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Belief–Efficiency in Markets

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Outline

1. Heterogeneous Beliefs and Unjustified Betting
2. Risk, Uncertainty, and Heterogeneous Beliefs
3. The Betting Example
4. Belief-Efficient Knight-Walras Equilibria
5. Implementing Belief Efficient Markets

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Volume of Trade and "Excessive Trade"

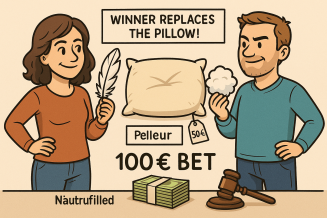
- in stationary worlds with common beliefs, rational agents do not trade much or even not at all:
  - Judd, Kubler, Schmedders, J. Fin. 2003
  - Milgrom, Stokey 1982, Tirole 1982, De Castro, Châteuneuf 2011
- volume of trade is huge in actual stock markets
  - total trade in top 500 stocks 30 trillion $ in 2014, twice US GDP,
  - In 2024, the NYSE trades over a billion shares daily.
- Hedging motives do not explain the amount of observed trading (Shiller, 1982)

Heterogeneous Expectations

- heterogeneity of agents' beliefs can be used to explain the massive amount of trade,
- Behavioral Finance Point of View:
  Overconfidence, Overprecision,
  Odean, AER 1999, Scheinkman, Xiong, JPE, 2003, Daniel, Hirshleifer, J. Econ. Perspectives 2015,
- Posner, Weyl, Northwestern University Law Review, 2013 call for a regulatory authority, that would need to approve trade in new financial assets

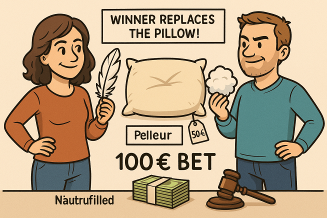
Beliefs and Unjustified Betting: The Pillow Example

- Maria and Peter have different opinions on the filling of a nice pillow worth 50 €
- Maria thinks it is natural feather-filled with probability 90 %
- Peter thinks it is polyester-filled with probability 90 %
- both are willing to bet 100 € to destroy the pillow and find out; the winner replaces the pillow



Beliefs and Unjustified Betting

- ex ante Maria:
  $0.9 \cdot (100 - 50) + 0.1 \cdot (-100) = 35$
- ex ante Peter:
  $0.1 \cdot (-100) + 0.9 \cdot (100 - 50) = 35$
- the bet is Pareto efficient (and a market outcome),
- from a social perspective, zero sum in money, and the pillow is destroyed



Heterogeneous Expectations and Efficiency

- The standard notion of efficiency might not be a suitable welfare measure with heterogeneous expectations (von Weizsäcker 1969, Drèze 1970, Starr, 1973, Harris 1978, Hammond 1981)
- We consideer the alternative notion of belief-neutral efficiency, Brunnermeier, Simsek, Xiong, QJE 2014
- See also No Betting Pareto Dominance, Gilboa, Samuelson, Schmeidler, ECMA 2014

Results

- we interpret heterogeneous beliefs as an instance of Knightian uncertainty
- we study markets in which agents agree on the probability distributions of fundamentals, yet disagree on some other events
- Theorem: Generically, markets allocate belief-neutral inefficiently with heterogeneous expectations
- Theorem: Suitable Knight-Walras equilibria (Beissner, Riedel, ECMA 2019) allocate belief-neutral efficiently
- Market design with suitable risk measures or transaction costs can implement belief-neutral efficient allocations and reduce unjustified betting

Outline

The Economy with Risk, Uncertainty, and Heterogeneous Beliefs

- there is a risky part of the economy where probabilities are commonly known, given by a σ–field $\mathcal{G}$
- the events in $\mathcal{G}$ have objective (intersubjectively agreed upon) probabilities; let Q be a probability measure that describes these objective probabilities on $\mathcal{G}$
- let $\mathcal{P}$ be a set of probability measures such that for all $P \in \mathcal{P}$ and all risky events $A \in \mathcal{G}$ we have

$$P(A) = Q(A)$$

- $\mathcal{P}$ represents all reasonable beliefs
- the agents have beliefs $P_i \in \mathcal{P}$ over the σ-field $\mathcal{F} \supset \mathcal{G}$
- assume that $\mathcal{P}$ is the convex hull of the subjective beliefs P_i of agents $i = 1, \ldots, I$

