf{X} + \boldsymbol{D}_{s_{t,i}}\mathbf{X}\boldsymbol{K}^\top\right)\boldsymbol{e}_i, \tag{A13}$$

$\forall i = 1, \dots, k$.

$\mathbf{Y}_T$ is a function of $\boldsymbol{G}_{J_0}$ and $\boldsymbol{Z}_{J_0}$, which are both treated as known.[28] Thus, $\mathbf{Y}_T$ is known and we can solve for $\mathbf{Y}_{T-1}$ from

$$\begin{aligned}\mathbf{0} = &\left(\boldsymbol{A}_{s_{T-1},i} + \boldsymbol{B}_{s_{T-1},i}\boldsymbol{G}_{J_0}\boldsymbol{K}^\top\boldsymbol{e}_i\boldsymbol{g}^\top\mathbf{Y}_{T-1}\boldsymbol{e}_i\right)\\ &+ \left(\boldsymbol{B}_{s_{T-1},i}\boldsymbol{Z}_{J_0}\boldsymbol{K}^\top + \boldsymbol{C}_{s_{T-1},i}\mathbf{X} + \boldsymbol{D}_{s_{T-1},i}\mathbf{X}\boldsymbol{K}^\top\right)\boldsymbol{e}_i + \boldsymbol{h}_{s_{T-1},i}y_{T-2}.\end{aligned}$$

For every $t \leq T$ the determinants relevant for E&U conditions are given by

$$|\mathcal{A}_{J_0J_1}| = \prod_i^k \det\left(\boldsymbol{A}_{s_{T-1},i} + \boldsymbol{B}_{s_{T-1},i}\boldsymbol{G}_{J_0}\boldsymbol{K}^\top\boldsymbol{e}_i\boldsymbol{g}^\top\right).$$

If $k = 2$, the determinants can be rewritten as

$$\begin{aligned}
|\mathcal{A}_{J_0J_1}| &= \det\left(A_1 + \boldsymbol{B}_1\boldsymbol{G}_{J_0}\boldsymbol{K}^\top\boldsymbol{e}_1\boldsymbol{g}^\top\right)\det\left(A_1 + \boldsymbol{B}_1\boldsymbol{G}_{J_0}\boldsymbol{K}^\top\boldsymbol{e}_2\boldsymbol{g}^\top\right), \quad J_1 = \{1,2\}\ (\text{PIR,PIR}),\\
|\mathcal{A}_{J_0J_1}| &= \det\left(A_0 + \boldsymbol{B}_0\boldsymbol{G}_{J_0}\boldsymbol{K}^\top\boldsymbol{e}_1\boldsymbol{g}^\top\right)\det\left(A_1 + \boldsymbol{B}_1\boldsymbol{G}_{J_0}\boldsymbol{K}^\top\boldsymbol{e}_2\boldsymbol{g}^\top\right), \quad J_1 = \{2\}\ (\text{ZIR,PIR}),\\
|\mathcal{A}_{J_0J_1}| &= \det\left(A_1 + \boldsymbol{B}_1\boldsymbol{G}_{J_0}\boldsymbol{K}^\top\boldsymbol{e}_1\boldsymbol{g}^\top\right)\det\left(A_0 + \boldsymbol{B}_0\boldsymbol{G}_{J_0}\boldsymbol{K}^\top\boldsymbol{e}_2\boldsymbol{g}^\top\right), \quad J_1 = \{1\}\ (\text{PIR,ZIR}),\\
|\mathcal{A}_{J_0J_1}| &= \det\left(A_0 + \boldsymbol{B}_0\boldsymbol{G}_{J_0}\boldsymbol{K}^\top\boldsymbol{e}_1\boldsymbol{g}^\top\right)\det\left(A_0 + \boldsymbol{B}_0\boldsymbol{G}_{J_0}\boldsymbol{K}^\top\boldsymbol{e}_2\boldsymbol{g}^\top\right), \quad J_1 = \{\emptyset\}\ (\text{ZIR,ZIR}).
\end{aligned} \tag{A14}$$

If the model has a unique MSV solution, use (A11) with (A12) and (A13), to solve for

28. In practice $\boldsymbol{G}_{J_0}$ and $\boldsymbol{Z}_{J_0}$ are precalculated as they are not time-varying per-se but are state dependent. For example, if J_0 always corresponds to the PIR case, then the ELB is never binding and $\boldsymbol{G}_{J_0}$ and $\boldsymbol{Z}_{J_0}$ can easily be obtained from the model policy function (Blanchard and Kahn, 1980).

