The Empirical Content of Expected Utility

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   - Possibility for testing continuity indirectly

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  - Decision maker consistent with prospect theory but not with expected utility violates continuity

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  - Makes understanding expected utility more difficult

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  - Multistage lotteries reduced to single stage
- **A data set** is $\mathcal{D} = (D, \succsim_{\mathcal{D}}, \succ_{\mathcal{D}})$ where $D \subset \Delta(X)$ is a **finite** set and $\succsim_{\mathcal{D}}$ and $\succ_{\mathcal{D}}$ are (possibly incomplete) preferences on D

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- **The theory of transitive order satisfying the strong independence axiom** $\mathcal{T}_{\text{Tr-Ind}}$ is the collection of pairs of preferences $(\succsim, \succ)$ on $\Delta(X)$ such that $\succ$ is the asymmetric part of $\succsim$ and the following two conditions hold

(1) $\succsim$ is transitive.

(2) For all $P, Q, R \in \Delta(X)$ and $\alpha \in (0, 1)$,

$$P \succsim Q \iff \alpha P + (1 - \alpha)R \succsim \alpha Q + (1 - \alpha)R.$$

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Definition

Let $\mathcal{T}$ be a theory and $\mathcal{D} = (D, \succsim_{\mathcal{D}}, \succ_{\mathcal{D}})$ be a data set. $\mathcal{T}$ **rationalizes** $\mathcal{D}$ if there exists $(\succsim, \succ) \in \mathcal{T}$ such that $\succsim_{\mathcal{D}} \subset \succsim$ and $\succ_{\mathcal{D}} \subset \succ$.

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Let $\mathcal{T}$ and $\mathcal{T}'$ be two theories. $\mathcal{T}$ **is data equivalent to** $\mathcal{T}'$ if for all data sets $\mathcal{D}$,

$$\mathcal{T} \text{ rationalizes } \mathcal{D} \iff \mathcal{T}' \text{ rationalizes } \mathcal{D}.$$

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- This suggests using the strong independence axiom for the expected utility theory
  - All the behavioral content of expected utility is contained in transitivity and strong independence axiom

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- Proof by examples of finite preference that satisfy weaker independence axiom but does not have a rationalization with EU

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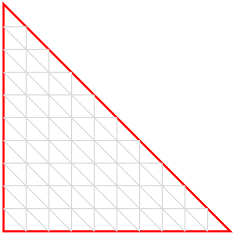

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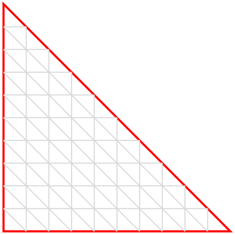

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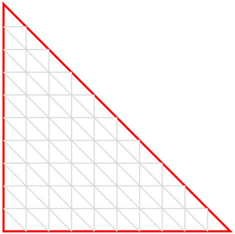

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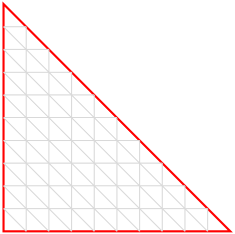

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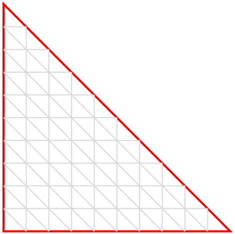

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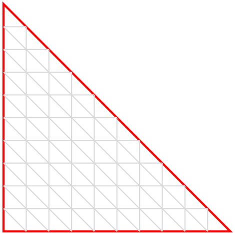

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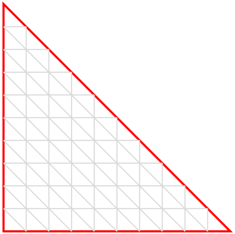

