

# Earnings dynamics and top-earnings inequality

Karl Hårmenberg <sup>1</sup>    Raysa Lizarraga <sup>2</sup>

<sup>1</sup> University of Oslo

<sup>2</sup> BI Norwegian Business School

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# Introduction

# Introduction

Canonical model of earnings dynamics:

$$y_t = z_t + \epsilon_t,$$
$$z_t = \rho z_{t-1} + \nu_t,$$

with  $\epsilon_t$  and  $\nu_t$  iid.

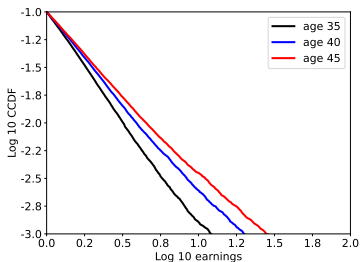
Advantages:

- ▶ Matches covariance structure of earnings
- ▶ Easy to use in consumption-savings models
- ▶ Generates a reasonable earnings distribution (log normal)

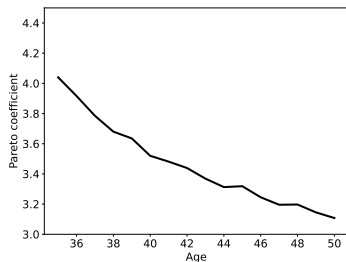
Disadvantage:

- ▶ Poor fit to top-earnings inequality dynamics

# Top-earnings facts (Norwegian male earnings)



(a) Top earnings are approximately Pareto distributed



(b) The age-specific Pareto coefficient is declining over the life cycle

- ▶ Top earnings  $\sim$  Pareto distribution
- ▶ Fast dynamics: top-earnings inequality increases substantially over the life cycle

# The canonical model and top-earnings inequality

- ▶ Canonical model  $\rightsquigarrow$  log-normal distribution  $\not\rightarrow$  Pareto tail
- ▶ + unit root and “reset”  $\rightsquigarrow$  Pareto distribution *but* slow dynamics
  - ▶ E.g., perpetual-youth households (McKay 2017; Carroll et al. 2017, 2020; Harmenberg and Öberg 2021)
  - ▶ Very slow dynamics of top-earnings inequality (Gabaix et al. 2016; Luttmer 2011)
- ▶ Ad hoc adjustment of earnings distribution (Hubmer et al. 2021)  
fast dynamics *but* not disciplined by micro-level moments

## Our model:

A generalization of the canonical model, a *square-root process*, which:

- ▶ Retains the parsimony of the canonical model (one state variable)
- ▶ Retains the excellent fit to the covariance structure of earnings
- ▶ Generates a Pareto tail in earnings
- ▶ Generates fast dynamics of top-earnings inequality

# Literature

## **Pareto distributions and earnings dynamics**

Pareto (1896); Champernowne (1953); Simon (1955); Nirei and Aoki (2016); Toda and Walsh (2015); Beare and Toda (2022)...

- ▶ Power laws: Gabaix (2009); Benhabib and Bisin (2018)

## **Canonical earnings process**

Friedman and Kuznets (1945); Lillard and Willis (1978); MaCurdy (1982); Abowd and Card (1989); Storesletten et al. (2004); Meghir and Pistaferri (2004); Low et al. (2010)...

## **Top earnings dynamics and macroeconomic outcomes**

- ▶ Savings behavior of the rich: Fagereng et al. (2019); Straub (2019)
- ▶ Optimal taxation of top earners: Guner et al. (2016); Badel et al. (2020); Kindermann and Krueger (2022)

## **Square-root processes in economics/finance** Cox et al. (1985)

Theoretical model

# Theoretical model

- ▶ We first introduce the earnings process in continuous time without transitory shocks
- ▶ Starting point:

$$dy_t = \theta (\bar{y} - y_t) dt + \Sigma dW_t. \quad (\text{Ornstein-Uhlenbeck process})$$

Continuous-time equivalent of  $AR(1)$  process

- ▶ Generalization of OU: *square-root process*
- ▶ Theory: square-root process  $\rightsquigarrow$  dynamics of top-earnings inequality

After theory: quantitatively calibrate and evaluate the model with Norwegian earnings data