$\mathbf{Y}_{T-1}$ as a function of y_{T-2}:

$$\begin{aligned}\mathbf{Y}_{T-1}\boldsymbol{e}_i = &-\left(\boldsymbol{A}_{s_{T-1},i} + \boldsymbol{B}_{s_{T-1},i}\boldsymbol{G}_{J_0}\boldsymbol{K}^\top \boldsymbol{e}_i \boldsymbol{g}^\top\right)^{-1} \\ &\left[\left(\boldsymbol{B}_{s_{T-1},i}\boldsymbol{Z}_{J_0}\boldsymbol{K}^\top + \boldsymbol{C}_{s_{T-1},i}\mathbf{X} + \boldsymbol{D}_{s_{T-1},i}\mathbf{X}\boldsymbol{K}^\top\right)\boldsymbol{e}_i + \boldsymbol{h}_{s_{T-1},i}y_{T-2}\right],\end{aligned}$$

$\forall i = 1, \ldots, k$.

B A New Keynesian Model with Fiscal Policy

B.1 The model setup

Households. The economy is populated with households indexed with i on a continuum of measure one. The households gain utility from consumption, dislike labour, and have access to one-period risk free bonds. The optimisation problem of the households is thus:

$$\max_{\{C_t, L_t, B_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\varphi}}{1+\varphi} \right) Z_t,$$

subject to the [nominal] period budget constraint given by

$$(1-\tau_t^c) P_t C_t + B_t = (1-\tau_t^w) W_t L_t + R_{t-1} B_{t-1} + P_t T_t,$$

where C_t is consumption, L_t is labour supply, B_t denotes bonds, R_t is nominal interest rate, P_t is the price level, τ_t^c is the consumption subsidy, τ_t^w is the wage tax rate, and T_t are lump-sum taxes.

The consumption bundle C_t consists of a continuum of differentiated goods, and is bundled by a CES aggregator of the form:

$$C_t = \left[\int_0^1 C_t(j)^{\frac{\epsilon-1}{\epsilon}} \, dj \right]^{\frac{\epsilon}{\epsilon-1}}.$$

The utility maximisation problem of the household results in the following intertemporal Euler equation:

$$\beta \mathbb{E}_t \frac{R_t}{\pi_{t+1}} \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{Z_{t+1}}{Z_t} = \mathbb{E}_t \frac{1-\tau_{t+1}^c}{1-\tau_t^c}.$$

The labour supply condition gives the following intratemporal Euler equation:

$$\frac{1-\tau_t^w}{1-\tau_t^c} w_t C_t^{-\sigma} = L_t^{\varphi}.$$

The intratemporal household problem of choosing a consumption bundle results in the following demand for good j:

$$C_t(j) = \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} C_t.$$

Production. Producers use labour as an input to produce differentiated consumption goods according to the following production technology:

$$Y_t(j) = L_t(j).$$

The price-setting problem of an individual firm j follows Rotemberg (1982) where firm j maximises the discounted value of profits,

$$\max_{\{P_t(i)\}} \mathbb{E}_t \sum_{T=t}^{\infty} Q_{t,T} \left[P_t(j) Y_{t,T}(j) - w_T L_T(j) - \frac{\Phi}{2} \left(\frac{P_{t,T}(j)}{P_{t-1,T}(j)} - 1 \right)^2 Y_{t,T} \right],$$

subject to:

$$Y_{t,T}(j) = \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t,$$

where Φ denotes a price adjustment cost parameter for the firms.[29] $Y_{t,T}(j)$ denotes demand at time T conditional on the price unchanged since period t. The firm maximises infinite discounted stream of profits, with revenues given by the first term and costs given by the second term. Households own firms, thus their revenues are discounted with the households' discount factor, $Q_{t,T}$:

$$Q_{t,T} = \beta \frac{P_t}{P_T} \left(\frac{C_T}{C_t} \right)^{-\sigma} \frac{Z_T}{Z_t}.$$

The solution to the firm problem results in the following equation for inflation:[30]

$$\pi_t(\pi_t - 1) = \frac{1}{\kappa} [\epsilon mc_t + 1 - \epsilon] + \mathbb{E}_t \left[Q_{t,t+1}(\pi_{t+1} - 1)\pi_{t+1} \frac{Y_{t+1}}{Y_t} \right].$$

Monetary authority. The monetary authority uses the [gross] nominal interest rate, R_t, as its policy instrument and sets it according to a TR of the form:

$$\frac{R_t}{\bar{R}} = \max \left\{ 1, \left(\frac{\pi_t}{\pi^*} \right)^{\phi_\pi} \left(\frac{Y_t}{Y_t^n} \right)^{\phi_y} \right\},$$

where ϕ_π and ϕ_y is the degree of reaction to contemporaneous inflation and the output deviations from natural level, respectively.