Individual Utilities

- agents' preferences are given by the subjective expected utility functional
$$U_i(c) = E^{P_i} u_i(c)$$
for standard Bernoulli utility functions u_i
- note that for $\mathcal{G}$–measurable consumption plans c, agents agree on the probability distribution of c
- the results can be generalized to ambiguity-averse preferences as long as the subjective beliefs diverge (Rigotti, Shannon, Strzalecki, ECMA 2008)

Fundamentals

- We consider an exchange economy
- all individual uncertainty is described by agents endowments $e_i : S \to \mathbb{R}_+$
- e_i is $\mathcal{G}$–measurable
- There is no Knightian uncertainty in fundamentals

Standard Efficiency and Equilibrium

- Let $e = \sum_i e_i$ be the aggregate endowment;

$$\Lambda(e) = \left\{ (x_i) \in \mathbb{X}_+^I : \sum_i x_i \leq e \right\}$$

  is the set of feasible allocations.
- A feasible allocation $(c_i) \in \Lambda(e)$ is efficient if there is no other feasible allocation $(y_i) \in \Lambda(e)$ that makes every agent better off in the sense that $U_i(y_i) > U_i(c_i)$.
- An (Arrow–Debreu or Walras) equilibrium consists of a linear price functional $\Psi : \mathbb{X} \to \mathbb{R}$ and a feasible allocation $(c_i) \in \Lambda(e)$ such that for all agents i and budget-feasible consumption plans y with $\Psi(y - e_i) \leq 0$ we have $U_i(y) \leq U_i(c_i)$.

Belief–Neutral Efficiency

Brunnermeier, Simsek, Xiong (QJE 2014) propose a belief-neutral welfare criterion

- from a welfare perspective, only the events in $\mathcal{G}$ have "correct" probabilities
- every belief $R \in \mathcal{P}$ might be a reasonable belief
- welfare should be evaluated in terms of a common reasonable belief $R \in \mathcal{P}$
- A feasible allocation $(c_i) \in \Lambda(e)$ is called *belief-neutral efficient* if it is efficient in every homogeneous belief economy with prior $R \in \mathcal{P}$ for all reasonable beliefs $R \in \mathcal{P}$.

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Unjustified Betting - the Example

- individual endowments have a known probability distribution
- there is no aggregate uncertainty
- If this was the whole economy, it is well known that markets would provide full insurance in equilibrium.
- Now assume that in addition to the fundamentals, there are additional, uncertain states that do not affect fundamentals (outcome of a sports event, e.g.)
- Beliefs on the odds differ among agents.
- In Arrow-Debreu equilibrium, agents start (unnecessary) betting on top of the full insurance allocation

Unjustified Betting - the Example

- four states $S = \{r_1, r_2, G, I\}$
- states r_1 and r_2 are risky with common probability 25% (fundamentals of the economy)
- states G and I are uncertain with G being "Germany wins"
- Maria has a probability of 12.5% for G, Peter 37.5%
- endowments $e_M = (3, 1, 2, 2)$, $e_P = (1, 3, 2, 2)$
- aggregate endowment always 4, individual risk is known, no uncertainty at the individual level
- $u_i = \log$

Unjustified Betting - the Example

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- states r_1 and r_2 are risky with common probability 25% (fundamentals of the economy)
- states G and I are uncertain with G being "Germany wins"
- Maria has a probability of 12.5% for G, Peter 37.5%
- equilibrium consumption $c_P = (2, 2, 3, 1)$, $c_M = (2, 2, 1, 3)$
- Agents have insured the risk of the first two states, but have created new, unnecessary risks on the sports events!
- The Arrow-Debreu allocation is not belief-neutral efficient because for any reasonable belief of the form $P^h = \sum_{i=1}^{2} h_i P_i$, the allocation c is dominated by the full insurance allocation

$$\bar{c}_i^h = E^{P^h} c_i$$

The General Case

Theorem

Generically in beliefs, Arrow-Debreu equilibrium allocations are belief-neutral inefficient.

Rigorous Formulation

We parametrize the set of economies by the belief profiles of the form $\mathfrak{B} = (P_1, \dots, P_I)$, keeping the other ingredients of the economy fixed.
A belief P with full support on $S = \{s_1, \dots, s_n\}$ can be equivalently described by a strictly positive vector $(P(s_1), \dots, P(s_{n-1})) \in \mathbb{R}^{n-1}$ with $\sum_{k=1}^{n-1} P(s_k) < 1$. Denote this open set by $\mathbb{B} \subset \mathbb{R}^{n-1}$.
Our set of economies

$$\mathfrak{E} = \{\mathcal{E}(P_1, \dots, P_I) : \text{ all } P_i \text{ have full support}\}$$

is isomorphic to $\mathbb{B}^I \subset \mathbb{R}^{(n-1)I}$.

