Impartial Social Rankings

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Introduction

- We model a situation where a set of individuals must rank themselves on the basis of their opinions.
    - Example: The department members must decide a ranking of them to choose tasks for the following years.
- We study social ranking functions that aggregate individual opinions to obtain a social ranking where the contribution of a single agent to the social outcome can have no consequence on any social binary comparison between her and someone else - *Impartiality*.
- In this situation an impartial function would be useful because no one would have a reason to hide her true valuation of others.

Related Literature

- Impartiality in the context of committees and elections.
  - How could partners share a divisible good, de Clippel et al. (2008).
  - A group of peers must choose one of them to receive a prize, Holzman and Moulin (2013) and Mackenzie (2015).
  - A group of peers must choose some of them (no restriction of cardinality) to receive a prize, Tamura and Ohseto (2013), Kurokawa et al. (2015) and Tamura (2016).

- Other papers, like Bossert and Storcken (1992) and Bossert and Sprumont (2014), analyze manipulability of aggregation rules (mapping each profile of individual rankings into a social ranking of the alternatives).

Contribution

Our model differ from the literature of Impartiality in two issues:

- We try to construct a social ranking of the agents with an axiomatic and deterministic approach. Another precedent: Kahng et al. (2018) focus on randomized algorithms with a different property of Impartiality.
- We allow each agent to express her opinion in any possible way (a nomination of an agent or a set of agents, a ranking of some or all agents, a utility function, etc.).

Our contribution is:

- We show the impossibility of combining Impartiality with a symmetry axiom, that we name Name Independence (similar result for (weak) Voter Anonymity).
- We provide a characterization of the impartial rules that also satisfy a weaker version of Name Independence.

Notation and Impartiality

- $N = \{1, \ldots, n\}$ is the finite set of agents.
- The set of possible messages each agent can declare has no structure in the paper. However, during this presentation I assume that this set is the set of all possible linear orders of the set of agents (including herself). $\mathcal{R}$ is the set of all possible rankings.
- A social ranking of the agents is a linear order over N.
- $f : \mathcal{R}^n \rightarrow \mathcal{R}$ is called a social ranking function.
- $U(f(R), i)$ is the upper contour set of i in the social ranking when the message profile has been $R = (R_1, \ldots, R_n)$.

Definition

We say that f is impartial if for any $i \in N$ and any message profile $R \in \mathcal{R}^n$,

$$U(f(R), i) = U(f(R_{-i}, R_i'), i) \text{ for all } R_i' \in \mathcal{R}.$$

An example

Example (Borda)

Let $n = 4$ and consider the Borda ranking f with tie-breaker $1 \succ 2 \succ 3 \succ 4$. Let $B_i(R)$ be the number of Borda points that agent i gets from profile R.

R_1	R_2	R_3	R_4
1	2	3	4
2	1	2	3
3	4	4	2
4	3	1	1

Then $B_1(R) = 9$, $B_2(R) = 12$, $B_3(R) = 10$, $B_4(R) = 9$ and hence the final ranking would be:

$$f(R) = (2, 3, 1, 4).$$

Let agent 3 now changes his message to R_3', where $3\, R_3'\, 1\, R_3'\, 2\, R_3'\, 4$. This modification results in $B_1(R\ _3, R_3') = 11$, $B_2(R_{-3}, R_3') = 11$, $B_3(R_{-3}, R_3') = 10$, $B_4(R_{-3}, R_3') = 8$. Therefore, the social ranking would be

$$f(R_{-3}, R_3') = (1, 2, 3, 4).$$

Observe that $U(f(R), 3) = \{2\} \neq \{1, 2\} = U(f(R_{-3}, R_3'), 3)$ and, thus, f violates Impartiality.