- For all $P, Q, R, S \in \Delta(X)$ and $\alpha \in (0,1)$,

(2′) $P \succ Q \implies \alpha P + (1-\alpha)R \succ \alpha Q + (1-\alpha)R.$

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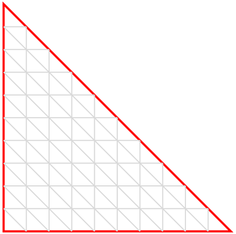
Outline

Empirical Content of Continuity under Strict Independence



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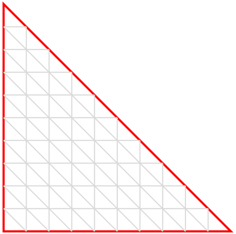

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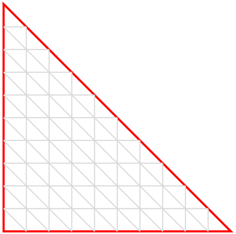

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- We can pinpoint the empirical content of continuity under strict independence.
- If we only observe strict preferences or only preferences in the interior of the set of lotteries, then strict independence and transitivity capture the empirical content of EU
- The empirical content of continuity is contained at the boundary and at indifference.

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Strict preferences

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(1) $\succsim$ is transitive.

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Which Independence Axioms Textbooks Use

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- Advanced microeconomics textbooks use the following independence axioms:
- (2): Mas-Colell et al. (1995), Rubinstein (2006), Gilboa (2009), and Muñoz-Garcia (2017)
- $(2')$: Fishburn (1970) and Kreps (1988; 1990)
- (2^*): Silberberg and Suen (2000) and Riley (2012)
- $(2^\dagger)$: Luenberger (1995), Gravelle and Rees (2004), Jehle and Reny (2011), Varian (2014), and Wang (2018).
- $(2^\ddagger)$: Kreps (2013)

Appendix

Proof Sketch: Part 1

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Assume that $(\succsim, \succ) \in \mathcal{T}_{\text{Tr-Ind}}$ and for each $i \in \{1, \ldots, n\}$, $P_i, Q_i \in \Delta(X)$,$\alpha_i > 0$, and $P_i \succsim Q_i$ with $\sum_{i=1}^{n} \alpha_i = 1$ and for some $k \in \{1, \ldots n\}$, $P_k \succ Q_k$, then $\sum_{i=1}^{n} \alpha_i P_i \succ \sum_{i=1}^{n} \alpha_i Q_i$.

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- By **Farkas' lemma** any finite collection of preferences either are rationalizable by expected utility or violate finite dominance axiom

Proof Sketch: Part 2

Since D is a finite set of simple lotteries, there exists a finite set $Y = \{y_1, \dots, y_n\} \subset X$ such that $D \subset \Delta(Y)$. Enumerate

$$\succsim^{\mathcal{D}} = \big\{(P_{1,1}, Q_{1,1}), \dots, (P_{p,1}, Q_{p,1})\big\} \text{ and } \succ^{\mathcal{D}} = \big\{(P_{1,2}, Q_{1,2}), \dots, (P_{q,2}, Q_{q,2})\big\}.$$

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By Farkas' lemma one and only one of the following two conditions holds:

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Proof Sketch: Part 3

(1) There exists $z \in \mathbb{R}^n$ such that $(B^P - B^Q)z \gg 0$ and $(A^P - A^Q)z \geq 0$.

Assume first that (1) holds and there exists $z \in \mathbb{R}^n$ such that $(B^P - B^Q)z \gg 0$ and $(A^P - A^Q)z \geq 0$, then preferences $(\succsim, \succ)$ defined by a utility function $u : X \to \mathbb{R}$ such that for each $i = 1, \ldots, n$, $u(y_i) = z_i$ rationalize $\mathcal{D}$ which is a contradiction.

Proof Sketch: Part 4

(2) There exist $z^1 \in \mathbb{R}^p$ and $z^2 \in \mathbb{R}^q$ such that $z^1(A^P - A^Q) + z^2(B^P - B^Q) = 0$, $z^1 \geq 0$, $z^2 > 0$, and $\sum_{i=1}^p z_i^1 + \sum_{i=1}^q z_i^2 = 1$.