# Theoretical model

A square-root process

Let log earnings  $y_t$  follow the process given by

$$dy_t = \theta (\bar{y} - y_t) dt + (\Sigma^2 + \Psi^2 y_t)^{1/2} dW_t.$$

where  $W_t$  is a standard Brownian motion

## Properties:

- ▶  $\theta$  is the mean-reversion parameter ( $\theta = 0 \leftrightarrow$  random walk)
- ▶  $\Psi = 0 \rightsquigarrow$  Ornstein-Uhlenbeck process (i.e., cts time  $AR(1)$ )
- ▶  $\Psi > 0 \rightsquigarrow$  state-dependent earnings risk (top earners  $\leftrightarrow$  higher risk)

Under  $\Psi \neq 0$ , a change of variable (unit of account) transforms the process into:

$$dy_t = \theta (\bar{y} - y_t) dt + \Psi \sqrt{y_t} dW_t.$$

(square-root process)

Cox-Ingersoll-Ross (CIR) process, use in finance to model interest rates.

# Properties of the square-root process

## Proposition

Let log earnings  $y_t$  follow the square-root process given by

$$dy_t = \theta (\bar{y} - y_t) dt + \Psi \sqrt{y_t} dW_t. \quad (1)$$

Let log earnings at time  $t = 0$  be given by the gamma distribution,

$$f_0(y) = \frac{(\eta_0)^{-n}}{\Gamma(n)} y^{n-1} \exp(-y/\eta_0) \quad (2)$$

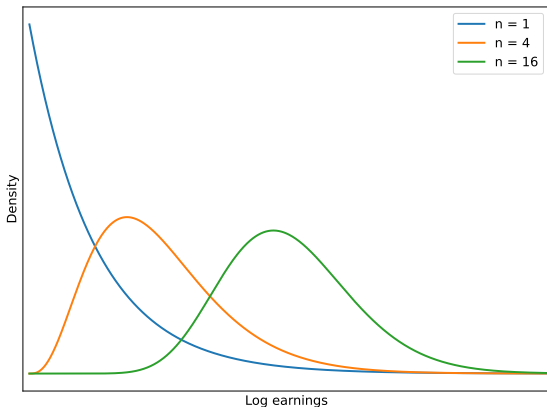
with  $n = \frac{2\theta\bar{y}}{\Psi^2}$ . Log earnings at time  $t$  is then also gamma distributed,

$$f_t(y) = \frac{(\eta_t)^{-n}}{\Gamma(n)} y^{n-1} \exp(-y/\eta_t) \quad (3)$$

where  $\eta_t$  evolves according to

$$d\eta_t = (\Psi^2/2 - \theta\eta_t) dt. \quad (4)$$

# Reminder: gamma distribution



- ▶ Gamma distribution with shape parameter  $n \sim$  sum of  $n$  i.i.d. exponential random variables
  - ▶  $n = 1 \leftrightarrow$  exponential distribution  $\rightsquigarrow$  Pareto distributed earnings
  - ▶  $n$  large  $\rightsquigarrow$  hump-shaped distribution with Pareto tail
  - ▶  $n = \infty \rightsquigarrow$  log-normal distribution

# Theoretical model

## Law of motion for earnings inequality:

$$d\eta_t = (\Psi^2/2 - \theta\eta_t)dt.$$

## Inequality measures over time:

- ▶  $\text{Var}(y_t) = n\eta_t^2$
- ▶ Asymptotic Pareto tail of earnings:  $\alpha_t = 1/\eta_t$

## Properties:

1. Asymptotic Pareto tail in earnings
2. Pareto tail in earnings throughout the life cycle
3. Closed-form dynamics of inequality measures

Proof of the proposition: Kolmogorov forward equation + guess and verify.

# Quantitative model evaluation

# Quantitative model evaluation

- ▶ The square-root process generates a Pareto tail in earnings
- ▶ The square-root process *can* generate dynamics of top inequality
- ▶ Can it generate *quantitatively* sufficiently fast dynamics of top inequality over the life cycle?
- ▶ *While* matching the covariance structure of earnings?

## Data:

- ▶ Norwegian male annual earnings data, age 35-50  
Sample as in Crawley et al. (2022)
- ▶ Baseline earnings measure: includes self-employment income  
Results are robust to excluding self-employment income

# Quantitative (discrete-time) models

## Square-root model

The square-root model is given by

$$y_t = z_t + \sigma \epsilon_t,$$

$$z_t = \max\{\theta \bar{z} + (1 - \theta)z_{t-1} + \Psi \sqrt{z_{t-1}} \nu_t, 0\}.$$

Max operator  $\rightsquigarrow z_t$  remains non-negative. Calibrated model: almost never binding.