Theorem

The set of economies $\mathcal{E}(P_1, \dots, P_I)$ that admit belief-neutral efficient Arrow-Debreu equilibria is both a Lebesgue null set and nowhere dense in $\mathfrak{E}$.

Proof Idea

- Belief-neutral efficient allocation equate agents' marginal rates of substitution state by state
- $$\frac{u_i'\left(c_i^*(s)\right)}{u_i'\left(c_i^*(t)\right)} = \frac{u_j'\left(c_j^*(s)\right)}{u_j'\left(c_j^*(t)\right)}$$

  for all agents i, j and states s, t.
- With heterogeneous beliefs, FOCs read as

  $$\frac{P_i(s)u_i'\left(c_i^*(s)\right)}{P_i(t)u_i'\left(c_i^*(t)\right)} = \frac{P_j(s)u_j'\left(c_j^*(s)\right)}{P_j(t)u_j'\left(c_j^*(t)\right)}$$

  for all agents i, j and states s, t.
- With distinct beliefs, the two systems of equations are not consistent with each other.

Outline

Knight–Walras Equilibria (Beissner, Riedel, ECMA 2019)

- For a closed convex set of probabilities $\mathcal{Q} \subset \Delta S$ introduce the sublinear expectation

$$\mathbb{E}^{\mathcal{Q}} c = \max_{P \in \mathcal{Q}} \mathbb{E}^{P} c$$

- A pair (ψ, c) consisting of a state–price $\psi : \Omega \to \mathbb{R}_+$ and a feasible allocation $c = (c_i)_{i=1,\ldots,I} \in \Lambda(e)$ is a Knight-Walras equilibrium (Beissner, Riedel, ECMA 2019) if all agents i maximize their utility subject to the budget constraint

$$\Psi^{\mathcal{Q}}(c - e_i) := \max_{Q \in \mathcal{Q}} E^{Q}[\psi(c - e_i)] \leq 0. \tag{1}$$

Knight–Walras Equilibria lead to Belief-Neutral Efficiency

Theorem

In every Knight-Walras equilibrium $(\psi^*, (c_i^*))$ with respect to the set of reasonable beliefs $\mathcal{P}$, the equilibrium allocation (c_i^*) is belief-neutral efficient.

Key idea

- The sublinear expectation $\mathbb{E}^{\mathcal{P}}$ is linear over the subspace of risky trades
- Hedging is still frictionless
- The sublinear expectation $\mathbb{E}^{\mathcal{P}}$ is nonlinear over the subspace of ambiguous trades
- Betting is too costly for one side for any price

Illustration in the Example

- The convex hull $\mathcal{P}$ of the two beliefs P_1 and P_2

$$\mathcal{P} = \left\{ \left(\frac{1}{4}, \frac{1}{4}, p, \frac{1}{2} - p \right) : p \in \left[\frac{1}{8}, \frac{3}{8} \right] \right\}.$$

- In Knight-Walras equilibrium, we have $\psi(s) = 1$ in all states and agents fully insure, consuming $c_i^*(s) = 2$ in all states, full insurance
- The Knight-Walras equilibrium price system is given by

$$\Psi^{\mathcal{P}}(x_1, x_2, x_G, x_I) = \frac{x_1 + x_2}{4} + \max_{p \in \left[\frac{1}{8}, \frac{3}{8}\right]} \left(p x_G + \left(\frac{1}{2} - p \right) x_I \right)$$

- The price system is linear in x_1 and x_2, the two risky states where beliefs agree, and nonlinear in the two states s_A and s_B where beliefs diverge.

Illustration in the Example

- The speculative Arrow-Debreu equilibrium consumption plan

$$c_1^* = (2, 2, 3, 1)$$

is not budget feasible with the sublinear Knight-Walras prices because we have

$$\begin{aligned}\Psi^{\mathcal{P}}(c_1^* - e_1) &= \frac{2-3+2-1}{4} \\ &\quad + \max_{p \in \left[\frac{1}{8}, \frac{3}{8}\right]} p \cdot (3-2) + \left(\frac{1}{2} - p\right) \cdot (1-2) \\ &= \max_{p \in \left[\frac{1}{8}, \frac{3}{8}\right]} p - \frac{1}{2} = \frac{1}{4} > 0.\end{aligned}$$

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Implementing Belief Efficient Markets

How to Implement Sublinear Prices in Practice?