Fiscal authority. The real flow budget constraint for the government is

$$\tau_t^w w_t L_t + T_t = G_t + \tau_t^c C_t. \tag{B1}$$

29. We calibrate Φ to the following:

$$\Phi = \frac{\epsilon \gamma}{(1-\gamma)(1-\beta\gamma)},$$

where γ is the probability of firm j being unable to optimally adjust its price in any given period as in a model with Calvo (1983) pricing.

30. Gross inflation is defined as $\pi_t = P_t / P_{t-1}$

Market clearing. Markets clear, hence all output is consumed or used for government expenditure,

$$Y_t = C_t + G_t + \frac{\Phi}{2}(\pi_t - 1)^2 Y_t.$$

B.2 Log Linearised Equilibrium Conditions

Log linearising the non-linear model equations about a non-inflation deterministic steady state yields the following:

Intertemporal Euler equation:[31]

$$\hat{c}_t = \mathbb{E}_t \hat{c}_{t+1} - \frac{1}{\sigma}\left(\hat{i}_t + \mathbb{E}_t\left[\varepsilon_t + \Psi^c \Delta\hat{\tau}^c_{t+1} - \hat{\pi}_{t+1}\right]\right), \tag{B2}$$

where $\varepsilon_t \equiv \Delta\hat{z}_{t+1}$ is preference shock.

Labour supply condition:[32]

$$\hat{w}_t = \sigma\hat{c}_t + \varphi\hat{l}_t + \Psi^w \hat{\tau}^w_t - \Psi^c \hat{\tau}^c_t. \tag{B3}$$

Output:

$$\hat{y}_t = \hat{l}_t. \tag{B4}$$

Inflation:

$$\hat{\pi}_t = \frac{1}{\Phi}\epsilon\hat{mc}_t + \beta\mathbb{E}_t\hat{\pi}_{t+1}. \tag{B5}$$

Marginal cost:

$$\hat{mc}_t = \hat{w}_t. \tag{B6}$$

Taylor rule:

$$i_t = \max\left\{-\mu, \phi_\pi\hat{\pi}_t + \phi_y\hat{y}_t\right\}. \tag{B7}$$

Government budget constraint

$$s_g\hat{g}_t + \tau^c s_c(\hat{\tau}^c_t + \hat{c}_t) = \frac{T}{Y}\hat{t}_t + \tau^w\frac{wL}{Y}(\hat{\tau}^w_t + \hat{w}_t + \hat{l}_t).$$

Aggregate resource constraint:

$$\hat{y}_t = s_c\hat{c}_t + s_g\hat{g}_t. \tag{B8}$$

31. We define $\Psi^c = \frac{\bar{\tau}^c}{1-\bar{\tau}^c}$ and $\Delta\hat{\tau}^c_{t+1} = \hat{\tau}^c_{t+1} - \hat{\tau}^c_t$.
32. We define $\Psi^w = \frac{\bar{\tau}^w}{1-\bar{\tau}^w}$.

Natural level of output. The natural level of output is attained when $\hat{mc}_t = 0$. Thence, combining (B8), (B6), and (B3) yields

$$\hat{y}_t^n = \frac{s_c}{\sigma + \varphi s_c}\left(\sigma \frac{s_g}{s_c}\hat{g}_t - \Psi^w \hat{\tau}_t^w + \Psi^c \hat{\tau}_t^c\right)$$

Aggregate supply. Combining (B5), (B3), (B4), and (B8) yields

$$\hat{\pi}_t = \underbrace{\frac{\epsilon(\sigma + \varphi s_c)}{\Phi s_c}}_{\kappa_y}\hat{x}_t + \beta \mathbb{E}_t \hat{\pi}_{t+1} \tag{B9}$$

Aggregate demand. Combining (B2) and (B8) yields

$$\hat{x}_t = \mathbb{E}_t \hat{x}_{t+1} - \frac{s_c}{\sigma}(\hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1} - \hat{r}_t^n), \tag{B10}$$

where the natural rate is given by

$$\hat{r}_t^n = -\Gamma \Delta \hat{g}_{t+1} + \sigma \varepsilon_t - \Psi^c \Delta \hat{\tau}_{t+1}^c + \frac{\sigma}{\sigma + \varphi s_c}\left(\Psi^c \Delta \hat{\tau}_{t+1}^c - \Psi^w \Delta \hat{\tau}_{t+1}^w\right) \tag{B11}$$

Absent of distortionary taxes, i.e. $\hat{\tau}_t^c = \hat{\tau}_t^w = 0$, the expression for $\hat{r}_t^n$ collapses to Equation (12d).