## Canonical model

The canonical model is given by

$$y_t = z_t + \sigma \epsilon_t,$$

$$z_t = (1 - \theta)z_{t-1} + \Sigma \nu_t.$$

In both models,  $\epsilon_t$  and  $\nu_t$  are i.i.d. Guided by proposition:  $\nu_t \sim N(0, 1)$ . We will consider  $\epsilon_t \sim N(0, 1)$  and  $\epsilon_t \sim \text{Laplace}(0, 1/\sqrt{2})$ .

# Calibration

## Parameters: (4 each)

- ▶ Square-root model: transitory shock std  $\sigma$ , persistence  $\theta$ , and  $\bar{z}$ ,  $\Psi$  determining the shock process for persistent earnings, initial permanent earnings top ineq
- ▶ Canonical model: transitory shock std  $\sigma$ , persistence  $\theta$ , and persistent shock std  $\Sigma$ , initial permanent earnings variance and top ineq

## Target moments:

- ▶ Variance covariance structure of log earnings from age 35-50
- ▶ Age-specific Pareto tail of earnings MLE of Pareto tail on top 10% of earners
- ▶ Empirical proxy for state-dependency of earnings risk  
reg coef  $b$  of  $(\Delta y_{it})^2 \sim a + by_{it-1}$  for above median earners

**Calibration details:** Minimize square distance between target moments and model moments. Ad hoc weights  $0.15^{-2}$ ,  $1.5^{-2}$ ,  $0.08^{-2}$  for the three sets of moments, to match the scale of figures to be shown. 100,000 simulated individuals, TikTak algorithm (Arnoud et al. 2022).

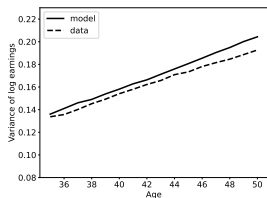
# Calibration Results

Model	Shocks	Objective	$\theta$	$\bar{z}$	$\Psi$	$\Sigma$	$\sigma$	Var( $z_0$ )	$z_0$ top ineq.
Square root	Laplace	0.125	0.015	1.451	0.110		0.162	0.110	0.173
Square root	Normal	0.361	0.014	1.495	0.106		0.197	0.108	0.172
Canonical	Laplace	0.539	0.001			0.074	0.174	0.097	0.246
Canonical	Normal	0.751	0.001			0.070	0.215	0.098	0.233

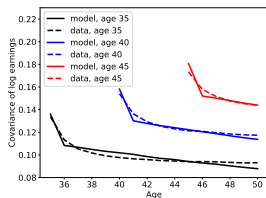
Table 1: Estimation results for the square-root model and the canonical model, using Laplace distributed and normally distributed transitory shocks.

- ▶ High persistence of permanent earnings  $\theta \approx 0$  in line with literature, transitory shock std  $\sigma$  in line with literature E.g., Crawley et al. (2022)
- ▶ Parameter  $\bar{z}$ :  $\bar{z} \rightarrow \infty \rightsquigarrow$  canonical model with log-normal earnings distribution,  $\bar{z} = \frac{\Psi^2}{2\theta} \approx 0.4 \rightsquigarrow$  Pareto distributed earnings
  - ▶ Calibrated value of  $\bar{z}$ : in between, bell-shaped distribution with Pareto tail
- ▶ Laplace distributed transitory shocks help match the proxy for state-dependency of earnings risk

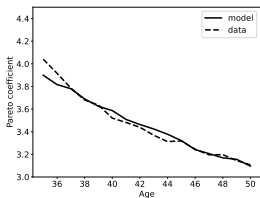
# Model fit – Square-root model



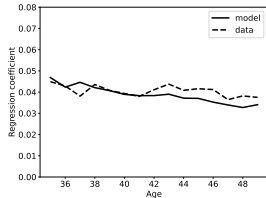
(a) The cross-sectional variance of log earnings for data and for the estimated model.



(b) The covariance of log earnings for data and for the estimated model.