A symmetry axiom: Name Independence

Anonymity and Neutrality state, respectively, that voters and candidates should be treated symmetrically in the procedure. Since voters and candidates are the same, we can capture an idea of symmetry across agents by one axiom:

Definition

We say that f is Name Independent if for any pair of agents $i,j \in N$ and the permutation $\sigma : N \to N$ such that $\sigma(k) = k$ for each $k \in N \backslash \{i,j\}$, $\sigma(i) = j$ and $\sigma(j) = i$, then

$$\sigma(f(R)) = f(\sigma(R_{\sigma(1)}), \ldots, \sigma(R_{\sigma(n)})) \text{ for any } R \in \mathcal{R}^n.$$

- $R_\sigma = (R_{\sigma(1)}, ..., R_{\sigma(n)})$ is a preference profile where agent i has the preferences agent $\sigma(i)$ had in R.
- Given a permutation σ of N and $R_k \in \mathcal{R}$, we define $\sigma(R_k)$ as a ranking such that $\sigma(i)\sigma(R_k)\sigma(j)$ whenever iR_kj for any $i,j \in N$.

A symmetry axiom: Name Independence

$\sigma(f(R)) = f(\sigma(R_{\sigma(1)}), \ldots, \sigma(R_{\sigma(n)}))$ for any $R \in \mathcal{R}^n$.

- Name Independence states that the position of agent i in the image under f after the permutation of the voters and the messages should be the same as the position of agent $\sigma(i) = j$ in $f(R)$.

A symmetry axiom: Name Independence

Example

Let $n = 4$ and consider the following profile:

R_1	R_2	R_3	R_4
1	2	3	4
2	1	2	3
3	4	4	2
4	3	1	1

Consider the permutation $\sigma(1) = 2$, $\sigma(2) = 1$, $\sigma(3) = 3$ and $\sigma(4) = 4$. Then, the "permuted" profile R' would be:

R'_1	R'_2	R'_3	R'_4
1	2	3	4
2	1	1	3
4	3	4	1
3	4	2	2

If, for instance, $f(R) = (3, 2, 1, 4)$ and f is Name Independent, the outcome in the "permuted" profile should be $f(R') = (3, 1, 2, 4)$.

An impossibility

Theorem

There is no social ranking function satisfying Impartiality and Name Independence.

Why?

- Name Independence implies that all rankings should be in the range of the social ranking function (Lemma 2).
- Impartiality implies that all agents that are situated in the first position in a ranking that appears in the range of the social ranking function never appears last in any other ranking of the range (Lemma 1).

Another property: Anonymity

Anonymity implies that voters should be treated symmetrically in the aggregation procedure.

This property states that the outcome in a message profile R should be the same as the outcome if we permute the voters, but not the messages.

Definition

We say that f is Anonymous if for any permutation $\sigma : N \to N$ and any $R \in \mathcal{R}^n$ we have that

$$f(R) = f(R_{\sigma(1)}, \dots, R_{\sigma(n)}).$$

Another property: Anonymity

Example

Let $n = 4$ and consider the following profile:

R_1	R_2	R_3	R_4
1	2	3	4
2	1	2	3
3	4	4	2
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Consider the permutation $\sigma(1) = 2$, $\sigma(2) = 1$, $\sigma(3) = 3$ and $\sigma(4) = 4$. Then, the “permuted" profile R' would be:

R'_1	R'_2	R'_3	R'_4
2	1	3	4
1	2	2	3
4	3	4	2
3	4	1	1

If f is Anonymous, then $f(R) = f(R')$.

An impossibility

We say that f is constant if for any message profiles $R, R' \in \mathcal{R}^n$,

$$f(R) = f(R').$$

Theorem

The unique Anonymous and Impartial social ranking functions are the constant ones.

An impossibility

This theorem can be also obtained if we weaken Anonymity and we only require that the position of each agent i should not change if we only permute the other agents.

Definition

We say that f is Weakly Anonymous if for any $i \in N$, any permutation $\sigma : N \rightarrow N$, with $\sigma(i) = i$, and any $R \in \mathcal{R}^n$ we have that

$$U(f(R), i) = U(f(R_{\sigma(1)}, \ldots, R_{\sigma(n)}), i).$$

Theorem

The unique Weakly Anonymous and Impartial social ranking functions are the constant ones.

Possibility results

All constant social ranking functions satisfy Impartiality.

Are there any non-constant impartial social ranking functions?