Under $\hat{\tau}_t^c = \hat{\tau}_t^w$ and $\hat{g}_t = 0$, the expression collapses to Equation (21).

B.3 Coefficient Matrices

The coefficients in the canonical representation of the model are:

$$\boldsymbol{A}_0 = \begin{bmatrix} 1 & -\kappa \\ s_c\sigma^{-1}\Gamma\psi_\pi & 1 + s_c\sigma^{-1}\Gamma\psi_y \end{bmatrix}, \quad \boldsymbol{A}_1 = \begin{bmatrix} 1 & -\kappa \\ s_c\sigma^{-1}\left(\phi_\pi + \Gamma\psi_\pi\right) & 1 + s_c\sigma^{-1}\left(\phi_y + \Gamma\psi_y\right) \end{bmatrix},$$

$$\boldsymbol{B}_0 = \boldsymbol{B}_1 = \begin{bmatrix} -\beta & 0 \\ -s_c\sigma^{-1} & -1 \end{bmatrix}, \quad \boldsymbol{C}_0 = \begin{bmatrix} 0 & 0 \\ -s_c & -s_c\sigma^{-1} \end{bmatrix}, \quad \boldsymbol{C}_1 = \begin{bmatrix} 0 & 0 \\ -s_c & 0 \end{bmatrix}, \quad \boldsymbol{D}_0 = \boldsymbol{D}_1 = 0_{2\times 2}$$

$$\boldsymbol{a} = \left[\phi_\pi, \phi_y\right]^\top, \boldsymbol{c} = \left[0, 1\right]\top, \quad \boldsymbol{b} = 0_{2\times 1}, \quad \boldsymbol{d} = 0_{2\times 1}.$$

B.4 Proof of Proposition 1

We can write the piecewise linear function $F(\mathbf{Y})$, where the mappings follow the same formulation as in Equation (6). If all the determinants $(\det \mathcal{A}_J, J \subset 1, \ldots, k)$ have the same sign, then the model exists a unique MSV solution. Solving the determinants gives

$$\begin{aligned}
\det A_{J_1} &= \left(\frac{\kappa s_c}{\sigma}\right)^2 \left(\Gamma\psi_\pi + \phi_\pi - 1\right)\left(\Gamma\psi_\pi + \phi_\pi - \Phi^g_{p,q,\beta,\kappa}\right), \\
\det A_{J_2} &= \left(\frac{\kappa s_c}{\sigma}\right)^2 \left(\Gamma\psi_\pi + \phi_\pi - 1\right)\left(\Gamma\psi_\pi - \Phi^g_{p,q,\beta,\kappa}\right) \\
&\quad - \frac{\kappa s_c}{\sigma^2}\phi_\pi (1-q)\left[\kappa s_c + \sigma(1-\beta p - \beta q + \beta)\right], \\
\det A_{J_3} &= \left(\frac{\kappa s_c}{\sigma}\right)^2 \left(\Gamma\psi_\pi - 1\right)\left(\Gamma\psi_\pi + \phi_\pi - \Phi^g_{p,q,\beta,\kappa}\right) \\
&\quad + \left(\frac{\kappa s_c}{\sigma}\right)^2 (1-q)\left[\frac{\phi_\pi \sigma(1-\beta p - \beta q + \beta)}{\kappa s_c} + \phi_\pi\right], \\
\det A_{J_4} &= \left(\frac{\kappa s_c}{\sigma}\right)^2 \left(\Gamma\psi_\pi - 1\right)\left(\Gamma\psi_\pi - \Phi^g_{p,q,\beta,\kappa}\right),
\end{aligned}$$

with

$$\Phi^g_{p,q,\beta,\kappa} \equiv p + q - 1 - \frac{(2-p-q)\left[1-\beta p - \beta q + \beta\right]\sigma}{\kappa s_c}.$$

The coefficient $\Phi^g_{p,q,\beta,\kappa}$ cannot be greater than 1 as $0 \le p, q \le 1$.