Now,

$$z^1 A^P + z^2 B^P = z^1 A^Q + z^2 B^Q \Longrightarrow \sum_{k=1}^p z_k^1 P_{i,1} + \sum_{k=1}^q z_k^2 P_{i,2} = \sum_{k=1}^p z_k^1 Q_{i,1} + \sum_{k=1}^q z_k^2 Q_{i,2}.$$

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Assume, per contra, that there exists $(\succsim', \succ') \in \mathcal{T}_{\text{Tr-Ind}}$ that rationalizes $\mathcal{D}$. By finite dominance axiom, since $z^2 > 0$,

$$\sum_{k=1}^{p} z_k^1 P_{i,1} + \sum_{k=1}^{q} z_k^2 P_{i,2} \succ' \sum_{k=1}^{p} z_k^1 Q_{i,1} + \sum_{k=1}^{q} z_k^2 Q_{i,2}.$$

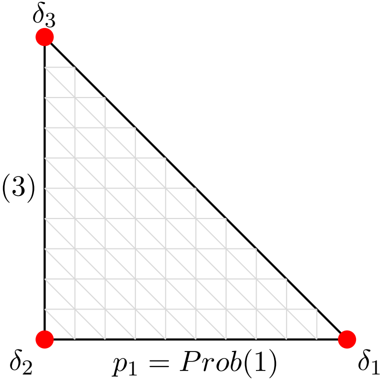
Denoting $R = \sum_{k=1}^{p} z_k^1 P_{i,1} + \sum_{k=1}^{q} z_k^2 P_{i,2}$, we have by the definition of $\succ'$, $R \succsim R$ and $R \not\succsim R$ that is a contradiction.

Counterexample for $(2')$

$(2')$ For all $P, Q, R \in \Delta(X)$ and $\alpha \in (0,1)$

$$P \succ Q \implies \alpha P + (1-\alpha)R \succ \alpha Q + (1-\alpha)R.$$

- $X = \{1,2,3\}$.



- Consider preferences $\succsim$ satisfying $(2')$ such that
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- Counterexample requires indifference at the boundary.
- Without either of these, this captures the empirical content.

Counterexample for $(2')$

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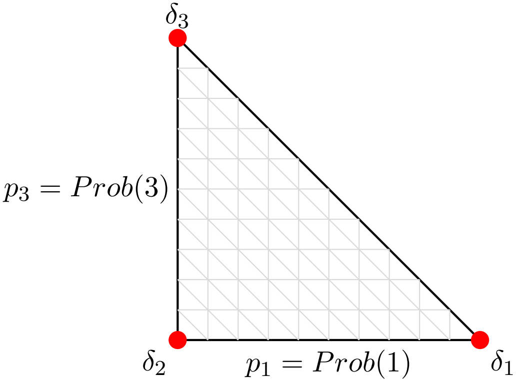

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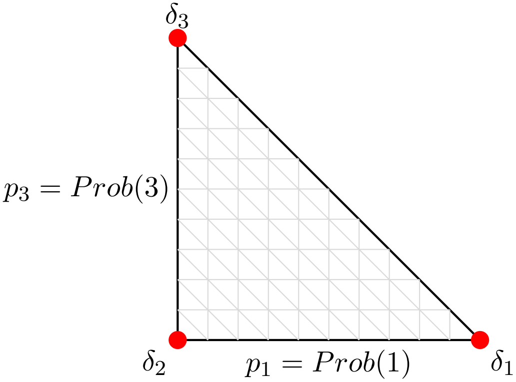

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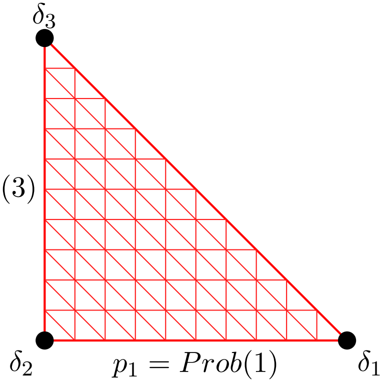
Counterexample for (2^*)

(2^*) For all $P, Q, R \in \Delta(X)$ and $\alpha \in (0,1)$

$$P \succsim Q \implies \alpha P + (1-\alpha)R \succsim \alpha Q + (1-\alpha)R.$$

- $X = \{1,2,3\}$.