Example

Suppose $n = 3$. Let f be such that:

1. Agent 3 is always ranked last.
2. To choose which agent occupies the first position in each profile, f looks at the opinion of agent 3 between 1 and 2: if 3 has put 1 above 2, the social ranking is $(1, 2, 3)$; otherwise, the social ranking is $(2, 1, 3)$.

Observe that f is impartial: the messages of agents 1 and 2 are irrelevant and the message of agent 3 only determines the ranking of a pair of agents that are both always situated above her.

Partition orderings

Definition

A social ranking function f is a **partition ordering** if there is an ordered partition of N, $\{N_1, \ldots, N_t\}$ and a set of t functions, one for each N_i, $f_{N_i}^{-N_i} : \mathcal{R}^{N \setminus N_i} \rightarrow \mathcal{R}_{|N_i}$ such that for all $i, j \in N$, with $i \in N_p$ and $j \in N_q$, and for any $R \in \mathcal{R}^n$:

$$i \in U(f(R), j) \text{ if } p < q \text{ or } [p = q \text{ and } i \in U(f_{N_p}^{-N_p}(R_{-N_p}), j)].$$

- Agents in N_1 are always ranked above agents in N_2, and agents in N_2 always ranked above agents in N_3, and so on.
- Moreover, the social ranking of the agents inside each set N_p of the partition is determined *only* by the messages of agents outside this set N_p.

In the above example, $N_1 = \{1, 2\}$ and $N_2 = \{3\}$.

Possibility results

Theorem

Let $n < 4$. Then, a social ranking function f satisfies Impartiality if and only if f is a partition ordering.

And if $n \geq 4$?

Are there other social ranking functions that satisfy Impartiality apart from the family of partition orderings? **Yes**

An Impartial SRF that is not a partition ordering

Example

Suppose $n = 4$.

Let f be such that:

1. Agent 4 is always ranked last.
2. The message of agent 4 determines who is ranked third.
3. The messages of the agents that have been determined before that are in the last two positions (agent 4 and the one agent 4 has determined to be third) are aggregated to obtain who is in first position and who is in second position.

Observe that f is impartial since the message of each agent is only considered to determine the ranking of the agents that are going to be situated higher than her.

However, it is not a partition ordering because, although all agents of $N_1 = \{1, 2, 3\}$ are always ranked higher than $N_2 = \{4\}$, the ranking of the agents of N_1 does not depend exclusively on the message of the agent outside N_1.

Impartial orderings

The example belongs to a bigger family of impartial SRF($n > 3$):

Definition

A social ranking function f is an **impartial ordering** if there is an ordered partition of N, $\{N_1, \ldots, N_t\}$ and a set of impartial functions $f_{R_{-N_p}}$, one for each set N_p of the ordered partition and each R_{-N_p}, such that for any $i, j \in N$, with $i \in N_p$, $j \in N_q$, and any $R \in \mathcal{R}^n$:

$$i \in U(f(R), j) \text{ if } p < q \text{ or } [p = q \text{ and } i \in U(f_{R_{-N_p}}(R_{N_p}), j)].$$

That is, each f in this subclass defines for each preference subprofile of the agents outside N_p, say R_{-Np} , an impartial function, $f_{R_{-N_p}}$, that determines the final ranking of the agents of N_p for each preference subprofile of the agents inside N_p, say R_{N_p}.

Another symmetry axiom: Weak Name Independent

- Name Independence implies full range of the social ranking function and this directly creates an impossibility with Impartiality.
- Thus, to obtain some possibility results, we need to relax Name Independence maintaining its spirit but that does not necessarily imply full range.
- A possibility is to require symmetry *only* to pairs of agents whose comparison is not fixed by the social ranking function, called *free pairs*.

Weak Name Independence

- We say that $\{i, j\}$ are **a free pair at** f if there exist $R, R' \in \mathcal{R}^n$ such that
$$i \in U(f(R), j) \text{ and } j \in U(f(R'), i).$$

Definition

A social ranking function f is Weakly Name Independent if for all of its free pairs $\{i, j\} \in N$ and for the permutation of N such that $\sigma(k) = k$ for all $k \in N \setminus \{i, j\}$, $\sigma(i) = j$ and $\sigma(j) = i$, then

$$\sigma(f(R)) = f(\sigma(R_{\sigma(1)}), \ldots, \sigma(R_{\sigma(n)})) \text{ for any } R \in \mathcal{R}^n.$$

Characterization of Impartial orderings

Theorem

A social ranking function f satisfies Impartiality and Weak Name Independence if and only if f is an impartial ordering.