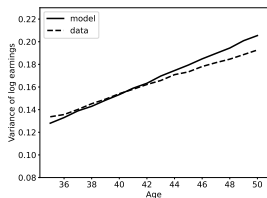
(c) The age-specific Pareto coefficient, computed by maximum likelihood on the top ten percent, for data and for the estimated model.



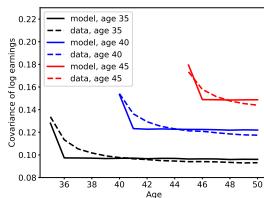
(d) The age-specific regression coefficient  $b$  for the regression  $(\Delta y_{it+1})^2 \sim a + by_{it}$ , for data and for the estimated model.

Figure 2: Model fit for the square-root model with Laplace distributed transitory shocks.

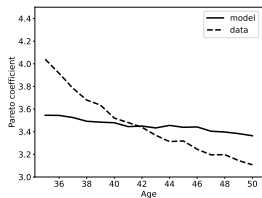
# Model fit – Canonical model



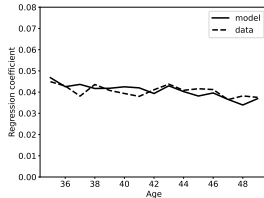
(a) The cross-sectional variance of log earnings for data and for the estimated model.



(b) The covariance of log earnings for data and for the estimated model.



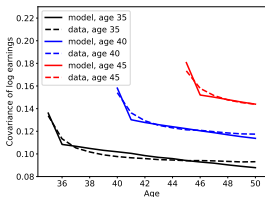
(c) The age-specific Pareto coefficient, computed by maximum likelihood on the top ten percent, for data and for the estimated model.



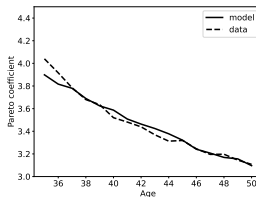
(d) The age-specific regression coefficient  $b$  for the regression  $(\Delta y_{it+1})^2 \sim a + by_{it}$ , for data and for the estimated model.

Figure 3: Model fit for the canonical model with Laplace distributed transitory shocks.

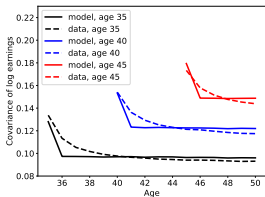
# Square-root model vs Canonical model



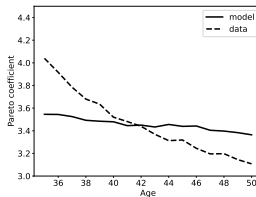
(a) The covariance of log earnings for the square-root model.



(b) The age-specific Pareto coefficient for the square-root model.



(c) The covariance of log earnings for the canonical model.



(d) The age-specific Pareto coefficient for the canonical model.

Figure 4: Model fit for the square-root model and the canonical model with Laplace distributed transitory shocks.

# Model fit

- ▶ The square-root process does not only match the variance-covariance moments, but also the dynamics of top earnings inequality, characterized by the Pareto tail coefficient (parameter values within the literature).
- ▶ The canonical model cannot match the substantial increase in top-earnings inequality over the life cycle. It yields a Pareto tail coefficient that is essentially constant over time.

# Identification heuristics

## Law of motion for earnings inequality:

$$d\eta_t = (\Psi^2/2 - \theta\eta_t)dt.$$

## Permanent earnings dynamics:

- ▶ Inequality steadily increasing over life cycle  $\rightsquigarrow \theta$  close to 0
- ▶ Parameter  $\Psi$ : rate of increase in both the variance of log earnings and decrease in the tail coefficient
- ▶ Parameter  $\bar{z}$ : initial relative level of variance vs tail coefficient

## Transitory shock dynamics:

- ▶ Parameter  $\sigma$ : drop in covariance from contemp. to future earnings
- ▶ Laplace distributed transitory shocks: match the proxy for state-dependency of earnings risk

Taking stock

# Taking stock

The square-root model is a one-step generalization of the canonical model which

- ▶ Retains an excellent fit to the covariance structure of earnings
- ▶ Generates a Pareto tail in earnings
- ▶ Generates dynamics of top-earnings inequality over the life cycle
- ▶ Remains parsimonious with only one state variable

The model is thus ready to be deployed for quantitative analyses where top-earnings inequality plays a role. Examples:

- ▶ Savings behavior of the rich: Fagereng et al. (2019); Straub (2019)
- ▶ Optimal taxation of top earners: Guner et al. (2016); Badel et al. (2020); Kindermann and Krueger (2022)

Focused on life cycle today. Future work: quantitatively applying the framework to the rise in US top-earnings inequality since the 1980s.