- Consider preferences $\succsim$ satisfying (2^*) such that
  - $\delta_3 \succ \delta_2 \succ \delta_1 \succ P$ for all other P
  - Otherwise ● indifferent.

Counterexample for (2^*)

(2^*) For all $P, Q, R \in \Delta(X)$ and $\alpha \in (0,1)$

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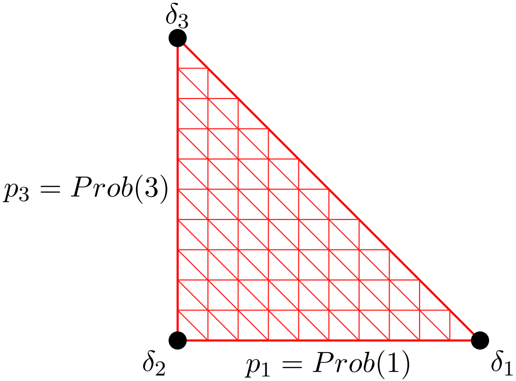

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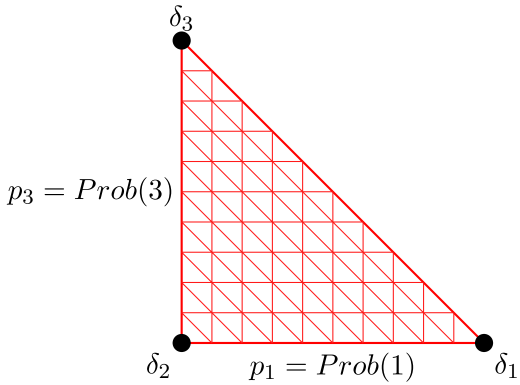

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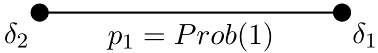
Counterexample for $(2^\dagger)$

$(2^\dagger)$ For all $P, Q, R \in \Delta(X)$ and $\alpha \in (0,1)$

$$P \sim Q \iff \alpha P + (1-\alpha)R \sim \alpha Q + (1-\alpha)R.$$

- $X = \{1, 2\}$.



- Consider preferences $\succsim$ satisfying $(2^\dagger)$ such that
  - $\delta_1 \succ P$ for all $P \neq \delta_1$
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Counterexample for $(2^{\dagger})$

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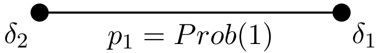

- Consider preferences $\succsim$ satisfying $(2^{\dagger})$ such that
  - $\delta_1 \succ P$ for all $P \neq \delta_1$
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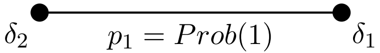

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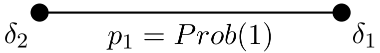

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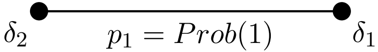

- Let $f : \mathbb{R} \to \mathbb{R}$ be a non-monotone solution to the Cauchy functional equation: for all $x, y \in \mathbb{R}$

$$f(x+y) = f(x) + f(y)$$

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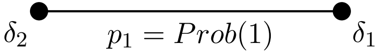

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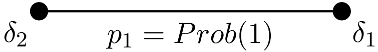

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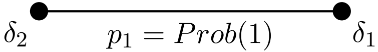

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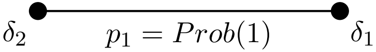

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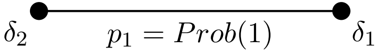

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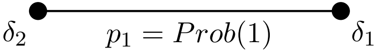

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