Suppose the local determinacy is satisfied (BK condition), i.e.,

$$\phi_\pi + \Gamma\psi_\pi > 1 \tag{B12}$$

When Equation (B12) is satisfied, the determinant of $\mathcal{A}_{J_1}$ is always positive as $\Phi^g_{p,q,\beta,\kappa} \le 1$. Therefore, equilibrium uniqueness requires determinant of $\mathcal{A}_{J_2}$, $\mathcal{A}_{J_3}$ and $\mathcal{A}_{J_4}$ are positive.

$\det \mathcal{A}_{J_2} > 0$ requires that

$$\Gamma\psi_\pi - \Phi^g_{p,q,\beta,\kappa} > \frac{\phi_\pi (1-q)\left[1 + \frac{\sigma(1-\beta p - \beta q + \beta)}{\kappa s_c}\right]}{\Gamma\psi_\pi + \phi_\pi - 1}, \tag{B13}$$

where the term on the RHS is always positive, from which we can conclude that $\Gamma\psi_\pi - \Phi^g_{p,q,\beta,\kappa} > 0$.

$\det \mathcal{A}_{J_4} > 0$ requires $\Gamma\psi_\pi - 1$ and $\Gamma\psi_\pi - \Phi^g_{p,q,\beta,\kappa}$ to have the same sign. From Equation (B13), $\Gamma\psi_\pi - \Phi^g_{p,q,\beta,\kappa} > 0$, therefore, E&U requires

$$(\Gamma\psi_\pi - 1) > 0. \tag{B14}$$

For $\det \mathcal{A}_{J_3} > 0$, we require

$$\left(\Gamma\psi_\pi - 1\right)\left(\Gamma\psi_\pi + \phi_\pi - \Phi^g_{p,q,\beta,\kappa}\right) > -(1-q)\left[\frac{\phi_\pi \sigma(1-\beta p - \beta q + \beta)}{\kappa s_c} + \phi_\pi\right] \tag{B15}$$

This is always positive when Equation (B14) holds.

Therefore, for all determinants to be positive, we require both Equation (B13) and (B14) to hold, which is

$$\Gamma\psi_\pi > \max\left\{1, \Phi^g_{p,q,\beta,\kappa} + \frac{\phi_\pi(1-q)\left[1+\frac{\sigma(1-\beta p-\beta q+\beta)}{\kappa s_c}\right]}{\Gamma\psi_\pi+\phi_\pi-1}\right\}$$

B.5 The Unconventional Fiscal Policy Case

Consider the special case of the government spending rule in Equation (13): Fiscal policy activates only when monetary policy is constrained following

$$\mathbb{E}_t\Delta\hat{g}_{t+1} = \mathbb{1}\left(\hat{i}_t = -\mu\right)(\psi^u_\pi\hat{\pi}_t + \psi^u_y\hat{x}_t), \tag{B16}$$

where ψ^u_π and ψ^u_y denote the coefficients of reaction to inflation and the output gap, respectively.

The presence of the FP instrument in the DISE allows the piecewise linear system to satisfy the E&U conditions, despite the presence of the ELB constraint on $\hat{i}_t$ and an active TR. The instrument $\mathbb{E}_t\Delta\hat{g}_{t+1}$ has the same effect in the NK model as the monetary policy instrument and, hence, it can be set to render the DISE in (12a) linear. The E&U conditions are satisfied so long as:

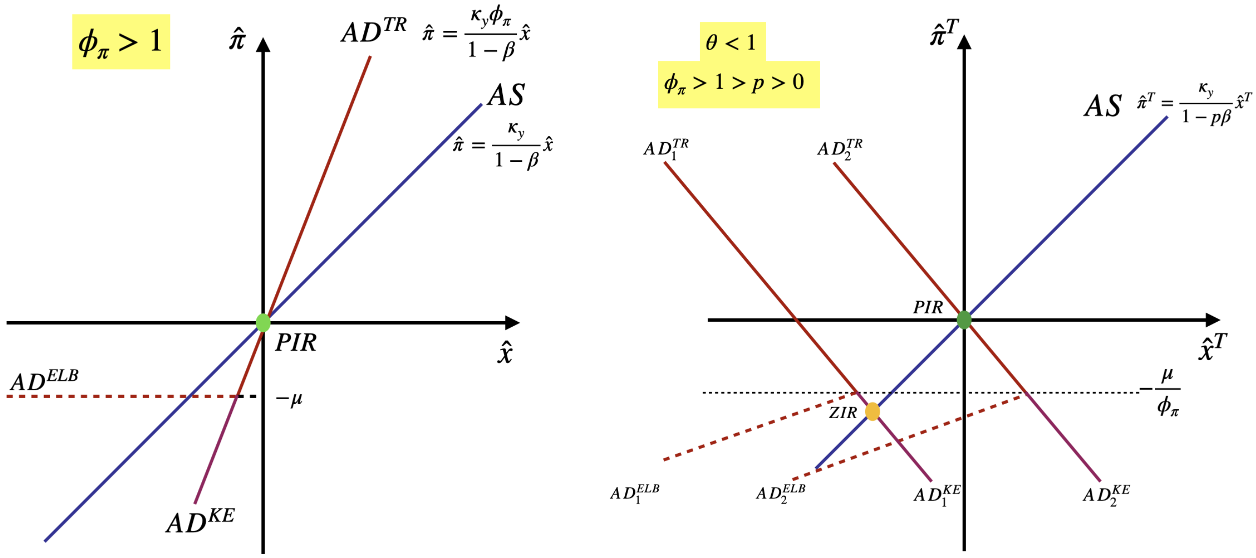
$$\psi^u_\pi = \Gamma^{-1}\phi_\pi, \quad \psi^u_y = \Gamma^{-1}\phi_y, \tag{B17}$$

which also allows (12c) to follow an active TR ($\phi_\pi > 1$). It is straightforward to see that since the model is now linear, it implies a unique MSV solution. The rule embeds the mechanism of the simple model in Correia et al. (2013) and Seidl and Seyrich (2023), which showed that a set of tax instruments can replicate monetary policy when the interest rate subject to the ELB constraint.

Uniqueness of an MSV solution in this case is illustrated in Figure 8 for the special case where $\phi_y = \psi^u_y = 0$. We plot AD and AS for both the absorbing (steady state) case where $\varepsilon_t = 0$ (Figure 8, Left) and the transitory state with a PIR absorbing state (Figure 8, Right).

In the absence of active FP, the AD curve is illustrated, as before, with a piecewise red line, which may not intersect AS as shown with $AD^{ELB,TR}$ in the left panel of Figure 8 and AD_1 in the right panel. Once FP is activate at the ELB, as in the fiscal rule (B16), it fully mimics monetary policy as if the latter were unconstrained. Thus, AD is a linear relation composed of the red AD^{TR} line and the purple AD^u line. In other words, in the presence of active FP stemming from the rule, the model always has a unique solution.

Figure 8: Existence and Uniqueness with Unconventional Fiscal Policy Rule



Note: Left panel illustrates the steady-state equilibrium. Right panel illustrates the transitory state equilibrium with a PIR absorbing state.

C NK Model with Less Persistent Fiscal Policy

C.1 Coefficient Matrices

The coefficients in the canonical representation of the model are:

$$\boldsymbol{A}_0 = \begin{bmatrix} 1 & -\kappa_y & 0 \\ 0 & 1 & s_c\sigma^{-1}\Gamma \\ -\left(1-\rho_g\right)\psi_\pi^* & -\left(1-\rho_g\right)\psi_y^* & 1 \end{bmatrix}, \quad \boldsymbol{A}_1 = \begin{bmatrix} 1 & -\kappa_y & 0 \\ s_c\sigma^{-1}\phi_\pi & 1+s_c\sigma^{-1}\phi_y & s_c\sigma^{-1}\Gamma \\ -\left(1-\rho_g\right)\psi_\pi^* & -\left(1-\rho_g\right)\psi_y^* & 1 \end{bmatrix},$$

$$\boldsymbol{B}_0 = \boldsymbol{B}_1 = \begin{bmatrix} -\beta & 0 & 0 \\ -s_c\sigma^{-1} & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{C}_0 = \begin{bmatrix} 0 & 0 \\ -s_c & -s_c\sigma^{-1} \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{C}_1 = \begin{bmatrix} 0 & 0 \\ -s_c & 0 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{D}_0 = \boldsymbol{D}_1 = 0_{3\times 2},$$

$$\boldsymbol{H}_0 = \boldsymbol{H}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\rho_g \end{bmatrix}, \quad \boldsymbol{a} = \begin{bmatrix} \phi_\pi \\ \phi_y \\ 0 \end{bmatrix}, \boldsymbol{c} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{b} = 0_{3\times 1}, \quad \boldsymbol{d} = 0_{2\times 1}, \quad \boldsymbol{h} = 0_{3\times 1}.$$