Thank you

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# Earnings process: dynamics

## Proof.

1. Guess that the cross-sectional distribution of log earnings at a given point in time follows a gamma distribution

$$f_t(y) = \frac{(\eta_t)^{-n}}{\Gamma(n)} y^{n-1} \exp(-y/\eta_t)$$

with  $n = \frac{2\theta\bar{y}}{\sigma^2}$  where  $\eta_t$  evolves according to

$$d\eta_t = (\sigma^2/2 - \theta\eta_t)dt.$$

2. Derive the Kolmogorov forward equation. Plugging in  $f_t(y)$

$$\frac{df_t}{dt} \frac{1}{f_t} = \underbrace{\left(-\frac{n}{\eta_t} + \frac{y}{\eta_t^2}\right)}_{\frac{df_t}{d\eta_t} \frac{1}{f_t}} \underbrace{\left(\frac{\sigma^2}{2} - \theta\eta_t\right)}_{\frac{d\eta_t}{dt}}.$$

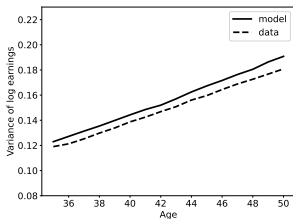
3. Therefore

$$\frac{df_t}{dt} = \frac{df_t}{d\eta_t} \frac{d\eta_t}{dt}$$

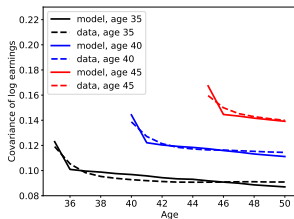
when  $\frac{d\eta_t}{dt} = \frac{\sigma^2}{2} - \theta\eta_t$



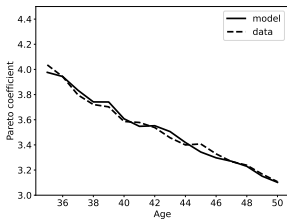
# Wage income results



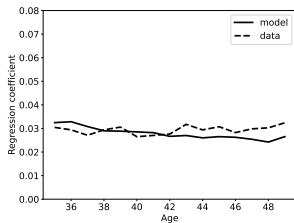
(a) Variance of log earnings



(b) Covariance of log earnings

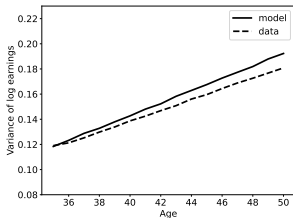


(c) Age-specific Pareto coefficient

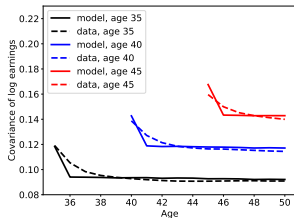


(d) State-dependency of earnings risk

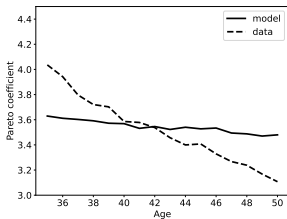
# Wage income results (canonical model)



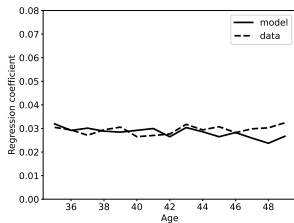
(a) Variance of log earnings



(b) Covariance of log earnings



(c) Age-specific Pareto coefficient



(d) State-dependency of earnings risk

# Wage income results (calibration)

Model	Shocks	Objective	$\theta$	$\bar{z}$	$\Psi$	$\Sigma$	$\sigma$	Var( $z_0$ )	$z_0$ top ineq.
Square root	Laplace	0.108	0.011	1.713	0.105		0.144	0.102	0.176
Square root	Normal	0.268	0.010	1.781	0.106		0.167	0.100	0.180
Canonical	Laplace	0.459	0.002			0.073	0.155	0.094	0.242
Canonical	Normal	0.650	0.000			0.070	0.196	0.092	0.255

- ▶ Similar calibration results
- ▶ Slightly better objective function value using wage income as our measure of earnings