Subclass of impartial orderings?

- As above mentioned the family of impartial orderings incorporate Impartiality as a property in its own definition. It says that, fixing the messages of the agents outside a set of the ordered partition, the ranking of the agents inside the set should be done aggregating the messages of the agents inside the set with a subrule that is impartial.
- Thus, the class of impartial orderings is not closely defined and requires a recursive argument.
- Can we define a **subfamily in a closed form**?

Subclass of impartial orderings?

- First, for each N_p, consider a rule $g_{N_p} : \mathcal{R}^{N \setminus N_p} \rightarrow \mathcal{R}^*_{N_p}$ that establishes an ordered partition of N_p using the preferences of the agents outside N_p.
- That is, for any preference profile $R \in \times_{i \in N} \mathcal{R}_i$ we use $g_{N_p}(R_{-N_p})$ to partition N_p in smaller subsets ordered as $(N_{(p,1)}, ..., N_{(p,v)})$.
- If any of these subsets $N_{(p,s)}(R)$ is not a singleton yet, then we use $g_{N_{(p,s)}}(R_{-N_{(p,s)}})$ to partition it into small subsets and ordered as $(N_{(p,s,1)}, ..., N_{(p,s,z)})$.

Subclass of impartial orderings?

- Proceeding similarly and in a recursive way, given an initial ordered partition $(N_1, ..., N_t)$ and a social ranking partition g_A for each $A \subseteq N_p$, with $p \in \{1, ..., t\}$, we assign for each preference profile $R \in \times_{i \in N} \mathcal{R}_i$ an identification of each agent $k \in N$ as a vector of numbers $\overrightarrow{n}_k(R)$ such that $N_{\overrightarrow{n}_k(R)} = \{k\}$.
- Finally, we denote by $<^L$the lexicographic order of vectors of natural numbers such that $\overrightarrow{n} <^L \overrightarrow{n'}$ if there is $l \in \mathbb{N}$ such that $n_i = n'_i$ for all $i < l$ and $n_l < n'_l$.

Sequential impartial orderings

Definition

A social ranking function f is a **sequential impartial ordering** if there is an ordered partition of N, $\{N_1, \ldots, N_t\}$ and a social ranking partition g_A for each $A \subseteq N_p$ and each $p \in \{1, ..., t\}$, such that for any $i, j \in N$ and $R \in \mathcal{R}^n$:

$$i \in U(f(R), j) \Leftrightarrow \overrightarrow{n}_i(R) <^L \overrightarrow{n}_j(R).$$

Sequential impartial orderings vs impartial orderings

- Observe also that the sequential impartial orderings are a subfamily of the impartial orderings. Then, they satisfy Impartiality and Weak Name Independence. In fact, they also satisfy a property stronger than Weak Name Independence.
- This property is going to require that when we fix the preferences of some agents and we have a subproblem in which the preferences of the remaining agents should decide a ranking of them, the subrule used satisfies Weak Name Independence.
- To define the property in formal terms, we need additional notation. We say that a pair of agents $\{i,j\} \subseteq N$ is a free pair at f for the message subprofile $R_A \in \times_{k\in A}\mathcal{R}_k$, with $i,j \notin A$, if there exist $m'_{-A}, m''_{-A} \in \times_{k\in N\setminus A}\mathcal{R}_k$ such that $j \in U(f(m_A, m'_{-A}), i)$ and $i \in U(f(m_A, m''_{-A}), j)$. Observe that, when $A = \varnothing$, the definition of being a free pair for a preference subprofile coincides with our previous definition of being a free pair.