- Coherent Risk Measures
- Transaction Costs

Implementing Belief Efficient Markets via Risk Measures

- sublinear expectations are coherent risk measures
- the paper thus gives an additional argument why such risk measures should be used
- not only for determining liquidity cushions
- but also for pricing by market makers

Implementing Belief Efficient Markets via Transaction Costs

- in the paper: construction of transaction cost equilibria with identical outcomes
- mathematically, sublinear prices can also be achieved via transaction cost
-

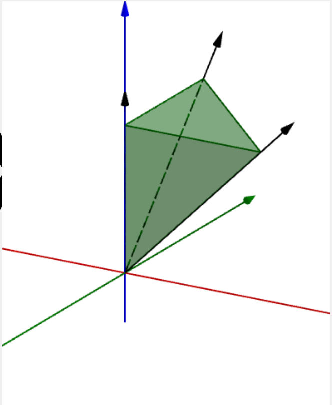
Theorem

Let $(\psi^*, (c_i^*))$ be a Knight-Walras equilibrium with respect to the set of reasonable beliefs $\mathcal{P}$ such that the equilibrium allocation (c_i^*) is belief-neutral efficient. There exists an efficient complete market with transaction costs $\mathcal{M}$ and a financial equilibrium with spot price $\phi = \psi^*$, portfolios $\theta_i = \left(\theta_i^A, \theta_i^B\right)$, and allocation (c_i^*).

- a cone $K \subset \mathbb{R}^S$ is finitely generated if there are finitely many vectors $x_k \in \mathbb{R}^S, k = 1, \ldots, n$ such that

$$K = \left\{ \sum_{k=1}^{n} \beta_k x_k : \beta_k \geq 0, k = 1, \ldots, n \right\}$$

- a polyhedral cone $K \subset \mathbb{R}^S$ is the intersection of finitely many half–spaces
- Minkowski–Weyl duality: every polyhedral cone is finitely generated



Constructing the Market - Convex Geometry

- convex cones of assets x priced by the belief of agent i:

$$V_i = \{x \in \mathbb{R}^S : \mathbb{E}^{\mathcal{P}} x = \mathbb{E}^{P_i} x\}.$$

- polyhedral because it is the intersection of finitely many half-spaces with normal vectors $P_j - P_i$.
- By the Minkowski–Weyl duality, we know that it is generated by finitely many payoffs $x_{i,1}, \ldots, x_{i,r_i}$:

$$V_i = \text{cone}(x_{i,1}, \ldots, x_{i,r_i}).$$

- We take these payoffs as the dividends for our assets and define the prices

$$q^B_{i,j} = \min_{l=1,\ldots,n} \mathbb{E}^P_l x_{i,j}, \quad q^A_{i,j} = \max_{l=1,\ldots,n} \mathbb{E}^P_l x_{i,j}, \quad j = 1, \ldots, r_i.$$

- riskless asset $x_0 = 1$,
- asset 1 pays off the endowment of agent 1 as a dividend, i.e. $x_1 = e_1 = (3, 1, 2, 2)$.
- two Arrow securities for the outcome of the sport event, i.e. $x_2 = (0, 0, 1, 0)$ and $x_3 = (0, 0, 0, 1)$.
- Define

$$q_j^B = \min\left(E^{P_1} x_j, E^{P_2} x_j\right), \quad q_j^A = \max\left(E^{P_1} x_j, E^{P_2} x_j\right), \quad j = 1, 2$$

- As the endowment e_1 is $\mathcal{G}$-measurable, the corresponding asset 1 is frictionless, with $q_1^B = q_1^A = 2$. T
- he Arrow securities exhibit transaction costs, with

$$q_j^B = \frac{1}{8} < q_j^A = \frac{3}{8}, \quad j = 2, 3.$$

- agents stop speculating on the sports event as they are no longer yielding positive expected profits under either belief.

Takeaway

- Model with excessive trade due to different beliefs
- Market equilibria are generically belief-neutral inefficient
- Knight–Walras equilibria are belief-neutral efficient
- we show how to design markets rules to dampen unjustified speculative trade, either via risk measures or transaction costs