Consistent Weak Name Independence

Definition

A social ranking function f is Consistent Weakly Name Independent if for any $A \subseteq N$, any $R_A \in \times_{k \in A} \mathcal{R}_k$ and any free pair $\{i, j\} \in N$ at f for the preference subprofile R_A and the permutation $\sigma : N \backslash A \to N \backslash A$ such that $\sigma(k) = k$ for each $k \in N \setminus (A \cup \{i, j\})$, $\sigma(i) = j$ and $\sigma(j) = i$, then

$$\sigma(f(R)_{|N \backslash A}) = f(R_A, \sigma(R_{\sigma(N \backslash A)})) \text{ for any } R \in \mathcal{R}^n.$$

Characterization of Sequential Impartial orderings

Theorem

If a social ranking function f satisfies Impartiality and Consistent Weak Name Independence, then f is a sequential impartial ordering.

Other Impartial social ranking functions?

- A natural question that arises from Theorem 1, 2 and 3 is if there are other impartial social ranking functions apart from those that satisfy Weak Name Independence with more than 3 agents.
- The answer is YES as the following example shows for four alternatives and a binary common preference space for all agents: there exist other impartial social ranking functions that do not belong to the family of impartial orderings.

Example of an impartial SRO violating WNI

Example

$N = \{1,2,3,4\}$ and $\mathcal{R} = \mathcal{R}^{\{1,2\}} \cup \mathcal{R}^{\{3,4\}}$ be a partition of the sets of all individual preferences over N where preferences in $\mathcal{R}^{\{1,2\}}$ have 1 or 2 as tops, while preferences in $\mathcal{R}^{\{3,4\}}$ have 3 or 4 as tops.
The range of the social ranking function that we are going to construct has 6 strict rankings: $1R^1 4R^1 2R^1 3$, $2R^2 1R^2 4R^2 3$, $1R^3 2R^3 4R^3 3$, $2R^4 1R^4 3R^4 4$, $1R^5 3R^5 2R^5 4$, and $1R^6 2R^6 3R^6 4$.
The social ranking function is such that, for any $R \in \times_{i\in N}\mathcal{R}_i$,
If $R_1, R_3 \in \mathcal{R}_i^N$, then $f(R) = R^1$.
If $R_1 \in \mathcal{R}_i^N$ and $R_3, R_4 \in \mathcal{R}_i^Y$, then $f(R) = R^2$.
If $R_1, R_4 \in \mathcal{R}_i^N$ and $R_3 \in \mathcal{R}_i^Y$, then $f(R) = R^3$.
If $R_1, R_3, R_4 \in \mathcal{R}_i^Y$, then $f(R) = R^4$.
If $R_1 \in \mathcal{R}_i^Y$ and $R_4 \in \mathcal{R}_i^N$, then $f(R) = R^5$.
Otherwise, $f(R) = R^6$.

Sequential impartial orderings vs partition orderings

- Like impartial orderings, sequential impartial orderings coincide with partition orderings in the fact that all of them initially partition the set of agents in such a way that all agents belonging to N_p are always ranked above any agent of N_q for all $q > p$.
- However, both classes differ from partition orderings in that the ranking of the agents inside N_p may not only depend on the messages of the agents outside that set but also on those of the agents inside N_p.
- In particular, for the sequential orderings, the preferences of the agents outside N_p only partition this set into ordered subsets such that the agents of $N_{(p,i)}$ are going to be ranked above the agents of $N_{(p,j)}$ with $i < j$, but the order of the agents of the same subset $N_{(p,i)}$ are not decided by only the preferences of the agents outside N_p, but also by the preferences of the agents of other subsets of N_p different from $N_{(p,i)}$.
- Then, this process is replicated iteratively until all subsets are singletons and a complete ranking has been reached.

Impartiality: Other frameworks, versions

- This axiom is in the same spirit as the usual strategy-proofness; however note that the difference in frameworks is crutial. For example, in ours we may not necessarily be asking for preferences, but rather for opinions that are unrelated to preferences.
- Different version may be used: We choose a version that guarantees that an agent is not able to change the candidates that will have in a better social position (her social upper contour).
- Choosing a single alternative: each agent's chance of winning is determined solely by the reports of his peers, removing incentives for a agent to submit a corrupted ballot.